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# ON THE SOLUTIONS OF THE COUPLED NONLINEAR PARABOLIC EQUATIONS *

Shen Longjun and Zhang Linghai
(Institute of Applied Physics and Computational Mathematics,
P.O. Box 8009, Beijing 100088, China)
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**Abstract** By means of the fixed point technique and integral estimation method, we study the solutions of periodic boundary value problem and initial value problem for the coupled nonlinear parabolic equations. The global classical solutions of the mentioned problems are shown to exists.

**Key Words** Nonlinear parabolic equation, existence and uniqueness global solution

**Classification** 35Q20.

## 1. Introduction and Preliminaries

The high resolution numerical simulation of plasma turbulence driven by ion temperature gradients in the presence of magnetic field inhomogeneities was performed in [1] with special attention to the behavior of the anomalous ion energy flux. The pressure gradient evolution is treated consistently with energy transport, allowing for the study of the saturated state in situations of relevance to Tokamak plasmas. Under some assumptions, the dynamical equations of plasma are reduced to the following coupled nonlinear parabolic equations [1]:

$$\Phi_t - \Delta\Phi_t + \alpha\Delta^2\Phi + J(\Delta\Phi, \Phi) + (1-\gamma)\Phi_y - \gamma\Psi_y = 0, \quad (x,y) \in \mathbf{R}^2,\ \mathbf{t} > \mathbf{0} \quad (1)$$

$$\Psi_t - \beta\Delta\Psi + J(\Phi, \Psi) + \gamma\Phi_y + \delta\Psi_y = f(x,y,t),$$
$$(x,y) \in \mathbf{R}^2,\ \mathbf{t} > \mathbf{0},\ \alpha > \mathbf{0},\ \beta > \mathbf{0} \quad (2)$$

with the periodic boundary value problem (PBVP)

$$\Phi(x+2D,y,t) = \Phi(x,y+2D,t) = \Phi(x,y,t), \quad (x,y) \in \mathbf{R}^2,\ \mathbf{t} > \mathbf{0} \quad (3)$$
$$\Psi(x+2D,y,t) = \Psi(x,y+2D,t) = \Psi(x,y,t), \quad (x,y) \in \mathbf{R}^2,\ \mathbf{t} > \mathbf{0} \quad (4)$$
$$\Phi(x,y,0) = \Phi_0(x,y), \quad (x,y) \in \mathbf{R}^2 \quad (5)$$
$$\Psi(x,y,0) = \Psi_0(x,y), \quad (x,y) \in \mathbf{R}^2 \quad (6)$$

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where the initial functions satisfy the periodic condition

$$\Phi_0(x+2D,y)=\Phi_0(x,y+2D)=\Phi_0(x,y),\quad (x,y)\in\mathbf{R}^2 \tag{7}$$

$$\Psi_0(x+2D,y)=\Psi_0(x,y+2D)=\Psi_0(x,y),\quad (x,y)\in\mathbf{R}^2 \tag{8}$$

and the initial value problem (IVP)

$$\Phi(x,y,0)=\Phi_0(x,y),\quad (x,y)\in\mathbf{R}^2 \tag{9}$$

$$\Psi(x,y,0)=\Psi_0(x,y),\quad (x,y)\in\mathbf{R}^2 \tag{10}$$

In Equations (1) and (2), $\Phi$ and $\Psi$ are the physical functions corresponding to the density and pressure, respectively. $\Delta$ is two-dimensional Laplace operator; $J(a,b)$ is the determinant of Jacobi matrix, namely, $J(a,b)=a_xb_y-a_yb_x$; $\alpha$ and $\beta$ are positive constants, $\gamma$ and $\delta$ are real constants; $f(x,y,t)$ is a suitable energy source. In Conditions (3–4) and (7–8), $D$ is a positive constant.

We remark that Guo Boling paid attention to the similar coupled nonlinear evolution equations [2]

$$\Delta\Psi_t+J(\Psi,\Delta\Psi)-\Delta^2\Psi+\alpha\theta_x=0,\quad (x,y)\in\mathbf{R}^2,\ \mathbf{t>0} \tag{11}$$

$$\theta_t+J(\Psi,\theta)-\beta\Delta\theta=0,\quad (x,y)\in\mathbf{R}^2,\ \mathbf{t>0},\ \alpha>0,\ \beta>0 \tag{12}$$

Zhou Yulin et al paid attention to the nonlinear evolution equation [3]

$$\Psi_t-\Delta\Psi_t+J(\Delta\Psi,\Psi)+\alpha\Delta\Psi_x+\beta\Delta\Psi_y+f(\Psi)_x+g(\Psi)_y=h(\Psi) \tag{13}$$

but there has been no contribution to this type of system up to now.

Let $T$ be a finite positive constant. Denote by $\Omega$ and $\Omega_T$ the domains

$$\Omega=\{(x,y)\in\mathbf{R}^2\mid \mathbf{0\le x,\ y\le 2D}\}$$

$$\Omega_T=\{(x,y,t)\in R^3\mid 0\le x,\ y\le 2D,\ 0\le t\le T\}$$

To simplify the notations, we will denote by $C$ all the positive constants appeared in the paper afterward, which do not depend on the solutions or their derivatives of any order, nor will they depend on $D$.

For any $f(x,y),g(x,y)\in C^1(\Omega)$, if they satisfy the periodic conditions (7) and (8), then we have the relation

$$\int_\Omega fJ(f,g)dxdy=0,\quad \int_\Omega gJ(f,g)dxdy=0$$

## 2. *A Priori* Estimate

In this section, we establish a series of integral estimates of the solutions of problem (1)–(6).

**Lemma 1** *Suppose that* $\Phi_0(x,y)\in H^1(\Omega)$, $\Psi_0(x,y)\in L^2(\Omega)$, $f(x,y,t)\in L^2(\Omega_T)$. *Then we have the integral estimation*

$$\sup_{0\le t\le T}\Big(\int_\Omega|\Phi|^2dxdy+\int_\Omega|\nabla\Phi|^2dxdy+\int_\Omega|\Psi|^2dxdy\Big)$$
$$+2\alpha\int_0^T\int_\Omega|\Delta\Phi|^2dxdydt+\beta\int_0^T\int_\Omega|\nabla\Psi|^2dxdydt$$
$$\le C\Big(\int_\Omega|\Phi_0|^2dxdy+\int_\Omega|\nabla\Phi_0|^2dxdy+\int_\Omega|\Psi_0|^2dxdy+\int_0^T\int_\Omega|f|^2dxdydt \tag{14}$$

**Proof** Multiply Equations (1) and (2) by $\Phi$ and $\Psi$ respectively, combine the product and integrate over $\Omega$ with respect to $(x,y)$, we get the integral identity

$$\frac{d}{dt}\int_\Omega(|\Phi|^2+|\nabla\Phi|^2+|\Psi|^2)dxdy+2\alpha\int_\Omega|\Delta\Phi|^2dxdy+2\beta\int_\Omega|\nabla\Psi|^2dxdy$$
$$=4\gamma\int_\Omega\Phi\Psi_ydxdy+2\int_\Omega f\Psi dxdy$$

The right hand side of this identity can be dominated by

$$\beta\int_\Omega|\nabla\Psi|^2dxdy+4\beta^{-1}\gamma^2\int_\Omega|\Phi|^2dxdy+\int_\Omega|f|^2dxdy+\int_\Omega|\Psi|^2dxdy$$

thus we get the inequality

$$\frac{d}{dt}\int_\Omega(|\Phi|^2+|\nabla\Phi|^2+|\Psi|^2)dxdy+2\alpha\int_\Omega|\Delta\Phi|^2dxdy+\beta\int_\Omega|\nabla\Psi|^2dxdy$$
$$\le 4\beta^{-1}\gamma^2\int_\Omega|\Phi|^2dxdy+\int_\Omega|f|^2dxdy+\int_\Omega|\Psi|^2dxdy$$

By means of Gronwall's inequality, we obtain (14).

**Lemma 2** *Assume that* $\Phi_0(x,y)\in H^2(\Omega)$, $\Psi_0(x,y)\in H^1(\Omega)$, $f(x,y,t)\in L^2(\Omega_T)$. *Then we have the integral estimation*

$$\sup_{0\le t\le T}\Big(\int_\Omega|\nabla\Phi|^2dxdy+\int_\Omega|\Delta\Phi|^2dxdy+\int_\Omega|\nabla\Psi|^2dxdy\Big)$$
$$+\alpha\int_0^T\int_\Omega|\nabla\Delta\Phi|^2dxdydt+\beta\int_0^T\int_\Omega|\Delta\Psi|^2dxdydt$$
$$\le C\Big(\int_\Omega|\Phi_0|^2dxdy+\int_\Omega|\nabla\Phi_0|^2dxdy+\int_\Omega|\Delta\Phi_0|^2dxdy$$
$$+\int_\Omega|\Psi_0|^2dxdy+\int_\Omega|\nabla\Psi_0|^2dxdy+\int_0^T\int_\Omega|f|^2dxdydt \tag{15}$$

**Proof** Multiply Equations (1) and (2) by $\Delta\Phi$ and $\Delta\Psi$ respectively, combine the product and integrate over $\Omega$ with respect to $(x,y)$, we get the integral identity

$$\frac{d}{dt}\int_\Omega(|\nabla\Phi|^2+|\Delta\Phi|^2+|\nabla\Psi|^2)dxdy+2\alpha\int_\Omega|\nabla\Delta\Phi|^2dxdy+2\beta\int_\Omega|\Delta\Psi|^2dxdy$$

$$= 4\gamma\int_\Omega \Phi_y \Delta\Psi dxdy + 2\int_\Omega \Delta\Psi J(\Phi,\Psi)dxdy - 2\int_\Omega f\Delta\Psi dxdy$$

The following simplifications are obviously true:

$$4\gamma\int_\Omega \Phi_y\Delta\Psi dxdy \le \frac{\beta}{3}\int_\Omega |\Delta\Psi|^2 dxdy + 12\beta^{-1}\gamma^2\int_\Omega |\nabla\Phi|^2 dxdy$$

$$2\int_\Omega \Delta\Psi J(\Phi,\Psi)dxdy \le \frac{\beta}{3}\int_\Omega |\Delta\Psi|^2 dxdy + \frac{3}{\beta}\int_\Omega |J(\Phi,\Psi)|^2 dxdy$$

$$\frac{3}{\beta}\int_\Omega |J(\Phi,\Psi)|^2 dxdy \le \frac{3}{\beta}\int_\Omega |\nabla\Phi|^2 |\nabla\Psi|^2 dxdy$$

$$\le \frac{3}{\beta}C\|\nabla\Phi(\cdot,\cdot,t)\|\,\|\nabla\Delta\Phi(\cdot,\cdot,t)\|\int_\Omega |\nabla\Psi|^2 dxdy$$

thus

$$2\int_\Omega \Delta\Psi J(\Phi,\Psi)dxdy \le \frac{\beta}{3}\int_\Omega |\Delta\Psi|^2 dxdy + \alpha\int_\Omega |\nabla\Delta\Phi|^2 dxdy$$
$$+ C\sup_{0\le t\le T}\int_\Omega |\nabla\Phi|^2 dxdy\Big(\int_\Omega |\nabla\Psi|^2 dxdy\Big)^2$$

$$-2\int_\Omega f\Delta\Psi dxdy \le \frac{\beta}{3}\int_\Omega |\Delta\Psi|^2 dxdy + \frac{3}{\beta}\int_\Omega |f|^2 dxdy$$

Now we get the integral inequality

$$\frac{d}{dt}\int_\Omega (|\nabla\Phi|^2 + |\Delta\Phi|^2 + |\nabla\Psi|^2)dxdy + \alpha\int_\Omega |\nabla\Delta\Phi|^2 dxdy + \beta\int_\Omega |\Delta\Psi|^2 dxdy$$
$$\le 12\beta^{-1}\gamma^2\int_\Omega |\nabla\Phi|^2 dxdy + C\Big(\int_\Omega |\nabla\Psi|^2 dxdy\Big)^2 + \frac{3}{\beta}\int_\Omega |f|^2 dxdy$$

By means of Lemma 1 and Gronwall's inequality, we obtain the estimate (15).

**Corollary 1** *Under the conditions of Lemma 2, we have the estimate*

$$\sup_{0\le t\le T}\|\Phi(\cdot,\cdot,t)\|_\infty \le C\Big(\|\Phi_0\|_2^2 + \|\Psi_0\|_1^2 + \int_0^T \|f(\cdot,\cdot,t)\|^2 dt\Big)^{\frac{1}{2}} \tag{16}$$

**Lemma 3** *Suppose that $\Phi_0(x,y)\in H^3(\Omega)$, $\Psi_0(x,y)\in H^1(\Omega)$, $f(x,y,t)\in L^2(\Omega_T)$. Then there is the integral estimation*

$$\alpha\sup_{0\le t\le T}\Big(\int_\Omega |\Delta\Phi|^2 dxdy + \int_\Omega |\nabla\Delta\Phi|^2 dxdy\Big) + \alpha^2\int_0^T\int_\Omega |\Delta^2\Phi|^2 dxdydt$$
$$+ \int_0^T\int_\Omega (|\Phi_t|^2 + 2|\nabla\Phi_t|^2 + |\Delta\Phi_t|^2)dxdydt$$
$$\le C\int_\Omega (|\Phi_0|^2 + |\nabla\Phi_0|^2 + |\Delta\Phi_0|^2 + |\nabla\Delta\Phi_0|^2)dxdy$$
$$+ C\int_\Omega (|\Psi_0|^2 + |\nabla\Psi_0|^2)dxdy + C\int_0^T\int_\Omega |f|^2 dxdydt \tag{17}$$

**Proof** From Equation (1), we can easily get the integral identity

$$\int_\Omega |\Phi_t - \Delta\Phi_t + \alpha\Delta^2\Phi|^2 dxdy = \int_\Omega |\Delta\Phi_x\Phi_y - \Delta\Phi_y\Phi_x + (1-\gamma)\Phi_y - \gamma\Psi_y|^2 dxdy$$

That is the identity

$$\begin{aligned}\alpha\frac{d}{dt}\int_\Omega(|\Delta\Phi|^2 + |\bigtriangledown\Delta\Phi|^2)dxdy + \int_\Omega(|\Phi_t|^2 + 2|\bigtriangledown\Phi_t|^2 + |\Delta\Phi_t|^2 + \alpha^2|\Delta^2\Phi|^2)dxdy \\ = \int_\Omega |\Delta\Phi_x\Phi_y - \Delta\Phi_y\Phi_x + (1-\gamma)\Phi_y - \gamma\Psi_y|^2 dxdy\end{aligned}$$

The right hand side of this identity can be estimated in the following way

$$\begin{aligned}&\int_\Omega |\Delta\Phi_x\Phi_y - \Delta\Phi_y\Phi_x + (1-\gamma)\Phi_y - \gamma\Psi_y|^2 dxdy \\ &\le C\|\bigtriangledown\Phi(\cdot,\cdot,t)\|_\infty^2 \int_\Omega |\bigtriangledown\Delta\Phi|^2 dxdy + C\int_\Omega |\bigtriangledown\Phi|^2 dxdy + C\int_\Omega|\bigtriangledown\Psi|^2 dxdy \\ &\le C\sup_{0\le t\le T}\int_\Omega|\bigtriangledown\Phi|^2dxdy\int_\Omega|\bigtriangledown\Delta\Phi|^2dxdy + C\Big(\int_\Omega|\bigtriangledown\Delta\Phi|^2dxdy\Big)^2 \\ &+ C\int_\Omega|\bigtriangledown\Phi|^2dxdy + C\int_\Omega|\bigtriangledown\Psi|^2dxdy\end{aligned}$$

Now we have the integral inequality

$$\begin{aligned}\alpha\frac{d}{dt}\int_\Omega(|\Delta\Phi|^2 + |\bigtriangledown\Delta\Phi|^2)dxdy + \int_\Omega(|\Phi_t|^2 + 2|\bigtriangledown\Phi_t|^2 + |\Delta\Phi_t|^2 + \alpha^2|\Delta^2\Phi|^2)dxdy \\ \le C\sup_{0\le t\le T}\int_\Omega|\bigtriangledown\Phi|^2dxdy\int_\Omega|\bigtriangledown\Delta\Phi|^2dxdy + C\Big(\int_\Omega|\bigtriangledown\Delta\Phi|^2dxdy\Big)^2 \\ + C\sup_{0\le t\le T}\int_\Omega|\bigtriangledown\Phi|^2dxdy + C\sup_{0\le t\le T}\int_\Omega|\bigtriangledown\Psi|^2dxdy\end{aligned}$$

By means of Lemma 2 and Gronwall's inequality, we obtain (17).

**Corollary 2** *Under the conditions of Lemma 3, we have the estimate*

$$\sup_{0\le t\le T}\|\bigtriangledown\Phi(\cdot,\cdot,t)\|_\infty \le C\Big(\|\Phi_0\|_3^2 + \|\Psi_0\|_1^2 + \int_0^T\|f(\cdot,\cdot,t)\|^2dt\Big)^{\frac{1}{2}} \tag{18}$$

**Corollary 3** *Under the conditions of Lemma 3, there is the estimate*

$$\begin{aligned}&\sup_{0\le t\le T}\Big(\int_\Omega|\Phi_t|^2dxdy + 2\int_\Omega|\bigtriangledown\Phi_t|^2dxdy + \int_\Omega|\Delta\Phi_t|^2dxdy\Big) \\ &\le C\int_\Omega(|\Phi_0|^2 + |\bigtriangledown\Phi_0|^2 + |\Delta\Phi_0|^2 + |\bigtriangledown\Delta\Phi_0|^2)dxdy \\ &+ C\int_\Omega(|\Psi_0|^2 + |\bigtriangledown\Psi_0|^2)dxdy + C\int_0^T\int_\Omega|f|^2dxdydt\end{aligned} \tag{19}$$

$$\sup_{0\le t\le T}\|\Phi_t(\cdot,\cdot,t)\|_{L^\infty(\Omega)} \le C\Big[\int_\Omega(|\Phi_0|^2+|\nabla\Phi_0|^2+|\Delta\Phi_0|^2+|\nabla\Delta\Phi_0|^2\Big)dxdy$$
$$+\int_\Omega(|\Psi_0|^2+|\nabla\Psi_0|^2)dxdy+\int_0^T\int_\Omega|f|^2dxdydt\Big]^{\frac{1}{2}} \tag{20}$$

**Lemma 4** *Assume that* $\Phi_0(x,y)\in H^3(\Omega)$, $\Psi_0(x,y)\in H^3(\Omega)$, $f(x,y,t)\in L^2(0,T;H^2(\Omega))$. *We have the integral estimation*

$$\beta\sup_{0\le t\le T}\int_\Omega|\nabla\Delta\Psi|^2dxdy+\int_0^T\int_\Omega|\Delta\Psi_t|^2dxdydt+\int_0^T\int_\Omega|\Delta^2\Psi|^2dxdydt$$
$$\le C\int_0\Big(|\Phi_0|^2+|\nabla\Phi_0|^2+|\Delta\Phi_0|^2+|\nabla\Delta\Phi_0|^2)dxdy$$
$$+C\int_\Omega(|\Psi_0|^2+|\nabla\Psi_0|^2+|\Delta\Psi_0|^2+|\nabla\Delta\Psi_0|^2)dxdy$$
$$+C\int_0^T\int_\Omega(|f|^2+|\Delta f|^2)dxdydt \tag{21}$$

**Proof** From Equation (2), we easily get the integral identity

$$\int_\Omega|\Delta\Psi_t-\beta\Delta^2\Psi|^2dxdy=\int_\Omega|\Delta(\Phi_x\Psi_y-\Phi_y\Psi_x)+\gamma\Delta\Phi_y+\delta\Delta\Psi_y-\Delta f|^2dxdy$$

therefore we get the relation

$$\beta\frac{d}{dt}\int_\Omega|\nabla\Delta\Psi|^2dxdy+\int_\Omega|\Delta\Psi_t|^2dxdy+\beta^2\int_\Omega|\Delta^2\Psi|^2dxdy$$
$$=\int_\Omega|\Delta(\Phi_x\Psi_y-\Phi_y\Psi_x)+\gamma\Delta\Phi_y+\delta\Delta\Psi_y-\Delta f|^2dxdy$$
$$\le C\|\nabla\Psi(\cdot,\cdot,t)\|_\infty^2\int_\Omega|\nabla\Delta\Phi|^2dxdy+C\|\nabla\Phi(\cdot,\cdot,t)\|_\infty^2\int_\Omega|\nabla\Delta\Psi|^2dxdy$$
$$+C(\|\Phi_{xx}(\cdot,\cdot,t)\|_\infty^2+\|\Phi_{xy}(\cdot,\cdot,t)\|_\infty^2+\|\Phi_{yy}(\cdot,\cdot,t)\|_\infty^2)\int_\Omega|\Delta\Psi|^2dxdy$$
$$+C\int_\Omega|\nabla\Delta\Phi|^2dxdy+C\int_\Omega|\nabla\Delta\Psi|^2dxdy+\int_\Omega|\Delta f|^2dxdy$$

The right hand side of this inequality can be estimated in the following way

$$\|\nabla\Psi(\cdot,\cdot,t)\|_\infty^2\int_\Omega|\nabla\Delta\Phi|^2dxdy$$
$$\le C\|\nabla\Psi(\cdot,\cdot,t)\|\,\|\nabla\Delta\Psi(\cdot,\cdot,t)\|\int_\Omega|\nabla\Delta\Phi|^2dxdy$$
$$\le C\int_\Omega|\nabla\Psi|^2dxdy\sup_{0\le t\le T}\int_\Omega|\nabla\Delta\Phi|^2dxdy$$
$$+C\int_\Omega|\nabla\Delta\Psi|^2dxdy\sup_{0\le t\le T}\int_\Omega|\nabla\Delta\Phi|^2dxdy,$$

$$\begin{aligned}
&\| \nabla \Phi(\cdot,\cdot,t)\|_\infty^2 \int_\Omega | \nabla \Delta\Psi|^2 dxdy \\
&\quad \le C\| \nabla \Phi(\cdot,\cdot,t)\| \, \| \nabla \Delta\Phi(\cdot,\cdot,t)\| \int_\Omega | \nabla \Delta\Psi|^2 dxdy \\
&\quad \le C \sup_{0\le t\le T} \int_\Omega | \nabla \Phi|^2 dxdy \int_\Omega | \nabla \Delta\Psi|^2 dxdy \\
&\qquad + C \sup_{0\le t\le T} \int_\Omega | \nabla \Delta\Phi|^2 dxdy \int_\Omega | \nabla \Delta\Psi|^2 dxdy
\end{aligned}$$

$$\begin{aligned}
&(\|\Phi_{xx}(\cdot,\cdot,t)\|_\infty^2 + \|\Phi_{xy}(\cdot,\cdot,t)\|_\infty^2 + \|\Phi_{yy}(\cdot,\cdot,t)\|_\infty^2) \int_\Omega |\Delta\Psi|^2 dxdy \\
&\quad \le C\|\Delta\Phi(\cdot,\cdot,t)\| \, \|\Delta^2\Phi(\cdot,\cdot,t)\| \, \| \nabla \Psi(\cdot,\cdot,t)\| \, \| \nabla \Delta\Psi(\cdot,\cdot,t)\| \\
&\quad \le C \int_\Omega |\Delta^2\Phi|^2 dxdy + C \int_\Omega | \nabla \Delta\Psi|^2 dxdy
\end{aligned}$$

At last, we obtain the integral inequality

$$\begin{aligned}
&\beta \frac{d}{dt} \int_\Omega | \nabla \Delta\Psi|^2 dxdy + \int_\Omega |\Delta\Psi_t|^2 dxdy + \beta^2 \int_\Omega |\Delta^2\Psi|^2 dxdy \\
&\quad \le C \int_\Omega | \nabla \Psi|^2 dxdy + C \int_\Omega | \nabla \Delta\Psi|^2 dxdy + C \int_\Omega |\Delta^2\Phi|^2 dxdy \\
&\qquad + C \int_\Omega | \nabla \Delta\Phi|^2 dxdy + C \int_\Omega |\Delta f|^2 dxdy
\end{aligned}$$

By means of Lemma 3 and Gronwall's inequality, we obtain (21).

**Corollary 4** *Under the conditions of Lemma 4, we have the estimate*

$$\sup_{0\le t\le T} (\|\Psi(\cdot,\cdot,t)\|_\infty + \| \nabla \Psi(\cdot,\cdot,t)\|_\infty) \le C\Big(\|\Phi_0\|_3^2 + \|\Psi_0\|_3^2 + \int_0^T \|f(\cdot,\cdot,t)\|_2^2 dt\Big)^{\frac{1}{2}} \quad (22)$$

**Corollary 5** *Under the conditions of Lemma 4, we have the estimate*

$$\begin{aligned}
&\sup_{0\le t\le T} \Big( \int_\Omega |\Psi_t|^2 dxdy + \int_\Omega |\Delta\Psi_t|^2 dxdy\Big) \le C \int_\Omega (|\Phi_0|^2 + | \nabla \Phi_0|^2 + |\Delta\Phi_0|^2 \\
&\quad + | \nabla \Delta\Phi_0|^2\Big) dxdy + C \int_\Omega (|\Psi_0|^2 + | \nabla \Psi_0|^2 + |\Delta\Psi_0|^2 + | \nabla \Delta\Psi_0|^2) dxdy \\
&\quad + C \int_0^T \int_\Omega (|f|^2 + |\Delta f|^2) dxdydt
\end{aligned} \quad (23)$$

$$\begin{aligned}
&\sup_{0\le t\le T} \|\Psi_t(\cdot,\cdot,t)\|_{L^\infty(\Omega)} \le C\Big[ \int_\Omega (|\Phi_0|^2 + | \nabla \Phi_0|^2 + |\Delta\Phi_0|^2 + | \nabla \Delta\Phi_0|^2\Big) dxdy \\
&\quad + \int_\Omega (|\Psi_0|^2 + | \nabla \Psi_0|^2 + |\Delta\Psi_0|^2 + | \nabla \Delta\Psi_0|^2) dxdy \\
&\quad + \int_0^T \int_\Omega (|f|^2 + |\Delta f|^2) dxdydt\Big]^{\frac{1}{2}}
\end{aligned} \quad (24)$$

**Lemma 5** *Suppose that $\Phi_0(x,y)\in H^5(\Omega)$, $\Psi_0(x,y)\in H^3(\Omega)$, $f(x,y,t)\in L^2(0,T;H^2(\Omega))$. We have the integral estimation*

$$\begin{aligned}
&\alpha\sup_{0\le t\le T}\Big(\int_\Omega|\nabla\Delta^2\Phi|^2dxdy+\int_\Omega|\Delta^2\Phi|^2dxdy)\\
&\qquad+\int_0^T\int_\Omega(|\Delta\Phi_t|^2+2|\nabla\Delta\Phi_t|^2+|\Delta^2\Phi_t|^2+\alpha^2|\Delta^3\Phi|^2)dxdydt\\
&\qquad\le C\Big(\|\Phi_0\|_5^2+\|\Psi_0\|_3^2+\int_0^T\|f(\cdot,\cdot,t)\|_2^2dt\Big)^{\frac12}
\end{aligned}\tag{25}$$

**Proof** From Equation (1), we can easily get the integral identity

$$\begin{aligned}
&\int_\Omega|\Delta\Phi_t-\Delta^2\Phi_t+\alpha\Delta^3\Phi|^2dxdy\\
&\quad=\int_\Omega|\Delta(\Delta\Phi_x\Phi_y-\Delta\Phi_y\Phi_x)+(1-\gamma)\Delta\Phi_y-\gamma\Delta\Psi_y|^2dxdy
\end{aligned}$$

That is the identity

$$\begin{aligned}
&\alpha\frac{d}{dt}\int_\Omega(|\Delta^2\Phi|^2+|\nabla\Delta^2\Phi|^2)dxdy\\
&\qquad+\int_\Omega(|\Delta\Phi_t|^2+2|\nabla\Delta\Phi_t|^2+|\Delta^2\Phi_t|^2+\alpha^2|\Delta^3\Phi|^2)dxdy\\
&\quad=\int_\Omega|\Delta(\Delta\Phi_x\Phi_y-\Delta\Phi_y\Phi_x+(1-\gamma)\Delta\Phi_y-\gamma\Delta\Psi_y|^2dxdy
\end{aligned}$$

The right hand side of this identity can be estimated in the following way

$$\begin{aligned}
&\int_\Omega|\Delta(\Delta\Phi_x\Phi_y-\Delta\Phi_y\Phi_x)+(1-\gamma)\Delta\Phi_y-\gamma\Delta\Psi_y|^2dxdy\\
&\le C\|\nabla\Phi(\cdot,\cdot,t)\|_\infty^2\int_\Omega|\nabla\Delta^2\Phi|^2dxdy+C\|\nabla\Delta\Phi(\cdot,\cdot,t)\|_\infty^2\int_\Omega|\nabla\Delta\Phi|^2dxdy\\
&\quad+C(\|\Phi_{xx}(\cdot,\cdot,t)\|_\infty^2+\|\Phi_{xy}(\cdot,\cdot,t)\|_\infty^2+\|\Phi_{yy}(\cdot,\cdot,t)\|_\infty^2)\int_\Omega|\Delta^2\Phi|^2dxdy\\
&\quad+C\int_\Omega|\nabla\Delta\Phi|^2dxdy+C\int_\Omega|\nabla\Delta\Psi|^2dxdy\\
&\le C\sup_{0\le t\le T}\|\nabla\Phi(\cdot,\cdot,t)\|_\infty^2\int_\Omega|\nabla\Delta^2\Phi|^2dxdy\\
&\quad+C\|\nabla\Delta\Phi(\cdot,\cdot,t)\|\,\|\nabla\Delta^2\Phi(\cdot,\cdot,t)\|\sup_{0\le t\le T}\int_\Omega|\nabla\Delta\Phi|^2dxdy\\
&\quad+C\|\Delta\Phi(\cdot,\cdot,t)\|\,\|\Delta^2\Phi(\cdot,\cdot,t)\|\,\|\nabla\Delta\Phi(\cdot,\cdot,t)\|\,\|\nabla\Delta^2\Phi(\cdot,\cdot,t)\|\\
&\quad+C\int_\Omega|\nabla\Delta\Phi|^2dxdy+C\int_\Omega|\nabla\Delta\Psi|^2dxdy\\
&\le C\int_\Omega|\nabla\Delta^2\Phi|^2dxdy+C\int_\Omega|\Delta^2\Phi|^2dxdy+C\int_\Omega|\nabla\Delta\Phi|^2dxdy
\end{aligned}$$

$$+C\int_\Omega|\nabla\Delta\Psi|^2dxdy$$

Now we have the integral inequality

$$\alpha\frac{d}{dt}\int_\Omega(|\Delta^2\Phi|^2+|\nabla\Delta^2\Phi|^2)dxdy+\int_\Omega(|\Delta\Phi_t|^2+2|\nabla\Delta\Phi_t|^2+|\Delta^2\Phi_t|^2+\alpha^2|\Delta^3\Phi|^2)dxdy$$
$$\le C\int_\Omega|\nabla\Delta^2\Phi|^2dxdy+C\int_\Omega|\Delta^2\Phi|^2dxdy+C\int_\Omega|\nabla\Delta\Phi|^2dxdy$$
$$+C\int_\Omega|\nabla\Delta\Psi|^2dxdy$$

By means of Lemma 4 and Gronwall's inequality, we obtain (25).

**Corollary 6** *Under the conditions of Lemma 5, we have the estimate*

$$\sup_{0\le t\le T}\|\nabla^3\Phi(\cdot,\cdot,t)\|_\infty\le C\Big(\|\Phi_0\|_5^2+\|\Psi_0\|_3^2+\int_0^T\|f(\cdot,\cdot,t)\|_2^2dt\Big)^{\frac12}\tag{26}$$

**Corollary 7** *Under the conditions of Lemma 5, we have the estimate*

$$\sup_{0\le t\le T}\int_\Omega(|\Delta\Phi_t|^2+2|\nabla\Delta\Phi_t|^2+|\Delta^2\Phi_t|^2)dxdy\le C\Big(\|\Phi_0\|_5^2+\|\Psi_0\|_3^2+\int_0^T\|f(\cdot,\cdot,t)\|_2^2dt\Big)\tag{27}$$

$$\sup_{0\le t\le T}\|\nabla^2\Phi_t(\cdot,\cdot,t)\|_\infty\le C\Big(\|\Phi_0\|_5^2+\|\Psi_0\|_3^2+\int_0^T\|f(\cdot,\cdot,t)\|_2^2dt\Big)\tag{28}$$

**Lemma 6** *Assume that* $\Phi_0(x,y)\in H^5(\Omega)$, $\Psi_0(x,y)\in H^5(\Omega)$, $f(x,y,t)\in L^2(0,T;H^4(\Omega))$. *We have the integral estimation*

$$\beta\sup_{0\le t\le T}\int_\Omega|\nabla\Delta^2\Psi|^2dxdy+\int_0^T\int_\Omega|\Delta^2\Psi_t|^2dxdydt+\beta^2\int_0^T\int_\Omega|\Delta^3\Psi|^2dxdydt$$
$$\le C\Big(\|\Phi_0\|_5^2+\|\Psi_0\|_5^2+\int_0^T\|f(\cdot,\cdot,t)\|_2^2dt\Big)^{\frac12}\tag{29}$$

**Proof** From Equation (2), we have the integral identity

$$\int_\Omega|\Delta^2\Psi_t-\beta\Delta^3\Psi|^2dxdy=\int_\Omega|\Delta^2(\Phi_x\Psi_y-\Phi_y\Psi_x)+\gamma\Delta^2\Phi_y+\delta\Delta^2\Psi_y-\Delta^2f|^2dxdy$$

therefore we get the relation

$$\beta\frac{d}{dt}\int_\Omega|\nabla\Delta^2\Psi|^2dxdy+\int_\Omega|\Delta^2\Psi_t|^2dxdy+\beta^2\int_\Omega|\Delta^3\Psi|^2dxdy$$
$$=\int_\Omega|\Delta^2(\Phi_x\Psi_y-\Phi_y\Psi_x)+\gamma\Delta^2\Phi_y+\delta\Delta^2\Psi_y-\Delta^2f|^2dxdy$$

$$\begin{aligned}
&\le C\|\nabla\Psi(\cdot,\cdot,t)\|_\infty^2\int_\Omega|\nabla\Delta^2\Phi|^2dxdy+C\|\nabla\Phi(\cdot,\cdot,t)\|_\infty^2\int_\Omega|\nabla\Delta^2\Psi|^2dxdy\\
&\quad+C\|\nabla\Delta\Phi(\cdot,\cdot,t)\|_\infty^2\int_\Omega|\nabla\Delta\Psi|^2dxdy+C\Big(\|\Psi_{xx}(\cdot,\cdot,t)\|_\infty^2+\|\Psi_{xy}(\cdot,\cdot,t)\|_\infty^2\\
&\quad+C(\|\Psi_{yy}(\cdot,\cdot,t)\|_\infty^2)\Big)\int_\Omega|\Delta^2\Phi|^2dxdy+C\Big(\|\Phi_{xx}(\cdot,\cdot,t)\|_\infty^2+\|\Phi_{xy}(\cdot,\cdot,t)\|_\infty^2\\
&\quad+\|\Phi_{yy}(\cdot,\cdot,t)\|_\infty^2\Big)\int_\Omega|\Delta^2\Psi|^2dxdy+C\int_\Omega|\nabla\Delta^2\Phi|^2dxdy\\
&\quad+C\int_\Omega|\nabla\Delta^2\Psi|^2dxdy+C\int_\Omega|\Delta^2 f|^2dxdy
\end{aligned}$$

The right hand side of this inequality can be estimated in the following way

$$\begin{aligned}
&\|\nabla\Psi(\cdot,\cdot,t)\|_\infty^2\int_\Omega|\nabla\Delta^2\Phi|^2dxdy\le C\int_\Omega|\nabla\Delta^2\Phi|^2dxdy\\
&\|\nabla\Phi(\cdot,\cdot,t)\|_\infty^2\int_\Omega|\nabla\Delta^2\Psi|^2dxdy\le C\int_\Omega|\nabla\Delta^2\Psi|^2dxdy\\
&\|\nabla\Delta\Phi(\cdot,\cdot,t)\|_\infty^2\int_\Omega|\nabla\Delta\Psi|^2dxdy\le C\int_\Omega|\nabla\Delta\Psi|^2dxdy\\
&(\|\Psi_{xx}(\cdot,\cdot,t)\|_\infty^2+\|\Psi_{xy}(\cdot,\cdot,t)\|_\infty^2+C(\|\Psi_{yy}(\cdot,\cdot,t)\|_\infty^2))\int_\Omega|\Delta^2\Phi|^2dxdy\\
&\qquad\le C\|\Delta\Psi(\cdot,\cdot,t)\|\,\|\Delta^2\Psi(\cdot,\cdot,t)\|\,\|\nabla\Delta\Phi(\cdot,\cdot,t)\|\,\|\nabla\Delta^2\Phi(\cdot,\cdot,t)\|\\
&\qquad\le C\int_\Omega|\Delta^2\Phi|^2dxdy+C\int_\Omega|\nabla\Delta^2\Phi|^2dxdy\\
&(\|\Phi_{xx}(\cdot,\cdot,t)\|_\infty^2+\|\Phi_{xy}(\cdot,\cdot,t)\|_\infty^2+\|\Phi_{yy}(\cdot,\cdot,t)\|_\infty^2)\int_\Omega|\Delta^2\Psi|^2dxdy\\
&\qquad\le C\int_\Omega|\Delta^2\Psi|^2dxdy
\end{aligned}$$

At last, we obtain the integral inequality

$$\begin{aligned}
&\beta\frac{d}{dt}\int_\Omega|\nabla\Delta^2\Psi|^2dxdy+\int_\Omega|\Delta^2\Psi_t|^2dxdy+\beta^2\int_\Omega|\Delta^3\Psi|^2dxdy\\
&\quad\le C\int_\Omega|\Delta^2\Psi|^2dxdy+C\int_\Omega|\Delta^2\Phi|^2dxdy\\
&\quad\quad+C\int_\Omega|\nabla\Delta^2\Phi|^2dxdy+C\int_\Omega|\nabla\Delta^2\Psi|^2dxdy+C\int_\Omega|\Delta^2 f|^2dxdy
\end{aligned}$$

By means of Lemma 5 and Gronwall's inequality, we obtain (29).

**Corollary 8** *Under the conditions of Lemma 6, we have the estimate*

$$\sup_{0\le t\le T}\|\nabla^3\Psi(\cdot,\cdot,t)\|_\infty\le C\Big(\|\Phi_0\|_5+\|\Psi_0\|_5+\Big(\int_0^T\|f(\cdot,\cdot,t)\|_4^2dt\Big)^{\frac{1}{2}}\tag{30}$$

**Corollary 9** *Under the conditions of Lemma 6, we have the estimate*

$$\sup_{0\le t\le T}\int_\Omega|\Delta^2\Psi_t|^2dxdy\le C\int_\Omega(|\Phi_0|^2+|\nabla\Phi_0|^2+|\Delta\Phi_0|^2+|\nabla\Delta\Phi_0|^2$$

$$+|\Delta^2\Phi_0|^2+|\nabla\Delta^2\Phi_0|^2)dxdy$$
$$+C\int_\Omega(|\Psi_0|^2+|\nabla\Psi_0|^2+|\Delta\Psi_0|^2+|\nabla\Delta\Psi_0|^2+|\Delta^2\Psi_0|^2+|\nabla\Delta^2\Psi_0|^2)dxdy$$
$$+C\int_0^T\int_\Omega(|f|^2+|\Delta f|^2+|\Delta^2 f|^2)dxdydt \tag{31}$$

$$\sup_{0\le t\le T}\|\nabla^2\Psi_t(\cdot,\cdot,t)\|_\infty\le C\Big[\int_\Omega(|\Phi_0|^2+|\nabla\Phi_0|^2+|\Delta\Phi_0|^2$$
$$+|\nabla\Delta\Phi_0|^2+|\Delta^2\Phi_0|^2+|\nabla\Delta^2\Phi_0|^2)dxdy$$
$$+\int_\Omega(|\Psi_0|^2+|\nabla\Psi_0|^2+|\Delta\Psi_0|^2+|\nabla\Delta\Psi_0|^2+|\Delta^2\Psi_0|^2+|\nabla\Delta^2\Psi_0|^2)dxdy$$
$$+\int_0^T\int_\Omega(|f|^2+|\Delta f|^2+|\Delta^2 f|^2)dxdydt\Big]^{\frac{1}{2}} \tag{32}$$

**Lemma 7** *Suppose that $\Phi_0(x,y)\in H^7(\Omega)$, $\Psi_0(x,y)\in H^5(\Omega)$, $f(x,y,t)\in L^2(0,T;H^4(\Omega_T))$. Then there is the integral estimation*

$$\alpha\sup_{0\le t\le T}\Big(\int_\Omega|\Delta^3\Phi|^2dxdy+\int_\Omega|\nabla\Delta^3\Phi|^2dxdy\Big)+\alpha^2\int_0^T\int_\Omega|\Delta^4\Phi|^2dxdydt$$
$$+\int_0^T\int_\Omega(|\Delta^2\Phi_t|^2+2|\nabla\Delta^2\Phi_t|^2+|\Delta^3\Phi_t|^2)dxdydt$$
$$\le C\Big(\|\Phi_0\|_7^2+\|\Psi_0\|_5^2+\int_0^T\|f(\cdot,\cdot,t)\|_4^2dt\Big) \tag{33}$$

**Corollary 10** *Under the conditions of Lemma 7, we have the estimate*

$$\sup_{0\le t\le T}\|\nabla^5\Phi(\cdot,\cdot,t)\|_\infty\le C\Big(\|\Phi_0\|_9^2+\|\Psi_0\|_7^2+\int_0^T\|f(\cdot,\cdot,t)\|^2\Big)^{\frac{1}{2}} \tag{34}$$

**Corollary 11** *Under the conditions of Lemma 7, we have the estimate*

$$\sup_{0\le t\le T}\int_\Omega(|\Delta^2\Phi_t|^2+2|\nabla\Delta^2\Phi_t|^2+|\Delta^3\Phi_t|^2)dxdy$$
$$\le C\Big(\|\Phi_0\|_7^2+\|\Psi_0\|_5^2+\int_0^T\|f(\cdot,\cdot,t)\|_4^2dt\Big) \tag{35}$$
$$\sup_{0\le t\le T}\|\nabla^4\Phi_t(\cdot,\cdot,t)\|_\infty\le C\Big(\|\Phi_0\|_7^2+\|\Psi_0\|_5^2+\int_0^T\|f(\cdot,\cdot,t)\|_4^2dt\Big)^{\frac{1}{2}} \tag{36}$$

## 3. Existence and Uniqueness

**Lemma 8** *Suppose that $\alpha$ is a positive constant, the initial function $\Phi_0(x,y)$ satisfies $\Phi_0(x,y)\in H^k(\Omega)$, $f(x,y,t)$ satisfies $f(x,y,t)\in L^2(\Omega_T)$. Then the periodic boundary value problem (3) and (5) for the following parabolic equation*

$$\Phi_t+\alpha(-1)^k\Delta^k\Phi=f(x,y,t)$$

*has a unique global solution* $\Phi(x,y,t) \in L^\infty(0,T;H^k(\Omega)) \cap L^2(0,T;H^{2k}(\Omega)) \cap W^{1,2}(0,T;L^2(\Omega))$, *where* $k=1,2$.

**Lemma 9** *Suppose that* $\alpha$ *and* $\beta$ *are positive constants, the initial functions* $\Phi_0(x,y)$ *and* $\Psi_0(x,y)$ *satisfy* $\Phi_0(x,y) \in H^3(\Omega)$, $\Psi_0(x,y) \in H^1(\Omega)$, $f(x,y,t)$ *and* $g(x,y,t)$ *satisfy* $f(x,y,t) \in L^2(\Omega_T)$, $g(x,y,t) \in L^2(\Omega_T)$. *Then the periodic boundary value problem* (3) *and* (5) *for the following coupled linear parabolic equation*

$$\Phi_t - \Delta\Phi_t + \alpha\Delta^2\Phi + (1-\gamma)\Phi_y - \gamma\Psi_y = g(x,y,t), \quad (x,y) \in R^2,\ t>0 \tag{37}$$

$$\Psi_t - \beta\Delta\Psi + \gamma\Phi_y + \Delta\Psi_y = f(x,y,t), \quad (x,y) \in R^2,\ t>0,\ \alpha>0, \beta>0 \tag{38}$$

*has a unique global solution* $\Phi(x,y,t) \in L^\infty(0,T;H^3(\Omega)) \cap L^2(0,T;H^4(\Omega)) \cap W^{1,2}(0,T;L^2(\Omega))$, $\Psi(x,y,t) \in L^\infty(0,T;H^1(\Omega)) \cap L^2(0,T;H^2(\Omega)) \cap W^{1,2}(0,T;L^2(\Omega))$.

**Proof** This Lemma can be proved with the technique of continuation of parameter.

**Theorem 1** *Suppose* $\Phi_0(x,y) \in H^7(\Omega)$, $\Psi_0(x,y) \in H^5(\Omega)$, $f(x,y,t) \in L^2(0,T;H^2(\Omega))$. *Then the periodic boundary value Problem* (3)–(6) *for the coupled nonlinear parabolic equations* (1) (2) *has unique classical global solution* $\Phi(x,y,t) \in L^\infty(0,T;H^7(\Omega)) \cap L^2(0,T;H^8(\Omega)) \cap W^{1,2}(0,T;H^6(\Omega)) \cap W^{1,\infty}(0,T;H^6(\Omega_T))$, $\Psi(x,y,t) \in L^\infty(0,T;H^5(\Omega)) \cap L^2(0,T;H^6(\Omega)) \cap W^{1,2}(0,T;H^4(\Omega)) \cap W^{1,\infty}(0,T;H^4(\Omega))$.

**Proof** It is well known now that the existence of global solutions to nonlinear evolution equations and systems can be established normally by using Leray-Schauder's fixed point principle. For reference we refer readers to Zhou and Guo [4]. The main difficulty in the study of the global existence is to verify *a priori* estimates which govern our strategy to prove the existence of the global solutions. Here, for the sake of shortness, we merely omit the detailed and standard procedures. The uniqueness of the solutions can also be proved normally and therefore is omitted.

## 4. IVP for the Coupled Equations (1) and (2)

The *a priori* estimates of the global solutions of the periodic boundary value problem (3)–(6) for the coupled nonlinear parabolic equations are independent of the period $2D$ of the domain $\Omega$, we can employ the usual method of limiting process $D \to \infty$ and the so-called diagonal selection to obtain the solution of the initial value problem.

**Theorem 2** *Suppose* $\Phi_0(x,y) \in H^7(R^2)$, $\Psi_0(x,y) \in H^5(R^2)$, $f(x,y,t) \in L^2(0,T;H^4(R^2))$. *Then the initial value Problem* (7)–(8) *for the coupled nonlinear parabolic equations* (1) (2) *has a unique classical global solutions*

$$\begin{aligned}\Phi(x,y,t) \in L^\infty(0,T;H^7(R^2)) \cap L^2(0,T;H^8(R^2)) \cap W^{1,2}(0,T;H^6(R^2))\\ \cap W^{1,\infty}(0,T;H^6(R^2)),\\ \Psi(x,y,t) \in L^\infty(0,T;H^5(R^2)) \cap L^2(0,T;H^6(R^2)) \cap W^{1,2}(0,T;H^4(R^2))\\ \cap W^{1,\infty}(0,T;H^4(R^2)).\end{aligned}$$

Although the method we employed in this paper is mainly integral estimate, it is somewhat different from the usual integral estimate, this is due to the contribution of the dissipative term $\alpha\Delta^2\Phi$ in Equation (1).

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