INTERNATIONAL JOURNAL OF
NUMERICAL ANALYSIS AND MODELING
Volume 10, Number 1, Pages 154–177

A PRIORI ERROR ESTIMATES FOR SEMI-DISCRETE DISCONTINUOUS GALERKIN METHODS SOLVING NONLINEAR HAMILTON-JACOBI EQUATIONS WITH SMOOTH SOLUTIONS

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Abstract. In this paper, we provide a priori L^2 error estimates for the semi-discrete discontinuous Galerkin method [3] and the local discontinuous Galerkin method [22] for one- and two-dimensional nonlinear Hamilton-Jacobi equations with smooth solutions. With a special Gauss-Radau projection, the optimal error estimates on rectangular meshes are obtained.

Key words. Hamilton-Jacobi equations, discontinuous Galerkin method, local discontinuous Galerkin method, a priori error estimates.

1. Introduction

In this paper, we are interested in the a priori L^2 error estimates of the semi-discrete discontinuous Galerkin (DG) and local discontinuous Galerkin (LDG) methods for smooth solutions of nonlinear Hamilton-Jacobi (HJ) equations in the one-dimensional case

$$\phi_t + H(\phi_x, x) = 0, \qquad \phi(x,0) = \phi^0(x) \tag{1}$$

and in the two-dimensional case:

$$\phi_t + H(\phi_x, \phi_y, x, y) = 0, \qquad \phi(x,y,0) = \phi^0(x,y). \tag{2}$$

The Hamiltonian H is assumed to be a smooth function of all the arguments. When there is no ambiguity, we also take the concise notation $H(\phi_x) = H(\phi_x, x)$ and $H(\phi_x, \phi_y) = H(\phi_x, \phi_y, x, y)$.

The DG method is a class of finite element methods using completely discontinuous piecewise polynomial space for the numerical solution in the spatial variables. It can be discretized in time by the explicit and nonlinearly stable high order Runge-Kutta time discretization [20], resulting in the so-called RKDG method. The RKDG method was first developed for nonlinear hyperbolic conservation laws by Cockburn et al. in [8, 7, 5, 9]. Later it was generalized to the LDG method for solving convection-diffusion equations by Cockburn and Shu [10].

The time-dependent Hamilton-Jacobi (HJ) equations (1) and (2) are closely related to the conservation laws. In the one-dimensional case, they are equivalent if one takes the spatial derivative in (1) and writes out the equation satisfied by $u = \phi_x$. It is thus not surprisingly that many successful numerical methods for the conservation laws have been adapted to solve the Hamilton-Jacobi equations. For finite difference schemes, the high order essentially non-oscillatory (ENO) and weighted ENO (WENO) schemes [18, 14, 25] are such examples. However, it is less straightforward to adapt DG schemes to solve the Hamilton-Jacobi equations, since

Received by the editors December 30, 2011.

2000 *Mathematics Subject Classification.* 65N30.

Research partially supported by DOE grant DE-FG02-08ER25863, NSF grant DMS-1112700 and NSFC grants 11071234 and 91130016.

the nonlinear Hamiltonian H prevents a direct integration by parts. Hu and Shu developed a DG scheme [13] for solving the nonlinear Hamilton-Jacobi equations, which is based on the Runge-Kutta discontinuous Galerkin (RKDG) method for solving conservation laws. They first solve the conservation law equation satisfied by $u = \phi_x$ with the standard DG method, which can determine ϕ for each element up to a constant, and then the missing constant is obtained by integration either in time or from the boundary. In two dimensions, this scheme involves a least square procedure to obtain ϕ from the numerical approximations of $u = \phi_x$ and $v = \phi_y$, as they may not satisfy the compatibility condition $u_y = v_x = \phi_{xy}$. Later, Li and Shu [16] reinterpreted the method in [13] by using a curl-free subspace for the discontinuous Galerkin method in the two-dimensional case to avoid the least squares procedure. The two algorithms in [13] and [16] are mathematically equivalent, however the latter avoids the least square procedure and also uses a smaller finite element space, resulting in a significant simplification in implementation with a reduced cost. The DG scheme in [13] achieves the optimal k-th order of accuracy for $u = \phi_x$ (and also $v = \phi_y$ in two dimensions), however the optimal $(k+1)$-th order accuracy for ϕ is not always observed numerically when k-th degree piecewise polynomial space is used. For the one-dimensional case, the error estimates for conservation laws in [23, 21, 24] can be directly applied, yielding k-th order error accuracy for the upwind fluxes and $(k-\frac{1}{2})$-th order error accuracy for general numerical fluxes for the derivative $u = \phi_x$ when k-th degree piecewise polynomial space is used. For the two-dimensional case, we can follow the a priori error estimates for $u = \phi_x$ and $v = \phi_y$ in the DG curl-free subspace, however only $(k-\frac{1}{2})$-th order accuracy can be obtained either for the upwind fluxes or for general fluxes, since the special projections need for the optimal error estimates in two dimensions cannot be defined in the curl-free subspace.

More recently, Cheng and Shu in [3] proposed a DG method for directly solving Hamilton-Jacobi equations without going through the derivatives $u = \phi_x$ and $v = \phi_y$. Also, Yan and Osher [22] designed a direct LDG method for solving Hamilton-Jacobi equations. Numerically, optimal order error accuracy has been observed for both of these two methods. For linear Hamiltonians, the DG and LDG methods in [3] and [22] are equivalent to those for solving conservation laws, hence stability and error estimates can be obtained following the techniques for conservation laws. However, for nonlinear Hamiltonians, the methods in [3] and [22] are distinct from the DG methods for conservation laws. In this paper, we follow and generalize the techniques in [23, 21, 24] to obtain a priori L^2 error estimates for the DG and LDG methods in [3] and [22] for directly solving nonlinear Hamilton-Jacobi equations with smooth solutions.

The paper is organized as follows. In Section 2, we introduce notations, definitions and auxiliary results used later in this paper. In Section 3, we obtain a priori error estimates for the one-dimensional Hamilton-Jacobi equations. In Section 4, we follow the same line as the one-dimensional case to obtain a priori error estimates for the two-dimensional Hamilton-Jacobi equations. Concluding remarks are given in Section 5.

2. Notations, definitions and auxiliary results

In this section, we follow [21, 24] to first introduce notations and definitions to be used later in this paper and also present some auxiliary results. We use a special Gauss-Radau projection as in [24], and present certain interpolation and inverse properties for the finite element spaces that will be used in the error analysis.

2.1. Basic notations.

2.1.1. One-dimensional case. We consider the one-dimensional Hamilton-Jacobi equation in the interval $I = (0, 1)$, which is divided into N cells as $0 = x_{\frac{1}{2}} < x_{\frac{3}{2}} < ... < x_{N+\frac{1}{2}} = 1$. We set $I_j = (x_{j-\frac{1}{2}}, x_{j+\frac{1}{2}})$, $\bar{I}_j = [x_{j-\frac{1}{2}}, x_{j+\frac{1}{2}}]$, $x_j = \frac{1}{2}(x_{j-\frac{1}{2}} + x_{j+\frac{1}{2}})$, $h_j = x_{j+\frac{1}{2}} - x_{j-\frac{1}{2}}$ and $\bar{I}_{j+\frac{1}{2}} = [x_j, x_{j+1}]$, for $j = 1, ..., N$, and we define the quantities

$$h = \max_{1\le j\le N} h_j \qquad \text{and} \qquad \rho = \min_{1\le j\le N} h_j \tag{3}$$

We assume the mesh is regular, namely the ratio of h over ρ stays bounded by a fixed positive constant ν^{-1} during mesh refinements, that is $\nu h \le \rho \le h$. The piecewise-polynomial approximation space is

$$V_h = \{v : v|_{I_j} \in P^k(I_j), j = 1, ..., N\} \tag{4}$$

where $P^k(I_j)$ denotes all polynomials of degree at most k on I_j.

The numerical solution is denoted by $\phi_h \in V_h$. As usual, we denote by $(\phi_h)^+_{j+\frac{1}{2}}$ and $(\phi_h)^-_{j+\frac{1}{2}}$ the values of ϕ_h at $x_{j+\frac{1}{2}}$ from the right cell I_{j+1} and the left cell I_j, respectively. We also use the notations $[\phi_h] = \phi_h^+ - \phi_h^-$ and $\bar{\phi}_h = \frac{1}{2}(\phi_h^+ + \phi_h^-)$ to denote the jump and the mean of the function ϕ_h at each element boundary point, respectively. Finally, we denote by $H_1(\phi_x, x) = \frac{\partial H}{\partial \phi_x}(\phi_x, x)$ and $H_{11}(\phi_x, x) = \frac{\partial^2 H}{\partial^2 \phi_x}(\phi_x, x)$, the first and second derivatives of H with respect to its first argument, respectively.

2.1.2. Two-dimensional case. We consider the two-dimensional Hamilton-Jacobi equation on the domain Ω. For a rectangular partition of $I \times J = [0, L_x] \times [0, L_y]$, we denote the mesh by $I_i \times J_j$ with $I_i = (x_{i-\frac{1}{2}}, x_{i+\frac{1}{2}})$ and $J_j = (y_{j-\frac{1}{2}}, y_{j+\frac{1}{2}})$, for $i = 1, ..., N_x$ and $j = 1, ..., N_y$. We set $x_i = \frac{1}{2}(x_{i-\frac{1}{2}} + x_{i+\frac{1}{2}})$ and $y_j = \frac{1}{2}(y_{j-\frac{1}{2}} + y_{j+\frac{1}{2}})$. The cell lengths are denoted by $h_i^x = x_{i+\frac{1}{2}} - x_{i-\frac{1}{2}}$ and $h_j^y = y_{j+\frac{1}{2}} - y_{j-\frac{1}{2}}$ with $h^x = \max_{1\le i\le N_x} h_i^x$, $h^y = \max_{1\le j\le N_y} h_j^y$ and $h = \max(h^x, h^y)$. We also assume the mesh is regular as in the one-dimensional case.

We define the space Z_h as the space of tensor product piecewise polynomials of degree at most k in each variable on every element, i.e.

$$Z_h = \{v : v \in Q^k(I_i \times J_j), \forall (x, y) \in I_i \times J_j, i = 1, ..., N_x, j = 1, ..., N_y\} \tag{5}$$

where Q^k is the space of tensor products of one-dimensional polynomials of degree up to k.

We denote by $(\phi_h)^+_{i+\frac{1}{2},y}$ and $(\phi_h)^-_{i+\frac{1}{2},y}$ the values of ϕ_h at $(x_{i+\frac{1}{2}}, y)$ from the right cell $I_{i+1} \times J_j$ and from the left cell $I_i \times J_j$, respectively, when $y \in J_j$ on all vertical edges. Also $[\phi_h]_{i+\frac{1}{2},y} = (\phi_h)^+_{i+\frac{1}{2},y} - (\phi_h)^-_{i+\frac{1}{2},y}$ and $(\bar{\phi}_h)_{i+\frac{1}{2},y} = \frac{1}{2}((\phi_h)^+_{i+\frac{1}{2},y} + (\phi_h)^-_{i+\frac{1}{2},y})$ denote the jump and the mean of the function ϕ_h at $(x_{i+\frac{1}{2}}, y)$ when $y \in J_j$. Similarly, we can define $(\phi_h)^+_{x,j+\frac{1}{2}}$, $(\phi_h)^-_{x,j+\frac{1}{2}}$, $[\phi_h]_{x,j+\frac{1}{2}}$ and $(\bar{\phi}_h)_{x,j+\frac{1}{2}}$. We denote by $H_1(\phi_x, \phi_y, x, y) = \frac{\partial H}{\partial \phi_x}(\phi_x, \phi_y, x, y)$ and $H_{11}(\phi_x, \phi_y, x, y) = \frac{\partial^2 H}{\partial^2 \phi_x}(\phi_x, \phi_y, x, y)$ the first and second derivatives of H with respect to its first argument, respectively. Similarly, we define $H_2(\phi_x, \phi_y, x, y) = \frac{\partial H}{\partial \phi_y}(\phi_x, \phi_y, x, y)$ and $H_{22}(\phi_x, \phi_y, x, y) = \frac{\partial^2 H}{\partial^2 \phi_y}(\phi_x, \phi_y, x, y)$. The mixed derivative is $H_{12}(\phi_x, \phi_y, x, y) = \frac{\partial^2 H}{\partial \phi_x \partial \phi_y}(\phi_x, \phi_y, x, y)$.

For an arbitrary unstructured triangulation, let $\mathbb{T}_h$ denote a tessellation of Ω with shape-regular elements K. Let Γ_h denotes the union of the boundary faces of elements $K \in \mathbb{T}_h$, i.e., $\Gamma_h = \cup_{K\in\mathbb{T}_h} \partial K$, and $\Gamma_0 = \Gamma_h \setminus \partial\Omega$. Let $P^k(K)$ be the space of polynomials of degree at most $k \geq 0$ on $K \in \mathbb{T}_h$. We denote the finite element space by

$$W_h = \{v : v \in P^k(K) \text{ for } (x,y) \in K, \forall K \in \mathbb{T}_h\} \tag{6}$$

Let e be an edge shared by the elements K and K'. Define the unit normal vectors n and n' on e pointing exterior to K and K', respectively. If ϕ_h is a function on K and K', but possibly discontinuous across e, let $(\phi_h)^{int_K}$ denote $((\phi_h)|_K)|_e$ and $(\phi_h)^{ext_K}$ denote $((\phi_h)|_{K'})|_e$.

2.2. Projection and interpolation properties.

2.2.1. One-dimensional case. We will consider the Gauss-Radau projection $\mathbb{R}_h$ that projects $\phi(x,t) \in L^2(0,1)$ into the finite element space V_h, which depends on the function $\phi(x,t)$ itself and is defined in each element as:

$$\mathbb{R}_h = \begin{cases} \mathbb{R}_h^+ & \text{if } H_1(\partial_x\phi) > 0 \text{ in the element } I_j, \\ \mathbb{R}_h^- & \text{if } H_1(\partial_x\phi) < 0 \text{ in the element } I_j, \\ \mathbb{P}_h & \text{if } H_1(\partial_x\phi) \text{ changes its sign on the element } \bar{I}_j. \end{cases} \tag{7}$$

where the L^2-projection $\mathbb{P}_h$ on the element I_j is,

$$\int_{I_j} (\mathbb{P}_h\phi(x) - \phi(x))v_h(x)dx = 0, \forall v_h \in P^k(I_j), \tag{8}$$

and the projection $\mathbb{R}_h^{\pm}$ on the element I_j is,

$$\int_{I_j} (\mathbb{R}_h^-\phi(x) - \phi(x))v_h(x)dx = 0, \forall v_h \in P^{k-1}(I_j), \tag{9}$$

with $\mathbb{R}_h^-\phi(x^+_{j-\frac{1}{2}}) - \phi(x^+_{j-\frac{1}{2}}) = 0$,

$$\int_{I_j} (\mathbb{R}_h^+\phi(x) - \phi(x))v_h(x)dx = 0, \forall v_h \in P^{k-1}(I_j), \tag{10}$$

with $\mathbb{R}_h^+\phi(x^-_{j+\frac{1}{2}}) - \phi(x^-_{j+\frac{1}{2}}) = 0$.

Denote by $\eta = \mathbb{R}_h\phi - \phi$ the projection error, by a standard scaling argument, for this projection, we can obtain [2, 4, 15]

$$\|\eta\| + h\|\eta_x\| + h^{1/2}\|\eta\|_{\Gamma_h} \leq Ch^{k+1} \tag{11}$$

and it follows from Sobolev's inequality that

$$\|\eta\|_\infty \leq Ch^{k+\frac{1}{2}}. \tag{12}$$

The positive constant C, solely depending on ϕ, is independent of h. $\|\cdot\|$ and $\|\cdot\|_\infty$ are the usual L^2-norm and L^∞-norm in Sobolev spaces, respectively. Γ_h is the union of all element interface points, and the L^2-norm on Γ_h is defined by

$$\|\eta\|_{\Gamma_h} = \left[\sum_{1\leq j\leq N} \left((\eta^+_{j+\frac{1}{2}})^2 + (\eta^-_{j+\frac{1}{2}})^2\right)\right]^{1/2} \tag{13}$$

From the projection (7), for each partition $\{I_j\}$ of I, we can accumulate the elements with the same projection into three classes, i.e., S_1 denotes the class of cells with projection $\mathbb{R}_h^+$, S_2 denotes the class of cells with projection $\mathbb{R}_h^-$ and S_3 denotes the class of cells with projection $\mathbb{P}_h$.

2.2.2. Two-dimensional case. For two-dimensional problems on Cartesian meshes, we also use the Gauss-Radau projection, which is the tensor product of the projections in the one-dimensional case. On a rectangle $I \times J = [0, L_x] \times [0, L_y]$, the projection is defined to be

$$\mathscr{P}\phi = (\mathbb{R}_h)_x \otimes (\mathbb{R}_h)_y \phi \tag{14}$$

where the subscripts indicate the application of the one-dimensional projection $\mathbb{R}_h$ with respect to the corresponding variable.

Denote by $\eta = \mathscr{P}\phi - \phi$ the projection error, there holds the similar approximation results [6, 12]

$$\|\eta\| + h\|\eta_x\| + h^{1/2}\|\eta\|_{\Gamma_h} \le Ch^{k+1} \tag{15}$$

where Γ_h denotes the set of intercell boundaries of all elements $I_i \times J_j$.

2.3. Notations for different constants. We will adopt the following convention for different constants. These constants may have a different value in each occurrence.

We will denote by C a positive constant independent of h, which may depend on the exact solution of the PDE. Especially, in the following we will denote by $\mathbb{C}$ a positive constant only for the a priori assumption. For problems considered in this paper, the exact solution is assumed to be smooth with periodic or compactly supported boundary condition. Also, $0 \le t \le T$ for a fixed T. Therefore, the exact solution is always bounded.

2.4. Inverse properties. Finally, we list some inverse properties of the finite element space V_h. For any $v_h \in V_h$, there exits a positive constant C, independent of h, such that

(16)

$$\text{(i) } \|(v_h)_x\| \le Ch^{-1}\|v_h\|, \quad \text{(ii) } \|v_h\|_{\Gamma_h} \le Ch^{-\frac{1}{2}}\|v_h\|, \quad \text{(iii) } \|v_h\|_\infty \le Ch^{-\frac{n}{2}}\|v_h\|$$

where $n = 1$ or 2 is the spatial dimension. More details of the inverse properties can be found in [4].

3. Error estimates for the HJ equations

3.1. One-dimensional case.

3.1.1. DG scheme for directly solving the HJ equations. In this section, we describe the DG scheme in [3] for directly solving the one-dimensional Hamilton-Jacobi equations. Here the DG scheme is formulated as: find $\phi_h(x,t) \in V_h$, such that for any $v_h \in V_h$,

$$\begin{aligned} &\int_{I_j} (\partial_t \phi_h(x,t) + H(\partial_x \phi_h(x,t), x)) v_h(x) dx \\ &\quad + \frac{1}{2}\Big(\min_{x \in I_{j+\frac{1}{2}}} H_1(\partial_x \phi_h, x_{j+\frac{1}{2}}) - \Big| \min_{x \in I_{j+\frac{1}{2}}} H_1(\partial_x \phi_h, x_{j+\frac{1}{2}}) \Big| \Big) [\phi_h]_{j+\frac{1}{2}} (v_h)^-_{j+\frac{1}{2}} \\ &\quad + \frac{1}{2}\Big(\max_{x \in I_{j-\frac{1}{2}}} H_1(\partial_x \phi_h, x_{j-\frac{1}{2}}) + \Big| \max_{x \in I_{j-\frac{1}{2}}} H_1(\partial_x \phi_h, x_{j-\frac{1}{2}}) \Big| \Big) [\phi_h]_{j-\frac{1}{2}} (v_h)^+_{j-\frac{1}{2}} \\ &= 0, \qquad j = 1, ..., N \end{aligned} \tag{17}$$

Noticing that, when taking the maximum or minimum, the scheme needs the reconstructed information of $\partial_x \phi_h$ on the cells $\bar{I}_{j-\frac{1}{2}}$ and $\bar{I}_{j+\frac{1}{2}}$, which contains the points $x_{j-\frac{1}{2}}$ and $x_{j+\frac{1}{2}}$ respectively, where the numerical solution $\phi_h(x,t)$ is discontinuous. A polynomial $w_{j+\frac{1}{2}}(x) \in P^{2k+1}$ is defined on $I_j \cup I_{j+1}$, such that

$$\int_{I_j} \phi_h v dx = \int_{I_j} w_{j+\frac{1}{2}} v dx \tag{18}$$

for any $v \in P^k$ on I_j, and

$$\int_{I_{j+1}} \phi_h v dx = \int_{I_{j+1}} w_{j+\frac{1}{2}} v dx \tag{19}$$

for any $v \in P^k$ on I_{j+1}. Then we use $\partial_x \phi_h = \partial_x w_{j+\frac{1}{2}}$ on $\bar{I}_{j+\frac{1}{2}}$ when taking the maximum or minimum in the scheme (17).

3.1.2. LDG scheme for directly solving the HJ equations. In this section, we describe the LDG scheme in [22] for directly solving the Hamilton-Jacobi equations. The LDG scheme is defined as follows: find $\phi_h \in V_h$ such that for all test function $u \in V_h$, we have

$$\int_{I_j} (\phi_h)_t u dx + \int_{I_j} \hat{H}(p_1, p_2) u dx = 0, \qquad j = 1, ..., N \tag{20}$$

where p_1 and p_2 are two auxiliary variables to approximate ϕ_x and $\hat{H}(p_1, p_2)$ is a monotone consistent numerical Hamiltonian [11] chosen to approximate $H(\phi_x)$. For any test function $v \in V_h$, $p_1 \in V_h$ is obtained by solving the following right upwind DG scheme

$$\int_{I_j} p_1 v dx + \int_{I_j} \phi_h v_x dx - (\phi_h)^+_{j+\frac{1}{2}} v^-_{j+\frac{1}{2}} + (\phi_h)^+_{j-\frac{1}{2}} v^+_{j-\frac{1}{2}} = 0 \tag{21}$$

and for any test function $w \in V_h$, $p_2 \in V_h$ is obtained by solving the following left upwind DG scheme

$$\int_{I_j} p_2 w dx + \int_{I_j} \phi_h w_x dx - (\phi_h)^-_{j+\frac{1}{2}} w^-_{j+\frac{1}{2}} + (\phi_h)^-_{j-\frac{1}{2}} w^+_{j-\frac{1}{2}} = 0. \tag{22}$$

We can use the simple Lax-Friedrichs numerical Hamiltonian

$$\hat{H}(p_1, p_2) = H\left(\frac{p_1 + p_2}{2}\right) - \frac{1}{2}\alpha(p_1 - p_2) \tag{23}$$

with $\alpha = \max_{p \in D} \left|\frac{\partial H(p)}{\partial p}\right|$. When D is taken as a local domain, which is evaluated locally as $D = [\min(p_1, p_2), \max(p_1, p_2)]|_{I_j}$, it is called a local Lax-Friedrichs Hamiltonian. With D taken as a global domain, namely D is evaluated over the whole computational domain and defined as $D = [\min(p_1, p_2), \max(p_1, p_2)]|_{\Omega}$, it is called a global Lax-Friedrichs Hamiltonian.

3.1.3. The main results. We state the main error estimates of the semi-discrete DG scheme (17) and the semi-discrete LDG scheme (20)-(22). Detailed proof will be given in the subsequent subsections.

Theorem 3.1. *Let ϕ be the exact solution of the problem (1), which is sufficiently smooth with bounded derivatives, and assume $H(\partial_x \phi, x) \in C^2$. Let ϕ_h be the numerical solution of the semi-discrete DG scheme (17) or the semi-discrete LDG scheme (20)-(22), and denote the corresponding numerical error by $e_\phi = \phi - \phi_h$.*

For regular triangulations of $I = (0,1)$, if the finite element space V_h is the piecewise polynomial space of degree $k \geq 2$, then for small enough h, there holds the following optimal error estimate

$$\|\phi - \phi_h\| \leq Ch^{k+1} \tag{24}$$

where the positive constant C depends on the final time T, k, $\|\phi\|_{L^\infty((0,T),H^{k+1}(I))}$ and the bounds on the m-th derivatives of $H(\partial_x\phi, x)$ with respect to its first argument, $m = 1, 2$. $\|\phi\|_{L^\infty((0,T),H^{k+1}(I))}$ is the maximum over $0 \leq t \leq T$ of the standard Sobolev $(k+1)$-norm in space.

3.1.4. Proof of Theorem 3.1 for the DG scheme. The cell error equation for this scheme is

$$\begin{aligned}
&\int_{I_j} \partial_t(\phi(x,t) - \phi_h(x,t))v_h(x)dx + \int_{I_j} (H(\partial_x\phi(x,t),x) - H(\partial_x\phi_h(x,t),x))v_h(x)dx \\
&\quad - \frac{1}{2}\Big(\min_{x\in I_{j+\frac{1}{2}}} H_1(\partial_x\phi_h, x_{j+\frac{1}{2}}) - \Big|\min_{x\in I_{j+\frac{1}{2}}} H_1(\partial_x\phi_h, x_{j+\frac{1}{2}})\Big|\Big)[\phi_h]_{j+\frac{1}{2}}(v_h)^-_{j+\frac{1}{2}} \\
&\quad - \frac{1}{2}\Big(\max_{x\in I_{j-\frac{1}{2}}} H_1(\partial_x\phi_h, x_{j-\frac{1}{2}}) + \Big|\max_{x\in I_{j-\frac{1}{2}}} H_1(\partial_x\phi_h, x_{j-\frac{1}{2}})\Big|\Big)[\phi_h]_{j-\frac{1}{2}}(v_h)^+_{j-\frac{1}{2}} \\
&\quad = 0
\end{aligned} \tag{25}$$

We take the Taylor expansion on $H(\partial_x\phi_h, x)$ with respect to its first argument

$$H(\partial_x\phi_h, x) = H(\partial_x\phi, x) - H_1(\partial_x\phi, x)\partial_x(\phi - \phi_h) + \frac{1}{2}\bar{H}_{11}(\partial_x(\phi - \phi_h))^2 \tag{26}$$

where the second derivative $\bar{H}_{11}$ is evaluated at some point between $\partial_x\phi$ and $\partial_x\phi_h$. We would like to adopt the following a priori assumption

$$\|\partial_x e_\phi\|_\infty = \|\partial_x(\phi - \phi_h)\|_\infty \leq \mathbb{C}h. \tag{27}$$

The reasonableness of this a priori assumption will be justified later. Denoting $\phi - \phi_h = e_\phi = (\mathbb{R}_h\phi - \phi_h) - (\mathbb{R}_h\phi - \phi) = \xi - \eta$ and taking the test function $v_h = \xi$, we obtain

$$\begin{aligned}
&\int_{I_j} \xi_t\xi dx + \int_{I_j} H_1\xi\partial_x\xi dx + (H_{1min})_{j+\frac{1}{2}}[\xi]_{j+\frac{1}{2}}(\xi)^-_{j+\frac{1}{2}} + (H_{1max})_{j-\frac{1}{2}}[\xi]_{j-\frac{1}{2}}(\xi)^+_{j-\frac{1}{2}} \\
&\qquad = \int_{I_j} \eta_t\xi dx + \int_{I_j} H_1\xi\partial_x\eta dx + (H_{1min})_{j+\frac{1}{2}}[\eta]_{j+\frac{1}{2}}(\xi)^-_{j+\frac{1}{2}} \\
&\qquad + (H_{1max})_{j-\frac{1}{2}}[\eta]_{j-\frac{1}{2}}(\xi)^+_{j-\frac{1}{2}} + \frac{1}{2}\int_{I_j} \xi\bar{H}_{11}(\partial_x(e_\phi))^2 dx
\end{aligned} \tag{28}$$

where we have taken the short-hand notations

$$H_1 = H_1(\partial_x\phi, x) \tag{29}$$

$$(H_{1min})_{j+\frac{1}{2}} = \frac{1}{2}\Big(\min_{x\in I_{j+\frac{1}{2}}} H_1(\partial_x\phi_h, x_{j+\frac{1}{2}}) - \Big|\min_{x\in I_{j+\frac{1}{2}}} H_1(\partial_x\phi_h, x_{j+\frac{1}{2}})\Big|\Big) \tag{30}$$

$$(H_{1max})_{j+\frac{1}{2}} = \frac{1}{2}\Big(\max_{x\in I_{j+\frac{1}{2}}} H_1(\partial_x\phi_h, x_{j+\frac{1}{2}}) + \Big|\max_{x\in I_{j+\frac{1}{2}}} H_1(\partial_x\phi_h, x_{j+\frac{1}{2}})\Big|\Big) \tag{31}$$

when there is no confusion, and $\bar{H}_{11}$ is a mean value from the Taylor expansion. For the second term on the left side of (28), we have

$$\int_{I_j} H_1\xi\partial_x\xi dx = -\frac{1}{2}\int_{I_j}\xi^2\partial_x H_1 dx + \frac{1}{2}\Big((H_1\xi^2)^-_{j+\frac{1}{2}} - (H_1\xi^2)^+_{j-\frac{1}{2}}\Big) \tag{32}$$

and for the second term on the right side of (28), we have

$$\begin{aligned}\int_{I_j} H_1\xi\partial_x\eta dx &= -\int_{I_j}(\xi\partial_x H_1 + H_1\partial_x\xi)\eta dx + (H_1\eta\xi)^-_{j+\frac{1}{2}} - (H_1\eta\xi)^+_{j-\frac{1}{2}}\\ &= -\int_{I_j}\xi\eta\partial_x H_1 dx - \int_{I_j}(H_1(\partial_x\phi,x) - H_1(\partial_x\phi,x)\Big|_{x=x_j})\eta\partial_x\xi dx\\ &\quad + (H_1\eta\xi)^-_{j+\frac{1}{2}} - (H_1\eta\xi)^+_{j-\frac{1}{2}}\end{aligned} \tag{33}$$

where we have used the property that $\mathbb{R}_h\phi - \phi$ is locally orthogonal to all polynomials of degree up to $k-1$, so

$$\int_{I_j} H_1(\partial_x\phi,x)\Big|_{x=x_j}\eta\partial_x\xi dx = 0. \tag{34}$$

Now (28) can be rewritten as

$$\begin{aligned}\int_{I_j}\xi_t\xi dx = &\int_{I_j}\eta_t\xi dx + \frac{1}{2}\int_{I_j}\xi^2\partial_x H_1 dx - \int_{I_j}\xi\eta\partial_x H_1 dx\\ &- \int_{I_j}(H_1(\partial_x\phi,x) - H_1(\partial_x\phi,x)\Big|_{x=x_j})\eta\partial_x\xi dx - \frac{1}{2}(H_1\xi^2)^-_{j+\frac{1}{2}}\\ &+ (H_1\eta\xi)^-_{j+\frac{1}{2}} - (H_{1min})_{j+\frac{1}{2}}[\xi]_{j+\frac{1}{2}}(\xi)^-_{j+\frac{1}{2}} + (H_{1min})_{j+\frac{1}{2}}[\eta]_{j+\frac{1}{2}}(\xi)^-_{j+\frac{1}{2}}\\ &+ \frac{1}{2}(H_1\xi^2)^+_{j-\frac{1}{2}} - (H_1\eta\xi)^+_{j-\frac{1}{2}} - (H_{1max})_{j-\frac{1}{2}}[\xi]_{j-\frac{1}{2}}(\xi)^+_{j-\frac{1}{2}}\\ &+ (H_{1max})_{j-\frac{1}{2}}[\eta]_{j-\frac{1}{2}}(\xi)^+_{j-\frac{1}{2}} + \frac{1}{2}\int_{I_j}\xi\bar{H}_{11}(\partial_x(e_\phi))^2 dx.\end{aligned} \tag{35}$$

Summing over j from j_1 to j_2, the error equation (35) becomes

$$\begin{aligned}\sum_{j=j_1}^{j_2}\int_{I_j}\xi_t\xi dx = &\sum_{j=j_1}^{j_2}\int_{I_j}\eta_t\xi dx + \sum_{j=j_1}^{j_2}\frac{1}{2}\int_{I_j}\xi^2\partial_x H_1 dx - \sum_{j=j_1}^{j_2}\int_{I_j}\xi\eta\partial_x H_1 dx\\ &- \sum_{j=j_1}^{j_2}\int_{I_j}(H_1(\partial_x\phi,x) - H_1(\partial_x\phi,x)\Big|_{x=x_j})\eta\partial_x\xi dx\\ &+ \sum_{j=j_1}^{j_2}\Big(-\frac{1}{2}(H_1\xi^2)^- + (H_1\eta\xi)^- - H_{1min}[\xi]\xi^- + H_{1min}[\eta]\xi^-\Big)_{j+\frac{1}{2}}\\ &+ \sum_{j=j_1}^{j_2}\Big(\frac{1}{2}(H_1\xi^2)^+ - (H_1\eta\xi)^+ - H_{1max}[\xi]\xi^+ + H_{1max}[\eta]\xi^+\Big)_{j-\frac{1}{2}}\\ &+ \sum_{j=j_1}^{j_2}\frac{1}{2}\int_{I_j}\xi\bar{H}_{11}(\partial_x(e_\phi))^2 dx.\end{aligned} \tag{36}$$

Since $H_1(\partial_x\phi,x)$ is continuous with respect to its first argument, we have

$$\max_j\left|H_1(\partial_x\phi,x) - H_1(\partial_x\phi,x)\Big|_{x=x_j}\right|_{x\in I_j} \le Ch \tag{37}$$

then for the first two lines of the right side of (36), we obtain

$$
\begin{aligned}
&\sum_{j=j_1}^{j_2}\int_{I_j}\eta_t\xi dx+\sum_{j=j_1}^{j_2}\frac{1}{2}\int_{I_j}\xi^2\partial_x H_1 dx-\sum_{j=j_1}^{j_2}\int_{I_j}\xi\eta\partial_x H_1 dx\\
&\quad-\sum_{j=j_1}^{j_2}\int_{I_j}(H_1(\partial_x\phi,x)-H_1(\partial_x\phi,x)\Big|_{x=x_j})\eta\partial_x\xi dx\\
&\quad\le C\sum_{j=j_1}^{j_2}\Big(\int_{I_j}|\eta_t\xi|dx+\int_{I_j}\xi^2 dx+\int_{I_j}|\xi\eta|dx+h\int_{I_j}|\eta\partial_x\xi|dx\Big).
\end{aligned}\tag{38}
$$

By the a priori assumption (27), for the last line of the right side of (36), we obtain

$$
\begin{aligned}
&\sum_{j=j_1}^{j_2}\frac{1}{2}\int_{I_j}\xi\bar{H}_{11}(\partial_x(e_\phi))^2 dx\le C\|\partial_x e_\phi\|_\infty\sum_j\frac{1}{2}\int_{I_j}|\xi\partial_x(\xi-\eta)|dx\\
&\quad\le Ch\sum_{j=j_1}^{j_2}\Big(\int_{I_j}|\xi\partial_x\xi|dx+\int_{I_j}|\xi\partial_x\eta|dx\Big).
\end{aligned}\tag{39}
$$

Now for the boundary terms of the third and fourth lines on the right side of (36), denoted by T_{bry}, first we have that $H_1(\partial_x\phi,x)$ is continuous at each cell boundary, which is $H_1=(H_1)^-=(H_1)^+$. If $H_1(\partial_x\phi,x)$ changes its sign in the cell $\bar{I}_j$ of S_3, then it has at least one zero point x^* in the cell $\bar{I}_j$. By taking a Taylor expansion at this zero point x^*, we easily obtain that $|H_1(\partial_x\phi,x)|\le Ch$ in such $\bar{I}_j$.

If we assume $\{I_j\}_{j=j_1}^{j_2}$ is a sequence of maximum consecutive cells in S_1 and $\bar{I}_s=\cup_{j=j_1}^{j_2}\bar{I}_j$, then in the interior open set I_s, we have $H_1>0$, so that $H_{1min}=0$ and $\max_{j_1\le j\le j_2}|H_1-H_{1max}|_{j+\frac{1}{2}}\le Ch$ (The proof of this inequality is given in the appendix). The projection is $\mathbb{R}_h^+$. The two boundary points of $\bar{I}_s$, $x_{j_1-\frac{1}{2}}$ and $x_{j_2+\frac{1}{2}}$, should belong to the cells $\bar{I}_{j_1-1}$ and $\bar{I}_{j_2+1}$ of S_3 respectively, otherwise we have $I_s=I=(0,1)$. In the following, we only consider the more general case that $\bar{I}_{j_1-1}\in S_3$ and $\bar{I}_{j_2+1}\in S_3$. In this case, we have $\left|H_1(\partial_x\phi,x)\Big|_{x=x_{j_1-\frac{1}{2}}}\right|\le Ch$ and

$\left| H_1(\partial_x \phi, x)\Big|_{x=x_{j_2+\frac{1}{2}}} \right| \le Ch$, and we can obtain

$$\begin{aligned}
T_{bry} =& \sum_{j=j_1}^{j_2-1} \Big(H_1(-\frac{1}{2}(\xi^2)^- + \frac{1}{2}(\xi^2)^+ - [\xi]\xi^+) \Big)_{j+\frac{1}{2}} \\
&+ \sum_{j=j_1}^{j_2-1} \Big(H_1((\eta\xi)^- - (\eta\xi)^+ + [\eta]\xi^+) \Big)_{j+\frac{1}{2}} \\
&+ (H_1)_{j_2+\frac{1}{2}} \Big(-\frac{1}{2}(\xi^2)^- + (\eta\xi)^- \Big)_{j_2+\frac{1}{2}} \\
&+ (H_1)_{j_1-\frac{1}{2}} \Big(\frac{1}{2}(\xi^2)^+ - (\eta\xi)^+ - [\xi]\xi^+ + [\eta]\xi^+ \Big)_{j_1-\frac{1}{2}} \\
&+ \sum_{j=j_1}^{j_2} \Big((H_{1max} - H_1)(-[\xi]\xi^+ + [\eta]\xi^+) \Big)_{j+\frac{1}{2}} \\
=& -\sum_{j=j_1}^{j_2-1} \frac{1}{2} \Big(H_1(\xi^+ - \xi^-)^2 \Big)_{j+\frac{1}{2}} + \sum_{j=j_1}^{j_2-1} \Big(H_1(\eta^-\xi^- - \eta^-\xi^+) \Big)_{j+\frac{1}{2}} \\
&+ (H_1)_{j_2+\frac{1}{2}} \Big(-\frac{1}{2}(\xi^2)^- + (\eta\xi)^- \Big)_{j_2+\frac{1}{2}} \\
&+ (H_1)_{j_1-\frac{1}{2}} \Big(\frac{1}{2}(\xi^2)^+ - (\eta\xi)^+ - [\xi]\xi^+ + [\eta]\xi^+ \Big)_{j_1-\frac{1}{2}} \\
&+ \sum_{j=j_1}^{j_2} \Big((H_{1max} - H_1)(-[\xi]\xi^+ + [\eta]\xi^+) \Big)_{j+\frac{1}{2}} \\
\le& Ch \sum_{j=j_1-1}^{j_2} \Big((\xi^+)^2 + (\xi^-)^2 + (\eta^+)^2 + (\eta^-)^2 \Big)_{j+\frac{1}{2}} \qquad (40)
\end{aligned}$$

where for the last inequality in (40), we have used the special interpolating property of the projection of $\mathbb{R}_h^+$.

Combining (36), (38), (39) and (40), we have

$$\begin{aligned}
\sum_{j=j_1}^{j_2} \int_{I_j} \xi_t \xi dx \le& C \sum_{j=j_1}^{j_2} \Big(\int_{I_j} |\eta_t \xi| dx + \int_{I_j} \xi^2 dx + \int_{I_j} |\xi\eta| dx \Big) \\
&+ Ch \sum_{j=j_1}^{j_2} \Big(\int_{I_j} |\eta \partial_x \xi| dx + \int_{I_j} |\xi \partial_x \xi| dx + \int_{I_j} |\xi \partial_x \eta| dx \Big) \\
&+ Ch \sum_{j=j_1-1}^{j_2} \Big((\xi^+)^2 + (\xi^-)^2 + (\eta^+)^2 + (\eta^-)^2 \Big)_{j+\frac{1}{2}}. \qquad (41)
\end{aligned}$$

Similarly, if $\{I_j\}_{j=j_1}^{j_2}$ is a sequence of consecutive cells in S_2, and the two boundary points $x_{j_1-\frac{1}{2}}$ and $x_{j_2+\frac{1}{2}}$ of $\bar{I}_s = \cup_{j=j_1}^{j_2} \bar{I}_j$ also belong to the cells $\bar{I}_{j_1-1}$ and $\bar{I}_{j_2+1}$ of S_3 respectively, then in I_s, we have $H_1 < 0$, so that $H_{1max} = 0$ and $\max_j |H_1 - H_{1min}|_{j+\frac{1}{2}} \le Ch$ (similar to the proof in the appendix), and the projection is $\mathbb{R}_h = \mathbb{R}_h^-$. With the special property of $\mathbb{R}_h^-$, we have the same results as (41).

Now if $\{I_j\}_{j=j_1}^{j_2}$ is a sequence of consecutive cells in S_3, the estimates of (38) and (39) are still the same. For the boundary term, first we have

$$\left| H_1(\partial_x \phi, x)\Big|_{x=x_{j+\frac{1}{2}}} \right| \leq Ch, \text{ for } j = j_1 - 1, ..., j_2. \tag{42}$$

and so $|H_{1min}| \leq Ch$ and $|H_{1max}| \leq Ch$ in each $\bar{I}_{j+\frac{1}{2}}$ $(j_1 \leq j \leq j_2)$ (also similar to the proof in the appendix). We can then obtain

$$\begin{aligned} T_{bry} =& \sum_{j=j_1}^{j_2} \Big(-\frac{1}{2}(H_1\xi^2)^- + (H_1\eta\xi)^- - H_{1min}[\xi]\xi^- + H_{1min}[\eta]\xi^- \Big)_{j+\frac{1}{2}} \\ &+ \sum_{j=j_1}^{j_2} \Big(\frac{1}{2}(H_1\xi^2)^+ - (H_1\eta\xi)^+ - H_{1max}[\xi]\xi^+ + H_{1max}[\eta]\xi^+ \Big)_{j-\frac{1}{2}} \\ \leq& Ch \sum_{j=j_1-1}^{j_2} \Big((\xi^+)^2 + (\xi^-)^2 + (\eta^+)^2 + (\eta^-)^2 \Big)_{j+\frac{1}{2}}. \end{aligned} \tag{43}$$

In this case, we also have the same estimate as (41).

Since each case in the classes of S_1, S_2 and S_3 has the same estimate (41), summing over all the sequences of consecutive cells in the three nonoverlapping classes, we have

$$\begin{aligned} \frac{d}{dt}\|\xi\|^2 \leq& C\Big(\int_0^1 (|\eta_t\xi| + \xi^2 + |\xi\eta|)dx \Big) \\ &+ Ch\Big(\int_0^1 (|\eta\partial_x\xi| + |\xi\partial_x\xi| + |\xi\partial_x\eta|)dx \Big) \\ &+ Ch \sum_j \Big((\xi^+)^2 + (\xi^-)^2 + (\eta^+)^2 + (\eta^-)^2 \Big)_{j+\frac{1}{2}} \end{aligned} \tag{44}$$

using Schwarz inequality, the inverse property (16) and the interpolating property (11), we have

$$\begin{aligned} \frac{d}{dt}\|\xi\|^2 \leq& C(\|\xi\|^2 + \|\eta_t\|^2 + \|\eta\|^2) + Ch^2(\|\partial_x\xi\|^2 + \|\partial_x\eta\|^2) + Ch(\|\xi\|^2_{\Gamma_h} + \|\eta\|^2_{\Gamma_h}) \\ \leq& C(\|\xi\|^2 + h^{2k+2}) \end{aligned} \tag{45}$$

applying the Gronwall's inequality and the triangle inequality, we get the optimal error estimate for the DG scheme.

Remark 3.1. *When summing over all the sequences for (41), each boundary term has been counted at most twice. This does not affect the optimal error estimate. This remark also applied for the following LDG scheme.*

3.1.5. Proof of Theorem 3.1 for the LDG scheme. In this subsection, we give the proof for the local Lax-Friedrichs LDG scheme of Theorem 3.1. We use α_j to denote the local α in the cell I_j. We also would like to make an a priori assumption that, for small enough h, there holds

$$\left\| \partial_x\phi - \frac{p_1+p_2}{2} \right\|_\infty \leq \mathbb{C}h \tag{46}$$

and leave its justification to the next subsection. Denoting $\phi - \phi_h = e_\phi = (\mathbb{R}_h\phi - \phi_h) - (\mathbb{R}_h\phi - \phi) = \xi - \eta$ and $z_h = \partial_x\phi_h - \frac{p_1+p_2}{2} \in V_h$, for the term z_h, we have

Lemma 3.2. *With the interpolation property (11), we have*

$$\|z_h\| \le Ch^{-1}(\|\xi\| + h^{k+1}). \tag{47}$$

Proof. *We have*

$$\int_{I_j} \partial_x \phi_h \zeta dx + \int_{I_j} \phi_h \zeta_x dx - (\phi_h)^-_{j+\frac{1}{2}} \zeta^-_{j+\frac{1}{2}} + (\phi_h)^+_{j-\frac{1}{2}} \zeta^+_{j-\frac{1}{2}} = 0. \tag{48}$$

Taking $v = w$ in (21) and $\zeta = w$ in (48), and combining Eqns. (21), (22) and (48), we obtain

$$\int_{I_j} z_h w dx + \frac{1}{2}[\phi_h]_{j+\frac{1}{2}} w^-_{j+\frac{1}{2}} + \frac{1}{2}[\phi_h]_{j-\frac{1}{2}} w^+_{j-\frac{1}{2}} = 0. \tag{49}$$

Summing over j, we get

$$\sum_j \int_{I_j} z_h w dx = -\sum_j ([\phi_h]\bar{w})_{j+\frac{1}{2}}. \tag{50}$$

Taking $w = z_h$, from the inverse property (16), the equation becomes

$$\begin{aligned} \|z_h\|^2 &= -\sum_j ([\phi_h]\bar{z}_h)_{j+\frac{1}{2}} \\ &= \sum_j ([\phi - \phi_h]\bar{z}_h)_{j+\frac{1}{2}} \\ &\le C\|\xi - \eta\|_{\Gamma_h} \|z_h\|_{\Gamma_h} \\ &\le Ch^{-1}(\|\xi\| + h^{k+1})\|z_h\| \end{aligned} \tag{51}$$

so we have $\|z_h\| \le Ch^{-1}(\|\xi\| + h^{k+1})$.

Now we are going to follow the main procedure as the proof for the DG scheme to get the error estimates for the LDG scheme. We first assume $\{I_j\}_{j=j_1}^{j_2}$ is a sequence of consecutive cells in S_1, with the two boundary points $x_{j_1-\frac{1}{2}}$ and $x_{j_2+\frac{1}{2}}$ of $\bar{I}_s = \cup_{j=j_1}^{j_2} \bar{I}_j$ belonging to the cells $\bar{I}_{j_1-1}$ and $\bar{I}_{j_2+1}$ of S_3 respectively. Then we have $H_1 > 0$ in I_s, so that $\|\alpha_j - H_1\|_\infty \le Ch$ in each cell $I_j \subset I_s$, and we have the following cell error equations:

$$\int_{I_j} (\phi - \phi_h)_t u dx + \int_{I_j} (H(\partial_x \phi) - \hat{H}(p_1, p_2)) u dx = 0, \tag{52}$$

$$\int_{I_j} (\partial_x \phi - p_2) w dx + \int_{I_j} (\phi - \phi_h) w_x dx - (\phi - \phi_h)^-_{j+\frac{1}{2}} w^-_{j+\frac{1}{2}} + (\phi - \phi_h)^-_{j-\frac{1}{2}} w^+_{j-\frac{1}{2}} = 0. \tag{53}$$

By taking a Taylor expansion for $H(\frac{p_1+p_2}{2})$, we have

$$H\left(\frac{p_1+p_2}{2}\right) = H(\partial_x \phi) - H_1(\partial_x \phi)\left(\partial_x \phi - \frac{p_1+p_2}{2}\right) + \frac{1}{2}\bar{H}_{11}\left(\partial_x \phi - \frac{p_1+p_2}{2}\right)^2. \tag{54}$$

Taking $u = \xi$ and $w = -\alpha_j \xi$, and adding (53) to (52), we obtain

$$\begin{aligned}
&\int_{I_j} \xi_t \xi dx + \int_{I_j} \xi H_1 \left(\partial_x \phi - \frac{p_1+p_2}{2}\right) dx + \frac{1}{2}\alpha_j \int_{I_j} \xi(p_1-p_2)dx \\
&\quad - \alpha_j \int_{I_j} \xi(\partial_x\phi - p_2)dx - \alpha_j \int_{I_j} \xi_x \xi dx + \alpha_j(\xi^-_{j+\frac12}\xi^-_{j+\frac12} - \xi^-_{j-\frac12}\xi^+_{j-\frac12}) \\
&= \int_{I_j} \eta_t \xi dx + \int_{I_j} \frac{1}{2}\xi \bar{H}_{11}\left(\partial_x\phi - \frac{p_1+p_2}{2}\right)^2 dx - \alpha_j \int_{I_j} \xi_x \eta dx \\
&\quad + \alpha_j(\eta^-_{j+\frac12}\xi^-_{j+\frac12} - \eta^-_{j-\frac12}\xi^+_{j-\frac12}).
\end{aligned} \tag{55}$$

Here H_1 is a short-hand notation for $H_1(\partial_x\phi)$. For the left hand side (LHS) of (55), we have

$$\begin{aligned}
LHS = &\int_{I_j} \xi_t \xi dx - \int_{I_j} \xi(\alpha_j - H_1)\left(\partial_x\phi - \frac{p_1+p_2}{2}\right) dx \\
&+ \frac{1}{2}\alpha_j(\xi^-_{j+\frac12}\xi^-_{j+\frac12} - 2\xi^-_{j-\frac12}\xi^+_{j-\frac12} + \xi^+_{j-\frac12}\xi^+_{j-\frac12}).
\end{aligned} \tag{56}$$

Since $H_1 > 0$, we have $\mathbb{R}_h = \mathbb{R}_h^+$. Applying the special property of $\mathbb{R}_h^+$, the right hand side (RHS) of (55) is

$$RHS = \int_{I_j} \eta_t \xi dx + \int_{I_j} \frac{1}{2}\xi\bar{H}_{11}\left(\partial_x\phi - \frac{p_1+p_2}{2}\right)^2 dx \tag{57}$$

We now sum over j for the LHS and RHS of (55) from j_1 to j_2. Since $I_{j_1-1} \in S_3$ and $I_{j_2+1} \in S_3$, in the adjacent cells, we have $|\alpha_{j_2}| \le Ch$ and $|\alpha_{j_1}| \le Ch$. Combining with (46), we obtain

$$\begin{aligned}
\sum_{j=j_1}^{j_2}\int_{I_j} \xi_t\xi dx = &\sum_{j=j_1}^{j_2}\int_{I_j} \xi(\alpha_j - H_1)\left(\partial_x\phi - \frac{p_1+p_2}{2}\right)dx - \sum_{j=j_1}^{j_2-1}\frac{1}{2}\alpha_j(\xi^- - \xi^+)^2_{j+\frac12} \\
&+ \frac12\alpha_{j_2}\xi^-_{j_2+\frac12}\xi^-_{j_2+\frac12} - \alpha_{j_1}\xi^-_{j_1-\frac12}\xi^+_{j_1-\frac12} + \frac12\alpha_{j_1}\xi^+_{j_1-\frac12}\xi^+_{j_1-\frac12} \\
&+ \sum_{j=j_1}^{j_2}\int_{I_j}\eta_t\xi dx + \sum_{j=j_1}^{j_2}\int_{I_j}\frac12\xi\bar{H}_{11}\left(\partial_x\phi - \frac{p_1+p_2}{2}\right)^2 dx \\
\le &\sum_{j=j_1}^{j_2}\int_{I_j}|\alpha_j - H_1||\xi(\partial_x(\phi-\phi_h) + z_h)|dx + \frac12\alpha_{j_2}\xi^-_{j_2+\frac12}\xi^-_{j_2+\frac12} \\
&- \alpha_{j_1}\xi^-_{j_1-\frac12}\xi^+_{j_1-\frac12} + \frac12\alpha_{j_1}\xi^+_{j_1-\frac12}\xi^+_{j_1-\frac12} + \sum_{j=j_1}^{j_2}\int_{I_j}\eta_t\xi dx \\
&+ C\|\partial_x\phi - \frac{p_1+p_2}{2}\|_\infty \sum_{j=j_1}^{j_2}\int_{I_j}|\xi(\partial_x(\phi-\phi_h)+z_h)|dx \\
\le &Ch\Big(\sum_{j=j_1}^{j_2}\int_{I_j}|\xi(\partial_x\xi + \partial_x\eta + z_h)|dx + (\xi^-_{j_2+\frac12})^2 + (\xi^-_{j_1-\frac12})^2 + (\xi^+_{j_1-\frac12})^2\Big) \\
&+ \sum_{j=j_1}^{j_2}\int_{I_j}|\eta_t\xi|dx.
\end{aligned} \tag{58}$$

Similarly, if $\{I_j\}_{j=j_1}^{j_2}$ is a sequence of consecutive cells in S_2, with the two boundary points $x_{j_1-\frac{1}{2}}$ and $x_{j_2+\frac{1}{2}}$ of $\bar{I}_s = \cup_{j=j_1}^{j_2} \bar{I}_j$ belonging to the cells $\bar{I}_{j_1-1}$ and $\bar{I}_{j_2+1}$ of S_3 respectively, then we have $H_1 < 0$ in I_s, so that $\|\alpha_j + H_1\|_\infty \le Ch$ in each cell $I_j \subset I_s$, and the cell error equations are (52) and the following:

(59)
$$\int_{I_j} (\partial_x\phi - p_1)v dx + \int_{I_j} (\phi-\phi_h)v_x dx - (\phi-\phi_h)^-_{j+\frac{1}{2}} v^+_{j+\frac{1}{2}} + (\phi-\phi_h)^+_{j-\frac{1}{2}} v^+_{j-\frac{1}{2}} = 0.$$

With the special property of $\mathbb{R}_h^-$, we can get the similar estimate as in (58), that is

$$\begin{aligned}\sum_{j=j_1}^{j_2}\int_{I_j}\xi_t\xi dx \le& Ch\Big(\sum_{j=j_1}^{j_2}\int_{I_j}|\xi(\partial_x\xi+\partial_x\eta+z_h)|dx + (\xi^-_{j_2+\frac{1}{2}})^2 + (\xi^+_{j_2+\frac{1}{2}})^2 + (\xi^+_{j_1-\frac{1}{2}})^2\Big)\\ &+\sum_{j=j_1}^{j_2}\int_{I_j}|\eta_t\xi|dx. \end{aligned} \tag{60}$$

Now if $\{I_j\}_{j=j_1}^{j_2}$ is a sequence of consecutive cells in S_3, in each cell $I_j \subset \bar{I}_s = \cup_{j=j_1}^{j_2} \bar{I}_j$, we directly have $|H_1(\partial_x\phi)| \le Ch$ and $\alpha_j \le Ch$. In this case, the cell error equations are (52), (53) and (59). By taking $u = 2\xi$, $v = \alpha_j\xi$, $w = -\alpha_j\xi$, and summing over j from j_1 to j_2, we can obtain that

$$\begin{aligned} 2\sum_{j=j_1}^{j_2}\int_{I_j}\xi_t\xi dx =& 2\sum_{j=j_1}^{j_2}\int_{I_j}\eta_t\xi dx - 2\sum_{j=j_1}^{j_2}\int_{I_j}\xi H_1(\partial_x\phi - \frac{p_1+p_2}{2})dx\\ &+\sum_{j=j_1}^{j_2}\alpha_j\Big(([\xi-\eta]\xi^+)_{j-\frac{1}{2}} - ([\xi-\eta]\xi^-)_{j+\frac{1}{2}}\Big)\\ &+\sum_{j=j_1}^{j_2}\int_{I_j}\xi\bar{H}_{11}(\partial_x\phi - \frac{p_1+p_2}{2})^2dx\\ \le& 2\sum_{j=j_1}^{j_2}\int_{I_j}|\eta_t\xi|dx + 2\sum_{j=j_1}^{j_2}\int_{I_j}|H_1||\xi(\partial_x(\phi-\phi_h)+z_h)|dx\\ &+\sum_{j=j_1}^{j_2}\alpha_j\Big|([\xi-\eta]\xi^+)_{j-\frac{1}{2}} - ([\xi-\eta]\xi^-)_{j+\frac{1}{2}}\Big|\\ &+C\|\partial_x\phi - \frac{p_1+p_2}{2}\|_\infty\sum_j\int_{I_j}|\xi(\partial_x(\phi-\phi_h)+z_h)|dx\\ \le& 2\sum_{j=j_1}^{j_2}\int_{I_j}|\eta_t\xi|dx + Ch\Big(\sum_{j=j_1}^{j_2}\int_{I_j}|\xi(\partial_x\xi+\partial_x\eta+z_h)|dx\\ &+\sum_{j=j_1-1}^{j_2}((\xi^+)^2+(\xi^-)^2+(\eta^+)^2+(\eta^-)^2)_{j+\frac{1}{2}}\Big). \end{aligned} \tag{61}$$

Combining (58), (60) and (61), for each sequence of consecutive cells in the classes of S_1, S_2 and S_3, we have

$$\sum_{j=j_1}^{j_2}\int_{I_j}\xi_t\xi dx \le \sum_{j=j_1}^{j_2}\int_{I_j}|\eta_t\xi|dx + Ch\Big(\sum_{j=j_1}^{j_2}\int_{I_j}|\xi(\partial_x\xi+\partial_x\eta+z_h)|dx + \sum_{j=j_1-1}^{j_2}((\xi^+)^2+(\xi^-)^2+(\eta^+)^2+(\eta^-)^2)_{j+\frac{1}{2}}\Big). \tag{62}$$

Summing over all the sequences in the three nonoverlapping classes, we have

$$\frac{d}{dt}\|\xi\|^2 \le \int_0^1|\eta_t\xi|dx + Ch\Big(\int_0^1|\xi(\partial_x\xi+\partial_x\eta+z_h)|dx + \|\xi\|^2_{\Gamma_h}+\|\eta\|^2_{\Gamma_h}\Big). \tag{63}$$

From the inverse property (16), the interpolation property (11) and the estimate for the term z_h (47), and using Schwarz inequality, (63) can be rewritten as

$$\begin{aligned}\frac{d}{dt}\|\xi\|^2 \le& C(\|\xi\|^2+\|\eta_t\|^2)+Ch^2(\|\partial_x\xi\|^2+\|\partial_x\eta\|^2+\|z_h\|^2)+Ch(\|\xi\|^2_{\Gamma_h}+\|\eta\|^2_{\Gamma_h})\\ \le& C(\|\xi\|^2+h^{2k+2}).\end{aligned} \tag{64}$$

Applying the Gronwall's inequality and the triangle inequality, we finally get the optimal error estimate for the LDG scheme.

Remark 3.2. *For the LDG scheme, we can only prove the optimal error estimate for the local, not the global, Lax-Friedrichs numerical Hamiltonian. This is because in the upwind case $H_1(\partial_x\phi) > 0$, we need $\hat{H}(p_2,p_1) = H(p_2)+O(h)(p_2-p_1)$. Besides the local Lax-Friedrichs Hamiltonian, other purely upwind numerical Hamiltonians, such as the Godunov* [18] *and Osher-Sethian* [19] *numerical Hamiltonians , would also yield optimal error estimates.*

3.1.6. Justification of the a priori assumption. Now, to complete the proof of Theorem 3.1, we follow [21, 24] to verify the a priori assumption (27) and (46). For (27), we have

$$\|\partial_x(\phi-\phi_h)\|_\infty \le \|\partial_x\xi\|_\infty+\|\partial_x\eta\|_\infty \le C(h^{-\frac{3}{2}}\|\xi\|+h^{k-\frac{1}{2}}) \tag{65}$$

and for (46), from the inverse property (16) and (47), we have

$$\begin{aligned}\|\partial_x\phi-\frac{p_1+p_2}{2}\|_\infty \le& \|\partial_x(\phi-\phi_h)+z_h\|_\infty\\ \le& \|\partial_x\xi\|_\infty+\|\partial_x\eta\|_\infty+\|z_h\|_\infty\\ \le& C(h^{-\frac{3}{2}}\|\xi\|+h^{k-\frac{1}{2}}+h^{-\frac{1}{2}}\|z_h\|)\\ \le& C(h^{-\frac{3}{2}}\|\xi\|+h^{k-\frac{1}{2}})\end{aligned} \tag{66}$$

Here $\|\partial_x\eta\|_\infty \le Ch^{k-\frac{1}{2}}$ can be referred to [1, 4].

From $\|\xi\| \le Ch^{k+1}$ and $k \ge 2$, we can get

$$\|\partial_x(\phi-\phi_h)\|_\infty \le Ch^{k-\frac{1}{2}} \le C_0h^{\frac{3}{2}} \tag{67}$$

where the positive constant C_0 depends on T but is independent of h. Certainly there exists a constant $h_0 > 0$ such that $C_0h^{\frac{1}{2}} < \mathbb{C}$, and consequently $\|\partial_x(\phi-\phi_h)\| \le \mathbb{C}h$ if $h \le h_0$. Thus the a priori assumptions (27) and (46) are justified.

3.2. Two-dimensional case.

3.2.1. DG scheme for directly solving HJ equations. The DG scheme for directly solving the two-dimensional HJ equations in [3] is defined on rectangular meshes as: find $\phi_h(x,y,t) \in Z_h$, such that for any test function $v_h \in Z_h$,

$$\begin{aligned}
&\int_{I_{i,j}} (\partial_t \phi_h(x,y,t) + H(\partial_x \phi_h(x,y,t), \partial_y \phi_h(x,y,t), x, y)) v_h(x,y) dxdy \\
&\qquad + \int_{J_j} H_{1min}(x_{i+\frac{1}{2}}, y)[\phi_h](x_{i+\frac{1}{2}}, y) v_h(x^-_{i+\frac{1}{2}}, y) dy \\
&\qquad + \int_{J_j} H_{1max}(x_{i-\frac{1}{2}}, y)[\phi_h](x_{i-\frac{1}{2}}, y) v_h(x^+_{i-\frac{1}{2}}, y) dy \\
&\qquad + \int_{I_i} H_{2min}(x, y_{j+\frac{1}{2}})[\phi_h](x, y_{j+\frac{1}{2}}) v_h(x, y^-_{j+\frac{1}{2}}) dy \\
&\qquad + \int_{I_i} H_{2max}(x, y_{j-\frac{1}{2}})[\phi_h](x, y_{j-\frac{1}{2}}) v_h(x, y^+_{j-\frac{1}{2}}) dy \\
&\qquad = 0
\end{aligned} \tag{68}$$

where

$$H_{1min}(x_{i+\frac{1}{2}}, y) = \frac{1}{2}\left(\min_{x \in I_{i+\frac{1}{2}}} H_1(\partial_x \phi_h, \overline{\partial_y \phi_h}, x_{i+\frac{1}{2}}, y) - \left| \min_{x \in I_{i+\frac{1}{2}}} H_1(\partial_x \phi_h, \overline{\partial_y \phi_h}, x_{i+\frac{1}{2}}, y) \right| \right)$$

$$H_{1max}(x_{i+\frac{1}{2}}, y) = \frac{1}{2}\left(\max_{x \in I_{i+\frac{1}{2}}} H_1(\partial_x \phi_h, \overline{\partial_y \phi_h}, x_{i+\frac{1}{2}}, y) + \left| \max_{x \in I_{i+\frac{1}{2}}} H_1(\partial_x \phi_h, \overline{\partial_y \phi_h}, x_{i+\frac{1}{2}}, y) \right| \right)$$

$$H_{2min}(x, y_{j+\frac{1}{2}}) = \frac{1}{2}\left(\min_{y \in J_{j+\frac{1}{2}}} H_2(\overline{\partial_x \phi_h}, \partial_y \phi_h, x, y_{j+\frac{1}{2}}) - \left| \min_{y \in J_{j+\frac{1}{2}}} H_2(\overline{\partial_x \phi_h}, \partial_y \phi_h, x, y_{j+\frac{1}{2}}) \right| \right)$$

$$H_{2max}(x, y_{j+\frac{1}{2}}) = \frac{1}{2}\left(\max_{y \in J_{j+\frac{1}{2}}} H_2(\overline{\partial_x \phi_h}, \partial_y \phi_h, x, y_{j+\frac{1}{2}}) + \left| \max_{y \in J_{j+\frac{1}{2}}} H_2(\overline{\partial_x \phi_h}, \partial_y \phi_h, x, y_{j+\frac{1}{2}}) \right| \right)$$

and in these formulae, we define

$$\overline{\partial_x \phi_h} = \frac{1}{2}((\partial_x \phi_h)^+ + (\partial_x \phi_h)^-), \quad \overline{\partial_y \phi_h} = \frac{1}{2}((\partial_y \phi_h)^+ + (\partial_y \phi_h)^-).$$

On the interfaces of the cells, along the normal direction, the reconstructed information of the partial derivatives as in the one-dimensional case is used, and tangential to the interface, the average of the partial derivatives from the two neighboring cells is used. The reconstruction process is the same as that in the one-dimensional case, except that we need to fix x or y, then perform the reconstruction on the other spatial variable.

3.2.2. LDG scheme for directly solving HJ equations. The LDG scheme for directly solving the two-dimensional HJ equations in [22] defined on arbitrary triangulation is: find $\phi_h \in W_h$, such that for any test function $u \in W_h$, we have

$$\int_K (\phi_h)_t u dxdy + \int_K \hat{H}(p_1, p_2, q_1, q_2) u dxdy = 0. \tag{69}$$

The variables p_1 and p_2 are used to approximate ϕ_x, and similar to the 1D case, p_1 and p_2 are obtained by solving two simple upwind DG schemes, that is: find $p_1 \in W_h$ and $p_2 \in W_h$, such that for any test functions $v_1 \in W_h$ and $v_2 \in W_h$, we have

$$\begin{cases} \int_K p_1 v_1 dxdy + \int_K \phi_h (v_1)_x dxdy - \int_{\partial K} (\phi_h)^+ n_x v_1^{int_K} ds = 0 \\ \int_K p_2 v_2 dxdy + \int_K \phi_h (v_2)_x dxdy - \int_{\partial K} (\phi_h)^- n_x v_2^{int_K} ds = 0 \end{cases} \tag{70}$$

with

$$(\phi_h)^+ = \begin{cases} (\phi_h)^{ext_K}, & \text{if } n_x \geq 0 \\ (\phi_h)^{int_K}, & \text{else} \end{cases}$$

and

$$(\phi_h)^- = \begin{cases} (\phi_h)^{int_K}, & \text{if } n_x \geq 0 \\ (\phi_h)^{ext_K}, & \text{else} \end{cases}$$

where ∂K is the boundary of element K, $n = (n_x, n_y)$ is the outward unit normal for element K along the element boundary ∂K. Here $(\phi_h)^{int_K}$ denotes the value of ϕ_h on ∂K evaluated from inside the element K, and correspondingly $(\phi_h)^{ext_K}$ denotes the value of ϕ_h on ∂K evaluated from the outside element K (inside the neighboring element K' which shares the same edge with K).

Two other variables q_1 and q_2 are used to approximate ϕ_y and are obtained by solving the following two upwind DG schemes: find $q_1 \in W_h$ and $q_2 \in W_h$, such that for any test functions $w_1 \in W_h$ and $w_2 \in W_h$, we have

$$\begin{cases} \int_K q_1 w_1 dxdy + \int_K \phi_h (w_1)_y dxdy - \int_{\partial K} (\phi_h)^+ n_y w_1^{int_K} ds = 0, \\ \int_K q_2 w_2 dxdy + \int_K \phi_h (w_2)_y dxdy - \int_{\partial K} (\phi_h)^- n_y w_2^{int_K} ds = 0 \end{cases} \tag{71}$$

with

$$(\phi_h)^+ = \begin{cases} (\phi_h)^{ext_K}, & \text{if } n_y \geq 0 \\ (\phi_h)^{int_K}, & \text{else} \end{cases}$$

and

$$(\phi_h)^- = \begin{cases} (\phi_h)^{int_K}, & \text{if } n_y \geq 0 \\ (\phi_h)^{ext_K}, & \text{else} \end{cases}$$

The Lax-Friedrichs type numerical Hamiltonian can be used, which is defined to be

$$\hat{H}(p_1, p_2, q_1, q_2) = H\left(\frac{p_1 + p_2}{2}, \frac{q_1 + q_2}{2}\right) - \frac{1}{2}\alpha(p_1 - p_2) - \frac{1}{2}\beta(q_1 - q_2) \tag{72}$$

with

$$\alpha = \max_{p \in D, q \in E} \left| \frac{\partial H(p,q)}{\partial p} \right|, \quad \beta = \max_{p \in D, q \in E} \left| \frac{\partial H(p,q)}{\partial q} \right| \tag{73}$$

here $H(u, v)$ is a short-hand notation for $H(u, v, x, y)$. With $D = [\min(p_1, p_2), \max(p_1, p_2)]|_K$ and $E = [\min(q_1, q_2), \max(q_1, q_2)]|_K$, it is called the local Lax-Friedrichs Hamiltonian. With $D = [\min(p_1, p_2), \max(p_1, p_2)]|_\Omega$ and $E = [\min(q_1, q_2), \max(q_1, q_2)]|_\Omega$, then it is called the global Lax-Friedrichs Hamiltonian.

The definition of the LDG scheme on rectangular meshes is that we only need to replace W_h with Z_h. In this case, our domain is $I \times J$ and each element $K = I_{i,j}$.

3.2.3. The main results. In this section, we state the main error estimate results for the semi-discrete DG scheme (68) and the semi-discrete LDG scheme (69)-(71) on Cartesian meshes.

Theorem 3.3. *Let ϕ be the exact solution of the problem (2), which is sufficiently smooth with bounded derivatives, and assume $H(u, v, x, y) \in C^2$. Let ϕ_h be the numerical solution of the semi-discrete DG scheme (68) or the semi-discrete LDG scheme (69), (70) and (71). Denote the corresponding numerical error as $e_\phi = \phi - \phi_h$. For a rectangular triangulation of $I \times J$, if the finite element space Z_h is* *the piecewise tensor product polynomials with degree $k \geq 3$, then for small enough h, there holds the following error estimates*

$$\|\phi - \phi_h\| \leq Ch^{k+1} \tag{74}$$

where the positive constant C depends on the final time T, k, $\|\phi\|_{L^\infty((0,T),H^{k+1}(I\times J))}$ and the bounds on the m-th derivatives of $H(\partial_x\phi, \partial_y\phi, x, y)$ with respect to its first and second arguments for $m = 1, 2$. $\|\phi\|_{L^\infty((0,T),H^{k+1}(I\times J))}$ is the maximum over $0 \leq t \leq T$ of the standard Sobolev $(k+1)$-norm in space.

Remark 3.3. *Notice that our proof for the optimal error estimate works only for the finite element space Z_h, not for the usual k-th degree polynomial space W_h. This is because the main technique is the special tensor product projection. However, numerical examples in [22] do verify the optimal order of accuracy for the LDG scheme defined on W_h.*

3.2.4. Sketch of the proof for the DG scheme. First we would like to adopt an a priori assumption for the two-dimensional DG scheme, that is

$$\|\partial_x(\phi - \phi_h)\|_\infty \leq \mathbb{C}h, \quad \|\partial_y(\phi - \phi_h)\|_\infty \leq \mathbb{C}h. \tag{75}$$

The cell error equation for the DG scheme is

$$\begin{aligned}
\int_{I_{i,j}} (\phi - \phi_h)_t v_h dxdy + &\int_{I_{i,j}} (H(\partial_x\phi, \partial_y\phi, x, y) - H(\partial_x\phi_h, \partial_y\phi_h, x, y))v_h dxdy \\
&- \int_{J_j} H_{1min}(x_{i+\frac{1}{2}}, y)[\phi_h](x_{i+\frac{1}{2}}, y)v_h(x^-_{i+\frac{1}{2}}, y)dy \\
&- \int_{J_j} H_{1max}(x_{i-\frac{1}{2}}, y)[\phi_h](x_{i-\frac{1}{2}}, y)v_h(x^+_{i-\frac{1}{2}}, y)dy \\
&- \int_{I_i} H_{2min}(x, y_{j+\frac{1}{2}})[\phi_h](x, y_{j+\frac{1}{2}})v_h(x, y^-_{j+\frac{1}{2}})dx \\
&- \int_{I_i} H_{2max}(x, y_{j-\frac{1}{2}})[\phi_h](x, y_{j-\frac{1}{2}})v_h(x, y^+_{j-\frac{1}{2}})dx \\
&= 0.
\end{aligned} \tag{76}$$

By taking a Taylor expansion

$$\begin{aligned}
H(\partial_x\phi, \partial_y\phi, x, y) - H(\partial_x\phi_h, \partial_y\phi_h, x, y) = H_1\partial_x(\phi - \phi_h) + H_2\partial_y(\phi - \phi_h) \\
- \frac{1}{2}\left(\bar{H}_{11}(\partial_x(\phi - \phi_h))^2 + 2\bar{H}_{12}\partial_x(\phi - \phi_h)\partial_y(\phi - \phi_h) + \bar{H}_{22}(\partial_y(\phi - \phi_h))^2\right)
\end{aligned} \tag{77}$$

we have

$$\int_{I_{i,j}} (\phi - \phi_h)_t v_h dxdy + \mathscr{J}_1 + \mathscr{J}_2 + \mathscr{J}_3 = 0 \tag{78}$$

where

$$\begin{aligned}
\mathscr{J}_1 =& \int_{I_{i,j}} (H_1\partial_x(\phi-\phi_h) - \frac{1}{2}\bar{H}_{11}(\partial_x(\phi-\phi_h))^2)v_h dxdy \\
&- \int_{J_j} H_{1min}(x_{i+\frac{1}{2}},y)[\phi_h](x_{i+\frac{1}{2}},y)v_h(x^-_{i+\frac{1}{2}},y)dy \\
&- \int_{J_j} H_{1max}(x_{i-\frac{1}{2}},y)[\phi_h](x_{i-\frac{1}{2}},y)v_h(x^+_{i-\frac{1}{2}},y)dy \\
\mathscr{J}_2 =& \int_{I_{i,j}} (H_2\partial_y(\phi-\phi_h) - \frac{1}{2}\bar{H}_{22}(\partial_y(\phi-\phi_h))^2)v_h dxdy \\
&- \int_{I_i} H_{2min}(x,y_{j+\frac{1}{2}})[\phi_h](x,y_{j+\frac{1}{2}})v_h(x,y^-_{j+\frac{1}{2}})dx \\
&- \int_{I_i} H_{2max}(x,y_{j-\frac{1}{2}})[\phi_h](x,y_{j-\frac{1}{2}})v_h(x,y^+_{j-\frac{1}{2}})dx \\
\mathscr{J}_3 =& -\int_{I_{i,j}} \bar{H}_{12}\partial_x(\phi-\phi_h)\partial_y(\phi-\phi_h)v_h dxdy.
\end{aligned}$$

$\mathscr{J}_1$ and $\mathscr{J}_2$ can be estimated similarly as in the one-dimensional case, and $\mathscr{J}_3$ can be estimated similarly as in (39). The optimal error estimate of the DG scheme for the two-dimensional case can then be obtained.

3.2.5. Sketch of the proof for the LDG scheme. In this section, we also give a sketch of the proof for the local Lax-Friedrichs LDG scheme on Cartesian meshes. An a priori assumption we adopt for the two-dimensional LDG scheme is

$$\|\partial_x\phi - \frac{p_1+p_2}{2}\|_\infty \le \mathbb{C}h, \quad \|\partial_y\phi - \frac{q_1+q_2}{2}\|_\infty \le \mathbb{C}h \tag{79}$$

The cell error equation for the LDG scheme is

$$\begin{aligned}
&\int_{I_{i,j}} (\phi-\phi_h)_t u dxdy + \int_{I_{i,j}} (H(\partial_x\phi,\partial_y\phi) - H(\frac{p_1+p_2}{2},\frac{q_1+q_2}{2}))u dxdy \\
&\qquad + \frac{1}{2}\alpha\int_{I_{i,j}} (p_1-p_2)u dxdy + \frac{1}{2}\beta\int_{I_{i,j}} (q_1-q_2)u dxdy = 0.
\end{aligned} \tag{80}$$

By taking a Taylor expansion

$$\begin{aligned}
&H(\partial_x\phi,\partial_y\phi) - H(\frac{p_1+p_2}{2},\frac{q_1+q_2}{2}) = H_1(\partial_x\phi - \frac{p_1+p_2}{2}) + H_2(\partial_y\phi - \frac{q_1+q_2}{2}) \\
&-\frac{1}{2}\left(\bar{H}_{11}(\partial_x\phi - \frac{p_1+p_2}{2})^2 - 2\bar{H}_{12}(\partial_x\phi - \frac{p_1+p_2}{2})(\partial_y\phi - \frac{q_1+q_2}{2}) + \bar{H}_{22}(\partial_y\phi - \frac{q_1+q_2}{2})^2\right)
\end{aligned}$$

we have

$$\int_{I_{i,j}} (\phi-\phi_h)_t u dxdy + \mathscr{J}_4 + \mathscr{J}_5 + \mathscr{J}_6 = 0 \tag{81}$$

where

$$\begin{aligned}
\mathscr{J}_4 &= \int_{I_{i,j}} u(H_1(\partial_x\phi - \frac{p_1+p_2}{2}) - \frac{1}{2}\bar{H}_{11}(\partial_x\phi - \frac{p_1+p_2}{2})^2)dxdy \\
&\quad + \frac{1}{2}\alpha \int_{I_{i,j}} (p_1 - p_2)udxdy \\
\mathscr{J}_5 &= \int_{I_{i,j}} u(H_2(\partial_y\phi - \frac{q_1+q_2}{2}) - \frac{1}{2}\bar{H}_{22}(\partial_y\phi - \frac{q_1+q_2}{2})^2)dxdy \\
&\quad + \frac{1}{2}\beta \int_{I_{i,j}} (q_1 - q_2)udxdy \\
\mathscr{J}_6 &= -\int_{I_{i,j}} \bar{H}_{12}(\partial_x\phi - \frac{p_1+p_2}{2})(\partial_y\phi - \frac{q_1+q_2}{2})udxdy.
\end{aligned}$$

With (70) and (71), $\mathscr{J}_4$ and $\mathscr{J}_5$ can be estimated similarly as in the one-dimensional case, and $\mathscr{J}_6$ can be estimated similarly as the term $\int_{I_{i,j}} \bar{H}_{11}(\partial_x\phi - \frac{p_1+p_2}{2})^2 udxdy$. The optimal error estimate of the LDG scheme for the two-dimensional case is thus obtained.

Remark 3.4. *For the two-dimensional LDG scheme on a rectangular mesh, another more general definition for the local Lax-Friedrichs Hamiltonian in (72) and (73) is that* [18], $D = [\min(p_1, p_2), \max(p_1, p_2)]|_{I_i}$ *but* $E = [\min(q_1, q_2), \max(q_1, q_2)]|_J$ *for* α, *and* $D = [\min(p_1, p_2), \max(p_1, p_2)]|_I$ *but* $E = [\min(q_1, q_2), \max(q_1, q_2)]|_{J_j}$ *for* β, *this Hamiltonian is monotone. However, our estimates for the two-dimensional case only works for the non monotone local Lax-Friedrichs Hamiltonian defined in (72) and (73), since we need* $|H_1 - \alpha| \le Ch$ *and* $|H_2 - \beta| \le Ch$ *in each cell* I_{ij}, *with the* α *and* β *contained in (80). But the monotone Godunov* [18] *and Osher-Sethian* [19] *numerical Hamiltonian, which do not contain* α *and* β *in (80), would also yield optimal error estimates for the two-dimensional case.*

3.2.6. Justification of the a priori assumption. For the two-dimensional case, the a priori assumptions (75) and (79) can be similarly verified as in the one-dimensional case. We need the restriction $k \ge 3$ in Theorem 3.3, due to the different bounds in (16). We omit the detailed proof here.

4. Concluding remarks

In this paper, we have obtained the optimal a priori L^2 error estimates for the semi-discrete DG scheme and the semi-discrete LDG scheme for directly solving the Hamilton-Jacobi equations in one- and two-dimensional cases on rectangular meshes. By using the regular L^2 projection rather than the special projections, we can prove sub-optimal a priori L^2 error estimates (half an order lower) for the same schemes on arbitrary triangulations in two dimensions. The proof follows the same lines and hence is not given in this paper. The sub-optimal a priori L^2 error estimates can also be obtained for the central DG scheme defined in [17] along the same lines.

As is well known, the viscosity solution to the Hamilton-Jacobi equation is generically only Lipschitz continuous, with possible discontinuous derivatives. Convergence and error estimates of the DG and LDG schemes for such cases are much more difficult to analyze and are worthy of future investigation.

5. Appendix

In this appendix, we are going to prove $\max_{j_1 \le j \le j_2} |H_1 - H_{1max}|_{j+\frac{1}{2}} \le Ch$ in Section 3.1.4. $H_1 = H_1(\partial_x \phi) > 0$ and $H_{1max} = \max_{x \in \bar{I}_{j+\frac{1}{2}}} H_1(\partial_x \phi_h)$, we omit the index x here. Although we use ϕ_h in the definition of H_{1max}, from (18) and (19), ϕ_h is actually the polynomial $w_{j+\frac{1}{2}}$, which leads to the proof of the inequality nontrivial. $H_{1max}(\partial_x w_{j+\frac{1}{2}})$ is actually the function $H_1(\partial_x w_{j+\frac{1}{2}})$ evaluated at some point x^* inside the cell $\bar{I}_{j+\frac{1}{2}}$, which we denote to be $H_1((\partial_x w_{j+\frac{1}{2}})^*)$.

First, we can define a similar projection $P\phi \in P^{2k+1}(I_j \cup I_{j+1})$ as $w_{j+\frac{1}{2}}$ but corresponding to the exact smooth solution ϕ, which is

$$\int_{I_s} \phi \zeta dx = \int_{I_s} P\phi \zeta dx, \tag{A.1}$$

for any $\zeta \in P^k(I_s)$, $s = j, j+1$. And from [4], we have the error estimate that

$$\|\partial_x(\phi - P\phi)\|_{L^\infty(I_j \cup I_{j+1})} \le Ch^{k-\frac{1}{2}}. \tag{A.2}$$

We adopt the a priori assumption similar to (27) in this section, that is

$$\|\phi - \phi_h\| \le \mathbb{C} h^{\frac{5}{2}} \tag{A.3}$$

the justification is easily followed from Section 3.1.6, since we have $k \ge 2$.

In the following, we are going to prove that $\|P\phi - w_{j+\frac{1}{2}}\|_{L^2(I_j \cup I_{j+1})} \le C\|\phi - \phi_h\|$. From (18), (19) and (A.1), we have

$$\int_{I_s} (P\phi - w_{j+\frac{1}{2}})\zeta dx = \int_{I_s} (\phi - \phi_h)\zeta dx, \tag{A.4}$$

for any $\zeta \in P^k(I_s)$, $s = j, j+1$. Denote $u = P\phi - w_{j+\frac{1}{2}} \in P^{2k+1}(I_j \cup I_{j+1})$, we can define the L^2-projection $u_1 \in P^k(I_j)$ of u in the cell I_j and $u_2 \in P^k(I_{j+1})$ of u in the cell I_{j+1}, which satisfy

$$\int_{I_j} u_1^2 dx = \int_{I_j} u_1 u dx \tag{A.5}$$

and

$$\int_{I_{j+1}} u_2^2 dx = \int_{I_{j+1}} u_2 u dx \tag{A.6}$$

we have $\|u_1\|_{L^2(I_j)} \le \|u\|_{L^2(I_j)}$ and $\|u_2\|_{L^2(I_{j+1})} \le \|u\|_{L^2(I_{j+1})}$. However, conversely, we also have

$$\|u\|_{L^2(I_j \cup I_{j+1})} \le C(\|u_1\|_{L^2(I_j)} + \|u_2\|_{L^2(I_{j+1})}). \tag{A.7}$$

We can first prove (A.7) on a standard cell $I = (0, 1)$, we define

$$\begin{aligned} V &= P^{2k+1}(I) = Span\{1, x, x^2, \cdots, x^{2k+1}\}, \\ V_1 &= Span\{1, x^2, x^4, \cdots, x^{2k}\}, \\ V_2 &= Span\{x, x^3, x^5, \cdots, x^{2k+1}\}, \\ W &= P^k(I) = Span\{1, x, x^2, \cdots, x^k\}. \end{aligned}$$

that is $V = V_1 + V_2$. For any $v \in V$, we have $v = v_1 + v_2$ where $v_1 \in V_1$ and $v_2 \in V_2$. Now we perform a L^2-projection P from V_1 to W, which is

$$\int_0^1 v_1 \zeta dx = \int_0^1 w_1 \zeta dx \tag{A.8}$$

for any $\zeta \in W$, here $w_1 = Pv_1$. Using the basis of V_1 and W, (A.8) can be written in the form $A_1\tilde{v}_1 = A\tilde{w}_1$, here $\tilde{v}_1 = (\tilde{v}_1^0, \tilde{v}_1^2, \cdots, \tilde{v}_1^{2k})$ and $\tilde{w}_1 = (\tilde{w}_1^0, \tilde{w}_1^1, \cdots, \tilde{w}_1^k)$ are the coefficients of v_1 and w_1 with the polynomial basis in the corresponding space, A_1 is a constant matrix which can be easily found to be invertible, and A is the mass matrix in W. So we have $\tilde{v}_1 = (A_1)^{-1}A\tilde{w}_1$, which means that

$$\|v_1\|^2_{L^2(V_1)} \le C\sum_{l=0}^{k}(\tilde{v}_1^l)^2 \le C|(A_1)^{-1}A|^2\sum_{l=0}^{k}(\tilde{w}_1^l)^2 \le C\|w_1\|^2_{L^2(W)} \tag{A.9}$$

Similarly, for $v_2 \in V_2$, we have the L^2-projection $w_2 = Pv_2$ in W, such that $\|v_2\|_{L^2(V_2)} \le C\|w_2\|_{L^2(W)}$, combining with (A.9), we have

$$\|v\|_{L^2(V)} = \|v_1 + v_2\|_{L^2(V)} \le \|v_1\|_{L^2(V_1)} + \|v_2\|_{L^2(V_2)} \le C(\|w_1\|_{L^2(W)} + \|w_2\|_{L^2(W)}) \tag{A.10}$$

using the scaling argument, where v, w_1 and w_2 corresponding to u, u_1 and u_2 respectively, we obtain (A.7).

In (A.4), taking $\zeta = u_1$ in I_j and $\zeta = u_2$ in I_{j+1}, combining (A.5) and (A.6), we obtain

$$\begin{aligned}\int_{I_j} u_1^2 dx + \int_{I_{j+1}} u_2^2 dx &= \int_{I_j} u_1 u dx + \int_{I_{j+1}} u_2 u dx \\ &= \int_{I_j} (\phi - \phi_h)u_1 dx + \int_{I_{j+1}} (\phi - \phi_h)u_2 dx \\ &\le \|\phi - \phi_h\|_{L^2(I_j)}\|u_1\|_{L^2(I_j)} + \|\phi - \phi_h\|_{L^2(I_{j+1})}\|u_2\|_{L^2(I_{j+1})} \\ &\le \|\phi - \phi_h\|\|u\|_{L^2(I_j \cup I_{j+1})}\end{aligned} \tag{A.11}$$

from (A.7) and (A.11), we obtain $\|P\phi - w_{j+\frac{1}{2}}\|_{L^2(I_j\cup I_{j+1})} \le C\|\phi - \phi_h\|$. Then with the inverse property (16) and the a priori assumption (A.3), we have

$$\begin{aligned}\|\partial_x(P\phi - w_{j+\frac{1}{2}})\|_{L^\infty(I_j\cup I_{j+1})} &\le Ch^{-\frac{3}{2}}\|P\phi - w_{j+\frac{1}{2}}\|_{L^2(I_j\cup I_{j+1})} \\ &\le Ch^{-\frac{3}{2}}\|\phi - \phi_h\| \\ &\le Ch\end{aligned} \tag{A.12}$$

Now we can get our result from

$$\begin{aligned}|H_1(\partial_x\phi) - H_{1max}(\partial_x w_{j+\frac{1}{2}})| \le& |H_1(\partial_x\phi) - H_1(\partial_x(P\phi))| \\ &+ |H_1(\partial_x(P\phi)) - H_1(\partial_x w_{j+\frac{1}{2}})| \\ &+ |H_1(\partial_x w_{j+\frac{1}{2}}) - H_1((\partial_x w_{j+\frac{1}{2}})^*)| \\ \le& C(|\partial_x(\phi - P\phi)| + |\partial_x(P\phi - w_{j+\frac{1}{2}})| \\ &+ |\partial_x w_{j+\frac{1}{2}} - (\partial_x w_{j+\frac{1}{2}})^*|) \\ \le& C(h^{k-\frac{1}{2}} + h + h) \\ \le& Ch\end{aligned} \tag{A.13}$$

here $k \ge 2$ and taking the maximum over $j_1 \le j \le j_2$ in (A.13), the inequality $\max_{j_1\le j\le j_2} |H_1 - H_{1max}|_{j+\frac{1}{2}} \le Ch$ has been obtained.

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