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A QUASISTEADY STEFAN PROBLEM WITH CURVATURE CORRECTION AND KINETIC UNDERCOOLING*

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Abstract A quasisteady Stefan problem with curvature correction and kinetic undercooling is considered. It is a problem with phase transition, in which not only the Stefan condition, but also the curvature correction and kinetic undercooling effects hold on the free boundary, and in phase regions elliptic equations are satisfied by the unknown temperature at each time. The existence and uniqueness of a local classical solution of this problem are obtained.
Key Words Stefan problem; curvature correction; kinetic undercooling.
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1. Introduction

Let Ω be a bounded domain in $\mathbf{R}^N$ and Γ^0 be a $(N-1)$-dimensional single-connected closed hypersurface in Ω. Our problem is to determine a function $u(x,t):\Omega\times[0,T]\to\mathbf{R}^1$ and a free boundary $\Gamma\equiv\bigcup_{0\le t\le T}(\Gamma_t\times\{t\})$, where $\Gamma_t(0\le t\le T)$ are $(N-1)$-dimensional single-connected closed hypersurfaces in Ω, such that

$$\begin{cases}\Delta u^i(\cdot,t)=0,\ \text{in } \Omega^i(t),\quad 0\le t\le T\quad i=1,2 & (1.1)\\ \dfrac{\partial u^1}{\partial n}=0\quad \text{on } \partial\Omega\times[0,T] & (1.2)\\ u^1=u^2=-\widehat{\alpha}\kappa-\widehat{\beta}V\quad \text{on } \Gamma_t,\quad 0\le t\le T & (1.3)\\ \dfrac{\partial u^1}{\partial\nu}-\dfrac{\partial u^2}{\partial\nu}=V\quad \text{on } \Gamma_t,\quad 0\le t\le T & (1.4)\\ \Gamma_0=\Gamma^0 & (1.5)\end{cases}$$

where, $u^i\equiv u$ in $\overline{\Omega^i(t)}\times[0,T]$, $i=1,2$; $\Omega^i(t)$ is the domain bounded by Γ_t for $i=1$, or by Γ_t and $\partial\Omega$ for $i=2$; n and ν are the unit outward normal vectors of Ω and $\Omega^1(t)$ respectively; κ is the mean curvature of Γ_t which takes positive value when $\Omega^1(t)$ protrudes into $\Omega^2(t)$; V is the velocity of Γ_t in the direction of ν; $\widehat{\alpha}$ and $\widehat{\beta}$ are positive constants.

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If (1.1) is replaced by the heat equation

$$u_t^i - \Delta u^i = 0 \text{ in } \bigcup_{0\le t\le T} (\Omega^i(t) \times \{t\}), \quad i = 1,2 \tag{1.6}$$

the system (1.2)–(1.6) is called the Stefan problem with the curvature correction and kinetic undercooling. This problem has been studied by many mathematicians. Chen, X. and Reitch, F. ([1]) established the existence and uniqueness of a local classical solution for $\widehat{\beta} > 0$ and $\widehat{\alpha} > 0$. Radkevich, E. ([2]) proved the existence of a local classical solution for $\widehat{\beta} \ge 0$ and $\widehat{\alpha} > 0$. Meimanov, A.M. ([3]) considered the problem in spherical symmetric case and obtained the existence and uniqueness of a global classical solution when $\widehat{\beta} > 0$ and $\widehat{\alpha} > 0$, and the nonexistence of a global classical solution when $\widehat{\beta} = 0$ and $\widehat{\alpha} > 0$. Liu, Z. ([4]) prove the existence and uniqueness of a global classical solution for $\widehat{\beta} > 0$ and $\widehat{\alpha} > 0$ in the two-dimensional axis symmetric case and he also got sufficient conditions for the convexness of the free boundary. And Luckhaus, S. ([5] [6]) set up a weak formulation for $\widehat{\beta} = 0$ and $\widehat{\alpha} > 0$, and proved the existence of a global weak solution of this problem.

For the quasisteady Stefan problem with curvature correction and kinetic undercooling, that is, the system (1.1)–(1.5), less results have been obtained. When $\widehat{\beta} = 0$, $\widehat{\alpha} > 0$ and $\partial\Omega$ being a chart of a Lipschitz function defined in $\mathbf{R}^1 (N = 2)$, Duchon, J. and Robert, R. ([7]) proved the existence and uniqueness of a local classical solution. When $\widehat{\beta} = 0$, $\widehat{\alpha} > 0$ and Ω being a bounded domain in $\mathbf{R}^2$, Chen, X. ([8]) established the existence of a local weak solution, and he also obtained the existence and asymptotic behaviors of a global weak solution when $\widehat{\beta} = 0$, $\widehat{\alpha} > 0$ and $\Omega = \mathbf{R}^2$.

In this paper we consider the system (1.1)–(1.5) in the case that $\widehat{\beta} > 0$ and $\widehat{\alpha} > 0$. We shall use the idea in [1] to formulate (1.3) as a parabolic equation on some smooth manifold. Our main results are the existence and uniqueness of a local classical solution. Without loss of generality, we might as well suppose $\widehat{\alpha} = \widehat{\beta} = 1$ in the sequel.

Now we state our main results as follows:

Theorem 1.1 *Assume that $\Gamma^0 \in C^{5+\alpha}$ and $\partial\Omega \in C^{3+\alpha}$ $(0 < \alpha < 1)$, then there exists a positive constant T, depending only on Γ^0 and Ω, such that, the system (1.1)–(1.5) has an unique classical solution $\{u, \Gamma\}$, satisfying*

$$u \in C^{2+\alpha,\frac{\alpha}{2}}(\overline{Q_T^1}) \cap C^{2+\alpha,\frac{\alpha}{2}}(\overline{Q_T^2}) \cap C(\overline{\Omega}_T) \tag{1.7}$$

$$u(\cdot,t) \in C^{3+\alpha}(\overline{\Omega^1(t)}) \cap C^{3+\alpha}(\overline{\Omega^2(t)}), \quad 0 \le t \le T \tag{1.8}$$

and

$$\Gamma \in C_{x,t}^{4+\alpha,\frac{2+\alpha}{2}}$$

where $\Omega_T \equiv \Omega \times (0,T)$, $Q_T^i \equiv \bigcup_{0\le t\le T} (\Omega^i(t) \times \{t\})$ $(i = 1,2)$

$$C_{x,t}^{2+\alpha,\frac{\alpha}{2}} \equiv \{v \mid v, v_x, v_{xx} \in C^{\alpha,\frac{\alpha}{2}}\} \tag{1.9}$$

and

$$C_{x,t}^{4+\alpha,\frac{2+\alpha}{2}} \equiv \{v \mid v, v_x, v_{xx}, v_{x^3}, v_{x^4}, v_t \in C^{\alpha,\frac{\alpha}{2}}\} \tag{1.10}$$

2. The Parabolic Equation on the Free Boundary

Let $\mathcal{M} \subset \mathbf{R}^N$ be a $(N-1)$-dimensional $C^{5+\alpha}$-manifold and the transformation

$$X^0 : \mathcal{M} \to \Gamma^0 \tag{2.1}$$

be a $C^{5+\alpha}$ diffeomorphism from $\mathcal{M}$ onto Γ^0.

Denote by s' the local coordinate of $\mathcal{M}$ and by $\nu^0(s')$ the unit outward normal vector of Γ^0 at $x = X^0(s')$. Set

$$X(s', s_N) = X^0(s') + s_N \nu^0(s') : \mathcal{M} \times [-L, L] \to \mathbf{R}^N \tag{2.2}$$

If L is small enough, then $X(s', s_N)$ is a $C^{5+\alpha}$-diffeomorphism from $\mathcal{M} \times [-L, L]$ onto some neighborhood of Γ^0 in $\mathbf{R}^N$.

Let $s' = (s_1, \cdots, s_{N-1})$, $s = (s', s_N)$ and $X(s) \equiv X(s', s_N)$. Denote the inverse of $x = X(s)$ by

$$S(x) = (S^1(x), \cdots, S^N(x)) \tag{2.3}$$

Now for any fixed positive constant T, we define a family of smooth hypersurfaces $\{\Gamma_t\}_{0 \le t \le T}$ by

$$\Gamma_t \equiv \{X(s', s_N) \mid s_N = \Lambda(s', t),\ s' \in \mathcal{M}\}, \quad 0 \le t \le T \tag{2.4}$$

where Λ is a C^2 function from $\mathcal{M} \times [0, T] \to [-L, L]$. Obviously, if L is small enough, then $\Gamma_t \subset\subset \Omega$. Thus, setting

$$v(s', t) \equiv u(X(s', \Lambda(s', t)), t), \quad s' \in \mathcal{M},\ 0 \le t \le T \tag{2.5}$$

(1.3) and (1.5) can be equivalently transformed into the following Cauchy problem of parabolic equations (see [1]):

$$\begin{cases} \dfrac{\partial \Lambda}{\partial t} = \displaystyle\sum_{i,j=1}^{N-1} a_{ij}(s', \Lambda, \nabla_{s'}\Lambda) \frac{\partial^2 \Lambda}{\partial s_i \partial s_j} + b(s', \Lambda, \nabla_{s'}\Lambda) \\ \qquad -c(s', \Lambda, \nabla_{s'}\Lambda) v(s', t), \quad s' \in \mathcal{M},\ 0 \le t \le T & (2.6) \\ \Lambda(s', 0) = 0, \quad s' \in \mathcal{M} & (2.7) \end{cases}$$

where

$$a_{ij}(s', s_N, p^1, \cdots, p^{N-1}) = \alpha^i \cdot \alpha^j - \frac{\displaystyle\sum_{k,l=1}^{N-1} (p^k \alpha^k \cdot \alpha^i)(p^l \alpha^l \cdot \alpha^j)}{1 + \left| \displaystyle\sum_{k=1}^{N-1} p^k \alpha^k \right|^2}$$

$$i,j=1,\cdots,N-1 \tag{2.8}$$

$$b(s',s_N,p^1,\cdots,p^{N-1})=\sum_{k=1}^{N-1}p^kTr(A^k)-Tr(A^N)-\frac{\sum_{k,l=1}^{N-1}p^kp^l(\alpha^k)^TA^N\alpha^l}{1+\left|\sum_{k=1}^{N-1}p^k\alpha^k\right|^2}$$
$$-\frac{\sum_{m,k,l=1}^{N-1}p^kp^lp^m(\alpha^k)^TA^m\alpha^l}{1+\left|\sum_{k=1}^{N-1}p^k\alpha^k\right|^2} \tag{2.9}$$

$$c(s',s_N,p^1,\cdots,p^{N-1})=\left(1+\left|\sum_{k=1}^{N-1}p^k\alpha^k\right|^2\right)^{\frac{1}{2}}\geq 1 \tag{2.10}$$

$$\alpha^i(s)=\left(\frac{\partial S^i(x)}{\partial x_1},\frac{\partial S^i(x)}{\partial x_2},\cdots,\frac{\partial S^i(x)}{\partial x_N}\right)^T_{x=X(s)},\quad i=1,\cdots,N-1 \tag{2.11}$$

$$A^i(s)=\begin{pmatrix}\frac{\partial^2S^i(x)}{\partial x_1\partial x_1}, & \cdots, & \frac{\partial^2S^i(x)}{\partial x_1\partial x_N}\\ \vdots & & \vdots\\ \frac{\partial^2S^i(x)}{\partial x_N\partial x_1}, & \cdots, & \frac{\partial^2S^i(x)}{\partial x_N\partial x_N}\end{pmatrix}_{x=X(s)},\quad i=1,\cdots,N \tag{2.12}$$

$$Tr(A^i)\equiv\text{the trace of }A^i,\quad i=1,\cdots,N \tag{2.13}$$

Lemma 2.1 *Let m and T be two constants, $0<T\leq 1$ and $m>0$. If*

$$\|v\|_{C^{2+\alpha,\frac{\alpha}{2}}(\mathcal{M}\times[0,T])}+\sup_{0\leq t\leq T}\|v(\cdot,t)\|_{C^3(\mathcal{M})}\leq m \tag{2.14}$$

then there exists a constant $T_m\in(0,T)$, such that, (2.7) and (2.6) has a unique solution $\Lambda(s',t)$ in $[0,T]$, which satisfies

$$\|\Lambda\|_{C^{4+\beta,\frac{\beta}{2}}(\mathcal{M}\times[0,T_m])}+\|\Lambda_t\|_{C^{\alpha,\frac{\alpha}{2}}(\mathcal{M}\times[0,T_m])}\leq C(m),\quad 0<\alpha<\beta<1 \tag{2.15}$$

where $\beta\in(\alpha,1)$ is a fixed constant, $C(m)$ is a constant independent of T and T_m depends only on m and T.

Proof It is easy to verify that, there exists a constant $C_0>0$, depending only on Γ^0, such that, if L is small enough, then

$$\frac{1}{C_0}|\xi|^2\leq\sum_{i,j=1}^{N-1}a_{ij}(s',s_N,p^1,\cdots,p^{N-1})\xi_i\xi_j\leq C_0|\xi|^2$$
$$\forall s'\in\mathcal{M},\ s_N\in[-L,L],\ p^i\in[-1,1],\ \xi\in\mathbf{R}^N$$

and

$$\|a_{ij}, b, c\|_{C^{3+\alpha}(\mathcal{M}\times[-L,L]\times[-1,1]^{N-1})} \le C_0, \quad i,j = 1,\cdots,N$$

Hence it follows from the theory of parabolic equations that there is a T_m, depending only on m and T, *s.t.*, (2.6) and (2.7) has a unique classical solution $\Lambda(s',t)$ in $[0,T_m]$. Moreover, by (2.14) and the Schauder's estimates, we have

$$\|\Lambda\|_{C^{4+\alpha,\frac{\alpha}{2}}(\mathcal{M}\times[0,T_m])} + \|\Lambda_t\|_{C^{\alpha,\frac{\alpha}{2}}(\mathcal{M}\times[0,T_m])} \le C(m)$$

which, together with (2.14) and the L^p theory of parabolic equations, implies

$$\|D^3_{s'}\Lambda\|_{W^{2,1}_p(\mathcal{M}\times[0,T_m])} \le C(m,p), \quad \forall 1 < p < +\infty$$

thus, (2.15) follows from Sobolev imbedding theorem.

If we continue v from $\mathcal{M}\times[0,T]$ to $\mathcal{M}\times[0,1]$ and preserve the inequality

$$\|v\|_{C^{2+\alpha,\frac{\alpha}{2}}}(\mathcal{M}\times[0,1]) + \sup_{0\le t\le 1}\|v(\cdot,t)\|_{C^3(\mathcal{M})} \le \widehat{C}(m)$$

where $\widehat{C}(m)$ is independent of T, then the constant $C(m)$ in (2.15) is obviously independent of T. Hence, we conclude the proof of Lemma 2.1.

3. An Approximating Elliptic Equation in the Bulk

For $0 < T \le 1$ and L small enough, we set

$$A_T \equiv \left\{\Lambda(s',t) \mid \|\Lambda\|_{C^{4+\alpha,\frac{\alpha}{2}}(\mathcal{M}\times[0,T])} \le \frac{L}{2}, \Lambda(s',0) = 0,\ s' \in \mathcal{M}\right\} \tag{3.1}$$

Arbitrarily taking $\Lambda \in A_T$ and $\varepsilon \in (0,1)$, we consider the following problem

$$\begin{cases} -\varepsilon u^i(\cdot,t) + \Delta u^i(\cdot,t) = 0 \text{ in } \Omega^i_\Lambda(t), \quad i = 1,2 & (3.2)\\ \left.\dfrac{\partial u^1(\cdot,t)}{\partial n}\right|_{\partial\Omega} = 0 & (3.3)\\ \dfrac{\partial u^1(\cdot,t)}{\partial \nu_\Lambda(t)} - \dfrac{\partial u^2(\cdot,t)}{\partial \nu_\Lambda(t)} = -\kappa_\Lambda(\cdot,t) - u^1(\cdot,t) \quad \text{on } \Gamma_\Lambda(t) & (3.4)\\ u^1(\cdot,t) = u^2(\cdot,t) \quad \text{on } \Gamma_\Lambda(t) & (3.5)\end{cases}$$

where $t \in [0,T]$, $\Gamma_\Lambda(t) \equiv \{x \in \widehat{\Omega} \mid S^N(x) = \Lambda(S^1(x),\cdots,S^{N-1}(x),t\}$, $\widehat{\Omega} \equiv \{x \in \Omega \mid \operatorname{dist}(x,\Gamma_0) \le L\}$, $\Omega^1_\Lambda(t)$ is the domain bounded by $\Gamma_\Lambda(t)$, $\Omega^2_\Lambda(t) \equiv \Omega\backslash\overline{\Omega^1_\Lambda(t)}$, $\nu_\Lambda(t)$ is the unit outward normal vector of $\Omega^1_\Lambda(t)$ and $\kappa_\Lambda(\cdot,t)$ is the mean curvature of $\Gamma_\Lambda(t)$ which takes positive values when $\Omega^1_\Lambda(t)$ protrudes into $\Omega^1_\Lambda(t)$. It is easy to verify that

$$\kappa_\Lambda(s',t) = -\frac{1}{c(s',\Lambda,\nabla_{s'}\Lambda)}\left[\sum_{i,j=1}^{N-1} a_{ij}(s',\Lambda,\nabla_{s'}\Lambda)\frac{\partial^2\Lambda}{\partial s_i\partial s_j} + b(s',\Lambda,\nabla_{s'}\Lambda)\right] \tag{3.6}$$

with $a_{ij}(s',\Lambda,\nabla_{s'}\Lambda)$ $(i,j=1,2,\cdots,N-1)$, $b(s',\Lambda,\nabla_{s'}\Lambda)$ and $c(s',\Lambda,\nabla_{s'}\Lambda)$ given by (2.8)–(2.10).

Lemma 3.1 *There exist constants T^0, C and $C(\varepsilon)$, s.t. for any $\Lambda\in A_{T^0}$, the system (3.2)–(3.5) has a unique classical solution $u^\varepsilon_\Lambda(x,t)$, $u^\varepsilon_\Lambda\equiv u^i$ for $x\in\Omega^i_\Lambda(t)$ $(i=1,2)$, satisfying*

$$\|u^\varepsilon_\Lambda\|_{C^{2+\alpha,\frac{\alpha}{2}}(\overline{Q^1_\Lambda})}+\|u^\varepsilon_\Lambda\|_{C^{2+\alpha,\frac{\alpha}{2}}(\overline{Q^2_\Lambda})}\le C\{1+\|u^\varepsilon_\Lambda\|_{C(\overline{\Omega}_{T^0})}\}\le C(\varepsilon) \tag{3.7}$$

and

$$\sup_{0\le t\le T^0}\|u^\varepsilon_\Lambda(\cdot,t)\|_{C^{3+\alpha}(\overline{Q^1_\Lambda})}+\sup_{0\le t\le T^0}\|u^\varepsilon_\Lambda(\cdot,t)\|_{C^{3+\alpha}(\overline{Q^2_\Lambda})}$$
$$\le C\{1+\|u^\varepsilon_\Lambda\|_{C(\overline{\Omega}_{T^0})}\}\le C(\varepsilon) \tag{3.8}$$

where $Q^i_\Lambda\equiv\bigcup_{0\le t\le T^0}(\Omega^i_\Lambda(t)\times\{t\})$ $(i=1,2)$, C and T^0 are independent of ε, and $C(\varepsilon)$ depends only on ε.

Proof For simplicity, in the following argument we rewrite u^ε_Λ, V_Λ and ν_Λ as u, V and ν respectively, and let the capital letter C denote constants independent of T and ε.

Set $h(\mu,\lambda)$ is such a function that

$$\begin{cases} h(\mu,\lambda)\in C^\infty([-L,L]\times\mathbf{R}^1)\\ h(\mu,\lambda)=\begin{cases}\mu & \text{for } |\mu|\ge\frac{3}{4}L\\ 0 & \text{for } \mu=\lambda \text{ and } |\mu|\le\frac{1}{2}L\end{cases}\\ \dfrac{\partial h}{\partial\mu}\ge C>0\end{cases}$$

Define a transformation $Y(x,t)$ from Ω_T into itself by

$$Y(x,t)=\begin{cases}(x,t) & \text{if dist}\,(x,\Gamma^0)\ge\frac{3}{4}L\\ (X^0(s')+h(S^N,\Lambda(s',t))\nu^0(s'),t)\,|_{(s',s^N)=(S^1(x),\cdots,S^N(x))} & \text{if dist}\,(x,\Gamma^0)\le\frac{3}{4}L\end{cases}$$

It is easy to verify that Y is a $C^{4+\alpha,\frac{\alpha}{2}}_{x,t}$ diffeomorphism from Ω_T onto Ω_T and, for any $t\in[0,T]$, Y maps $\Gamma_\Lambda(t)\times\{t\}$ onto $\Gamma^0\times\{t\}$. Moreover

$$\|Y\|_{C^{4+\alpha,\frac{\alpha}{2}}_{x,t}(\overline{\Omega_T})}\le C \tag{3.9}$$

$$\|Y^{-1}\|_{C^{4+\alpha,\frac{\alpha}{2}}_{x,t}(\overline{\Omega_T})}\le C \tag{3.10}$$

Letting $Z(y,t)=u(Y^{-1}(y,t))$, or $u(x,t)=z(Y(x,t))$, then (3.2)–(3.5) is transformed into

$$\begin{cases} -\varepsilon z^k + \sum_{i,j=1}^{N} \widehat{a}_{ij} \dfrac{\partial^2 z^k}{\partial y_i \partial y_j} + \sum_{i=1}^{N} \widehat{b}^i \dfrac{\partial z^k}{\partial y_i} = 0 \text{ in } \Omega_0^k \times \{t\}, \quad k = 1,2 & (3.11) \\ \dfrac{\partial z^1}{\partial n} = 0 \text{ on } \partial\Omega \times \{t\} & (3.12) \\ \widehat{\nu} \cdot [\nabla_y z^1 - \nabla_y z^2] = -\widehat{\kappa} - Z^1 \text{ on } \Gamma^0 \times \{t\} & (3.13) \\ z^1 = z^2 \text{ on } \Gamma^0 \times \{t\} & (3.14) \end{cases}$$

where $t \in [0,T]$, $z^k \equiv z$ in $\Omega_0^k \times \{t\}$ $(k = 1,2)$, Ω_2^1 is the domain bounded by $\Gamma^0, \Omega_0^2 = \Omega \backslash \overline{\Omega_0^1}$, and

$$\begin{cases} \widehat{a}_{ij} \equiv \widehat{a}_{ij}(y,t) = (\nabla_x Y^i \cdot \nabla_x Y^j)(x,t)|_{(x,t)=Y^{-1}(y,t)} \\ \widehat{b}_{ij} \equiv \widehat{b}_i(y,t) = (\Delta_x Y^i)(x,t)|_{(x,t)=Y^{-1}(y,t)} \\ Y(x,t) \equiv (Y^1(x,t), \cdots, Y^N(x,t)) \\ \widehat{\kappa} \equiv \widehat{\kappa}(y,t) = \kappa_\Lambda(x,t)\,|_{(x,t)=Y^{-1}(y,t)} \\ \widehat{\nu} \equiv \widehat{\nu}(y,t) = (\nu(x,t) \cdot \nabla_x) Y(x,t)\,|_{(x,t)=Y^{-1}(y,t)} \end{cases}$$

Denoting $\nu \equiv (\nu^1(y,t), \cdots, \nu^N(y,t))$, then after simple calculations we can get

$$\nu^i(y,t) = \left. \frac{\dfrac{\partial S^N}{\partial x_i} - \sum_{j=1}^{N-1} \dfrac{\partial \Lambda}{\partial S^j} \dfrac{\partial S^j}{\partial x_i}}{\sqrt{1 + \sum_{i=1}^{N} \left(\sum_{j=1}^{N-1} \dfrac{\partial \Lambda}{\partial S^j} \dfrac{\partial S^j}{\partial x_i} \right)^2}} \right|_{(x,t)=Y^{-1}(y,t)}, \quad i = 1, \cdots, N$$

Moreover, we can easily verify that

$$\|\widehat{a}_{ij}\|_{C^{3+\alpha,\frac{\alpha}{2}}(\overline{\Omega}_T)} \le C \tag{3.15}$$

$$\|\widehat{b}_i\|_{C^{2+\alpha,\frac{\alpha}{2}}(\overline{\Omega}_T)} \le C \tag{3.16}$$

and, after some computations,

$$\|\widehat{\kappa}\|_{C^{2+\alpha,\frac{\alpha}{2}}(\overline{\Omega}_T)} \le C \tag{3.17}$$

$$\|\widehat{\nu}\|_{C^{3+\alpha,\frac{\alpha}{2}}(\overline{\Omega}_T)} \le C \tag{3.18}$$

Since $\widehat{\nu}(y,0) = \nu^0(y)$, $y \in \Gamma^0$, and $\widehat{a}_{ij}(y,0) = I_{N\times N}$ (the $N \times N$ unit matrix), we can take T so small that

$$\widehat{\nu} \cdot \nu^0 \ge \frac{1}{2} \quad \text{on } \Gamma^0 \times [0,T] \tag{3.19}$$

Evidently, such choice of T depends only on Γ^0. Moreover, it is obvious that

$$\frac{1}{C}|\xi| \le \sum_{i,j=1}^{N} \widehat{a}_{ij}\xi_i\xi_j \le C|\xi|^2 \quad \text{on } \Omega \times [0,T] \quad \text{for } \forall \xi \in \mathbf{R}^N \tag{3.20}$$

Now we can apply the following lemma to the system (3.11)–(3.14) and obtain a unique solution $z(y,t)$ of (3.11)–(3.14), such that $z \in C(\overline{\Omega}_T)$,

$$\|z\|_{C^{2+\alpha,\frac{\alpha}{2}}(\overline{\Omega_0^1}\times[0,T])} + \|z\|_{C^{2+\alpha,\frac{\alpha}{2}}(\overline{\Omega_0^2}\times[0,T])} \le C\{1+\|z\|_{C(\overline{\Omega}_T)}\} \le C(\varepsilon)$$

and

$$\sup_{0\le t\le T}\|z(\cdot,t)\|_{C^{3+\alpha}(\overline{\Omega_0^1})} + \sup_{0\le t\le T}\|z(\cdot,t)\|_{C^{3+\alpha}(\overline{\Omega_0^2})} \le C\{1+\|z\|_{C(\overline{\Omega_T})}\} \le C(\varepsilon)$$

which, together with (3.9) and the relation $u(x,t) = Z(Y(x,t))$, gives (3.7) and (3.8).

The proof of Lemma 3.1 is completed.

Next we consider a more general problem: determine a function $\varphi(x,t)$, *s.t.*, for any fixed $t \in [0,T]$

$$\begin{cases} \sum_{i,j=1}^{N} A_{ij}\dfrac{\partial^2\varphi^k}{\partial y_i\partial y_j} + \sum_{i=1}^{N} B_i\dfrac{\partial\varphi^k}{\partial y_i} + H\varphi^k = E \quad \text{in } \Omega_0^k, \quad k=1,2 & (3.21)\\ \dfrac{\partial\varphi^1}{\partial n} = 0 \quad \text{on } \partial\Omega & (3.22)\\ \eta\cdot[\nabla\varphi^1 - \nabla\varphi^2] = G - \gamma\varphi \quad \text{on } \Gamma_0 & (3.23)\\ \varphi^1 = \varphi^2 \quad \text{on } \Gamma_0 & (3.24)\end{cases}$$

where $\varphi = \varphi^k$ in Ω_0^k $(k=1,2)$, γ is a positive constant and

$$\eta\cdot[\nabla\varphi^1 - \nabla\varphi^2] \equiv \frac{\partial\tilde{\varphi}^1}{\partial\nu_\Lambda} - \frac{\partial\tilde{\varphi}^2}{\partial\nu_\Lambda} \quad \text{on } \Gamma_\Lambda(t) \tag{3.25}$$

for $\tilde{\varphi}^i(x,t) \equiv \varphi^i(Y^{-1}(x,t))$ $(i=1,2)$ and $\Lambda \in A_T$.

Lemma 3.2 *Assume that*

$$M^l \equiv \max\{\|A_{ij}\|_{C^{l+1+\alpha,\frac{\alpha}{2}}(\overline{\Omega}_T)}, \|\eta\|_{C^{l+1+\alpha,\frac{\alpha}{2}}(\overline{\Omega}_T)}, \|B_i\|_{C^{l+\alpha,\frac{\alpha}{2}}(\overline{\Omega}_T)}, \|H\|_{C^{l+\alpha,\frac{\alpha}{2}}(\overline{\Omega}_T)},\ i,j=1,\cdots,N\} < +\infty \tag{3.26}$$

$$I^l \equiv \|G\|_{C^{l+1+\alpha,\frac{\alpha}{2}}(\overline{\Omega}_T)} + \|E\|_{C^{l+\alpha,\frac{\alpha}{2}}(\overline{\Omega}_0^1\times[0,T])} + \|E\|_{C^{l+\alpha,\frac{\alpha}{2}}(\overline{\Omega}_0^2\times[0,T])} < +\infty \tag{3.27}$$

$$\eta\cdot\nu^0 \ge \gamma_1 > 0 \quad on\ \Gamma^0\times[0,T] \tag{3.28}$$

$$H \le -\gamma_2 < 0 \quad in\ \overline{\Omega}_T \tag{3.29}$$

and

$$\gamma_3^{-1}|\xi|^2 \le \sum_{i,j=1}^{N} A_{ij}\xi_i\xi_j \le \gamma_3|\xi|^2 \quad \text{on } \overline{\Omega}_T,\ \forall\xi\in\mathbf{R}^N \tag{3.30}$$

where $l=0$ or 1, γ_i $(i=1,2,3)$ is positive constants, then the system (3.21)–(3.24) has a unique classical solution $\varphi \in C(\overline{\Omega}_T)$, s.t.,

$$\sum_{i=1}^{2}\|\varphi\|_{C^{l+1+\alpha,\frac{\alpha}{2}}(\overline{\Omega}_0^i\times[0,T])} \le C(M^l,\gamma_1,\gamma_3)\{\|\varphi\|_{C(\overline{\Omega}_T)} + I^l\}$$

$$\leq C(M^l, I^l, \gamma_1, \gamma_2, \gamma_3), \quad l = 0,1 \tag{3.31}$$

$$\sum_{i=1}^{2}\Big\{\sup_{0\leq t\leq T}\|\varphi(\cdot,t)\|_{C^{l+2+\alpha}(\overline{\Omega}_0^i)}\Big\} \leq C(M^l,\gamma_1,\gamma_3)\{\|\varphi\|_{C(\overline{\Omega}_T)} + I^l\}$$
$$\leq C(M^l, I^l, \gamma_1, \gamma_2, \gamma_3), l = 0,1 \tag{3.32}$$

where $C(M^l,\gamma_1,\gamma_3)$ depends only on M^l, γ_1 and γ_3, and $C(M^l, I^l, \gamma_1, \gamma_2, \gamma_3)$ only on $M^l, I^l, \gamma_1, \gamma_2$ and γ_3.

Proof Here we give the proof of this lemma only for $l = 0$ because in the other case the argument is similar. For simplicity, we might as well suppose $\gamma = 0$. For $\gamma > 0$, we can use the Leray-Schauder's fixed point theorem to obtain the same results as for $\gamma = 0$.

In terms of (3.27) and $\Gamma^0 \in C^{5+\alpha}$, we can find a function $\zeta(x,t) \in C_0(\overline{\Omega}_T)$, such that

$$\begin{cases} \eta\cdot[\nabla\zeta^1 - \nabla\zeta^2] = G \quad \text{on} \quad \Gamma^0, \zeta^i \equiv \zeta \quad \text{in } \overline{\Omega}_T^i \ i = 1,2, \\ \sup\limits_{0\leq t\leq T}\{\|\zeta(\cdot,t)\|_{C^2(\overline{\Omega}_0^1)} + \|\zeta(\cdot,t)\|_{C^2(\overline{\Omega}_0^2)}\} \leq C(M^0, l^0, \gamma_1) \end{cases}$$

where $C(M^0, l^0, \gamma_1)$ depends only on M^0, l^0 and γ_1.

Let $\widehat{\varphi} = \varphi - \zeta$. Noticing (3.25), we can find from (3.21)-(3.24) that

$$\begin{cases} \sum\limits_{i,j=1}^{N} A_{ij}\dfrac{\partial^2\widehat{\varphi}}{\partial y_i\partial y_j} + \sum\limits_{i=1}^{N} B_i\dfrac{\partial\widehat{\varphi}}{\partial y_i} + H\widehat{\varphi} = \widehat{E} \quad \text{in } \Omega_T & (3.33) \\ \dfrac{\partial\widehat{\varphi}}{\partial n} = 0 \quad \text{on } \partial\Omega\times[0,T] & (3.34) \end{cases}$$

where $\|\widehat{E}\|_{L^\infty(\Omega_T)} \leq C(M^0, I^0, \gamma_1, \gamma_3)$.

By the theory of elliptic equation, there exists an unique weak solution $\widehat{\varphi} \in C^\alpha(\overline{\Omega})$ $(0 \leq t \leq T)$ of (3.32) and (3.33) which satisfies $\|\widehat{\varphi}(\cdot,t)\|_{C(\overline{\Omega})} \leq C(M^0, I^0, \gamma_1, \gamma_2, \gamma_3)$ for $t \in [0,T]$. Therefore, for any $t \in [0,T]$, (3.21)-(3.24) has a unique weak solution $\varphi \in C(\overline{\Omega})$ $(0 \leq t \leq T)$ and the second inequalities in (3.30) and (3.31) hold for $l = 0$.

For any $\widehat{y} \in \Gamma^0$, we can transform (3.21)-(3.24) in a neighborhood of $\widehat{y}$ into a system of two elliptic equations by straightening Γ^0 in some neighborhood of $\widehat{y}$ and then making a reflection along the straightened boundary (see [1]). We can verify that this elliptic system satisfies the complementary condition and hence by the theory of elliptic systems (see [9]), we obtain the first inequality in (3.31).

For any $t_1, t_2 \in [0,T]$, set $\psi^k(y) = (\varphi^k(y,t_1) - \varphi^k(y,t_2))/|t_1 - t_2|^{\frac{\alpha}{2}}$, $k = 1,2$. After some calculations, we find from (3.21)-(3.24) that

$$\begin{cases} \sum\limits_{i,j=1}^{N} A_{ij}(y,t_1)\dfrac{\partial^2\psi^k}{\partial y_i\partial y_j} + \sum\limits_{i=1}^{N} B_{ij}(y,t_1)\dfrac{\partial\psi^k}{\partial y_i} + H(y,t_1)\psi^k = f^k \quad \text{in } \Omega_0^k & (3.35) \\ \dfrac{\partial\psi^1}{\partial n} = 0 \quad \text{on } \partial\Omega & (3.36) \\ \eta_1\cdot[\nabla\psi^1 - \nabla\psi^2] = g \quad \text{on } \Gamma^0 & (3.37) \\ \psi^1 = \psi^2 \quad \text{on } \Gamma^0 & (3.38) \end{cases}$$

where $\eta_1 = \eta(y, t_1)$,

$$\|f^k\|_{L^\infty(\Omega_T^k)} \le C(M^0, \gamma_1, \gamma_3)\{\|\varphi\|_{L^\infty(\Omega_T)} + I^0\}, \quad k = 1, 2 \tag{3.39}$$

and

$$\|g\|_{C^1(\overline{\Omega})} \le C(M^0, \gamma_1, \gamma_3)\{\|\varphi\|_{L^\infty(\Omega_T)} + I^0\} \tag{3.40}$$

As in the beginning of this proof, we can find a function $W \in C_0(\overline{\Omega})$, such that

$$\begin{cases} \eta_1 \cdot (\nabla W^1 - \nabla W^2) = g \text{ on } \Gamma^0, \ W^k = W \quad \text{in } \Omega_0^k, \ k = 1, 2 \\ \sup\limits_{0 \le t \le T} \left\{\|W(\cdot, t)\|_{C^2(\overline{\Omega}_0^1)} + \|W(\cdot, t)\|_{C^2(\overline{\Omega}_0^2)}\right\} \le C(\gamma_1, M^0)\|g\|_{C^1(\overline{\Omega})} \end{cases}$$

Setting $\widehat{\psi} = \psi - W$, then we have from (3.35)–(3.38) and (3.25) that

$$\begin{cases} \sum\limits_{i,j=1}^{N} A_{ij}(y, t_1)\dfrac{\partial^2 \widehat{\psi}}{\partial y_i \partial y_j} + \sum\limits_{i=1}^{N} B_{ij}(y, t_1)\dfrac{\partial \widehat{\psi}}{\partial y_i} + H(y, t_1)\widehat{\psi} = \widehat{f} \quad \text{in } \Omega & (3.41) \\ \dfrac{\partial \widehat{\psi}}{\partial n} = 0 \quad \text{on } \partial\Omega & (3.42) \end{cases}$$

where

$$\|\widehat{f}\|_{L^\infty(\Omega_T)} \le C(M^0, \gamma_1, \gamma_3)\{\|\varphi\|_{L^\infty(\Omega_T)} + I^0\} \tag{3.43}$$

So we apply that L^p theory of elliptic equations to (3.40) and (3.41) to obtain

$$\|\widehat{\psi}\|_{W^{2,p}(\Omega)} \le C(M^0, \gamma_1, \gamma_3, p)\{\|\widehat{f}\|_{L^\infty(\Omega)} + \|\widehat{\psi}\|_{L^\infty(\Omega)}\}, \quad 1 < p < +\infty$$

Hence,

$$\|\psi\|_{W^{2,p}(\Omega_0^1)} + \|\psi\|_{W^{2,p}(\Omega_0^2)} \le C(M^0, \gamma_1, \gamma_3)\{\|\varphi\|_{L^\infty(\Omega_T)} + I^0\}, \quad \forall 1 < p < +\infty$$

which implies from Sobolev imbedding theorems that

$$\|\psi\|_{C^1(\overline{\Omega}_0^1)} + \|\psi\|_{C^1(\overline{\Omega}_0^2)} \le C(M^0, \gamma_1, \gamma_3)\{\|\varphi\|_{L^\infty(\Omega_T)} + I^0\}$$

Therefore, the first inequality in (3.30) for $l = 0$ follows from the last inequality and the definition of ψ.

The proof of Lemma 3.2 is completed.

4. The Existence and Uniqueness of the Approximating Solutions for (1.1)–(1.5)

For any $\Lambda \in A_T$ defined in (3.1), let u_Λ^ε be the solution of (3.2)–(3.5) and Λ^ε be the solution of (2.6) and (2.7) in $\mathcal{M} \times [0, T_\varepsilon]$ for $v = u_\Lambda^\varepsilon(X(s', \Lambda(s', t)), t)$, here $T_\varepsilon \in (0, T^0)$ is determined from Lemma 2.1 and Lemma 3.1 which depends only on Γ^0 and ε.

Define an operator $\mathcal{F}_\varepsilon$ from A_T into $C(\mathcal{M}\times[0,T_\varepsilon])$ by $\mathcal{F}_\varepsilon\Lambda=\Lambda^\varepsilon$ for any $\Lambda\in A_T$. We must show $\mathcal{F}_\varepsilon: A_{T_\varepsilon}\to A_{T_\varepsilon}$ and $\mathcal{F}_\varepsilon$ has a unique fixed point in A_{T_ε}.

From Lemma 2.1 and Lemma 3.1, we have

$$\|\mathcal{F}_\varepsilon\Lambda\|_{C^{4+\beta,\frac{\beta}{2}}(\mathcal{M}\times[0,T_\varepsilon])}\le C(\varepsilon) \tag{4.1}$$

where $0<\alpha<\beta<1$ and $C(\varepsilon)$ is independent of T_ε.

Noting that $\Lambda^\varepsilon(s',0)=0$ for $s'\in\mathcal{M}$, we derive from (4.1) that

$$\|\Lambda^\varepsilon\|_{C^{4+\alpha,\frac{\alpha}{2}}(\mathcal{M}\times[0,T_\varepsilon])}\le C(\varepsilon)T_\varepsilon^{\frac{\beta-\alpha}{2}} \tag{4.2}$$

Hence if we let T_ε so small that the right hand side of (4.2) is less than $\frac{L}{2}$, then (4.1) and (4.2) imply that $\mathcal{F}_\varepsilon$ is a compact operator from A_{T_ε} into itself. The continuity of $\mathcal{F}_\varepsilon$ on A_{T_ε} follows from the compactness of $\mathcal{F}_\varepsilon$ and the uniqueness of the solution of (2.6) and (2.7) for any $\Lambda\in A_{T_\varepsilon}$. Thus, by the Schauder's fixed point theorem, $\mathcal{F}_\varepsilon$ has a fixed point in A_{T_ε}. Moreover, in terms of the same argument in Section 6, we can show that $\mathcal{F}_\varepsilon$ has only one fixed point in A_{T_ε}. Thus we have proved.

Lemma 4.1 *Assume that $\Gamma^0\in C^{5+\alpha}$ and $\partial\Omega\in C^{3+\alpha}$ $(0<\alpha<1)$, then there exists a positive constant T_ε, depending only on ε, Ω and Γ^0, such that, the system (2.6), (2.7) and (3.2)–(3.5) has a unique classical solution $\{\Lambda^\varepsilon,u^\varepsilon_{\Lambda^\varepsilon}\}$, which satisfies $\Lambda^\varepsilon\in C^{4+\beta,\frac{\beta}{2}}(\mathcal{M}\times[0,T_\varepsilon])$ $(\alpha<\beta<1)$, $\Lambda_t\in C^{\alpha,\frac{\alpha}{2}}$, $u^\varepsilon_{\Lambda^\varepsilon}\in C(\overline{\Omega}_T)\cap C^{2+\alpha,\frac{\alpha}{2}}(\overline{\Omega^1_{\Lambda^\varepsilon}(t)}\times[0,T_\varepsilon])\cap C^{2+\alpha,\frac{\alpha}{2}}(\overline{\Omega^2_{\Lambda^\varepsilon}(t)}\times[0,T_\varepsilon])$, $u^\varepsilon_{\Lambda^\varepsilon}(\cdot,t)\in C^{3+\alpha}(\overline{\Omega^1_{\Lambda^\varepsilon}(t)})\cap C^{3+\alpha}(\overline{\Omega^2_{\Lambda^\varepsilon}(t)})$ $(0\le t\le T_\varepsilon)$, and*

$$\|\Lambda^\varepsilon\|_{C^{4+\alpha,\frac{\alpha}{2}}(\mathcal{M}\times[0,T_\varepsilon])}\le\frac{L}{2} \tag{4.3}$$

where $\overline{\Omega^i_{\Lambda^\varepsilon}(t)}$ $(i=1,2)$ is defined in (3.2).

5. The Existence of a Local Classical Solution of (1.1)–(1.5)

Lemma 5.1 *Let $\{\Lambda^\varepsilon,u^\varepsilon_{\Lambda^\varepsilon}\}$ be the solution of (2.6), (2.7) and (3.2)–(3.5) provided by Lemma 4.1. Then we have the following estimates:*

$$\|\Lambda^\varepsilon\|_{C^{4+\beta,\frac{\beta}{2}}(\mathcal{M}\times[0,T_\varepsilon])}\le C \tag{5.1}$$

$$\sum_{i=1}^{2}\left\{\|u^\varepsilon_{\Lambda^\varepsilon}\|_{C^{2+\alpha,\frac{\alpha}{2}}(\overline{\Omega^i_{\Lambda^\varepsilon}(t)}\times[0,T_\varepsilon])}+\sup_{0\le t\le T}\|u^\varepsilon_{\Lambda^\varepsilon}\|_{C^{3+\alpha}(\overline{\Omega^i_{\Lambda^\varepsilon}(t)})}\right\}\le C \tag{5.2}$$

where the constant C is independent of ε and T_ε.

Proof We have from (2.6) and (4.3) that

$$\left|\int_{\mathcal{M}\times[0,T]}v^\varepsilon_{\Lambda^\varepsilon}(s',t)c(s',\Lambda_\varepsilon,\nabla_{s'}\Lambda_\varepsilon)ds'dt\right|\le C \tag{5.3}$$

which implies from (4.3) and the imbedding theorems that

$$\max_{\mathcal{M}\times[0,T]} |v^{\varepsilon}_{\Lambda^{\varepsilon}}(s',t)c(s',\Lambda_{\varepsilon},\nabla_{s'}\Lambda_{\varepsilon})| \le \rho\sum_{i=1}^{2}\|v^{\varepsilon}_{\Lambda^{\varepsilon}}\|_{C^{2+\alpha,\frac{\alpha}{2}}(\overline{\Omega^{i}_{\Lambda^{\varepsilon}}(t)}\times[0,T_{\varepsilon}])} + C_{\rho} \tag{5.4}$$

where C_{ρ} is a constant depending on ρ.

Noting that $c(s',\Lambda_{\varepsilon},\nabla_{s'}\Lambda) \ge 1$, we get

$$\|u^{\varepsilon}_{\Lambda_{\varepsilon}}\|_{C(\Gamma_{\Lambda}(t)\times[0,T_{\varepsilon}])} \le \rho\sum_{i=1}^{2}\|u^{\varepsilon}_{\Lambda^{\varepsilon}}\|_{C^{2+\alpha,\frac{\alpha}{2}}(\overline{\Omega^{i}_{\Lambda^{\varepsilon}}(t)}\times[0,T_{\varepsilon}])} + C_{\rho}. \tag{5.5}$$

Now applying the maximum principal to (3.2)–(3.5) for $\Lambda = \Lambda^{\varepsilon}$ and $u = u^{\varepsilon}_{\Lambda^{\varepsilon}}$ in $\Omega^{\varepsilon}_{\Lambda^{\varepsilon}} \times [0,T]$ and using (5.3), (3.7) and (3.8), we obtain

$$(1-C\rho)\sum_{i=1}^{2}\left[\|u^{\varepsilon}_{\Lambda^{\varepsilon}}\|_{C^{2+\alpha,\frac{\alpha}{2}}(\overline{\Omega^{i}_{\Lambda^{\varepsilon}}(t)}\times[0,T_{\varepsilon}])} + \sum_{0\le t\le T_{\varepsilon}}\|u^{\varepsilon}_{\Lambda^{\varepsilon}}\|_{C^{3+\alpha}(\overline{\Omega^{i}_{\Lambda^{\varepsilon}}(t)})}\right] \le C_{\rho}$$

Thus, by taking ρ small enough, we yield (5.2) from the above inequality, while (5.2) and Lemma 2.1 give (5.1). Hence we conclude the proof of Lemma 5.1.

Lemma 5.2 *Under the assumptions of Lemma 5.1, there exists a constant $\lambda > 0$, independent of ε, such that*

$$\widehat{T}_{\varepsilon} \ge \lambda > 0 \quad \text{for any } \varepsilon \in (0,1) \tag{5.6}$$

where $\widehat{T}_{\varepsilon}$ is a constant such that $[0,\widehat{T}_{\varepsilon}]$ is the maximum interval in which the solution $\{\Lambda^{\varepsilon}, u^{\varepsilon}_{\Lambda^{\varepsilon}}\}$ exists classically.

Proof If

$$\|\Lambda^{\varepsilon}\|_{C^{4+\alpha,\frac{\alpha}{2}}(\mathcal{M}\times[0,T_{\varepsilon}])} \ge \frac{L}{8} \tag{5.7}$$

Then, from (5.1) and (2.7), we have

$$\frac{L}{8} \le \|\Lambda^{\varepsilon}\|_{C^{4+\alpha,\frac{\alpha}{2}}(\mathcal{M}\times[0,T_{\varepsilon}])} \le CT_{\varepsilon}^{\frac{\beta-\alpha}{2}}$$

So

$$\widehat{T}_{\varepsilon} \ge T_{\varepsilon} \ge \left(\frac{L}{8C}\right)^{\frac{2}{\beta-\alpha}} \equiv \lambda_1 > 0 \tag{5.8}$$

Now we suppose

$$\|\Lambda^{\varepsilon}\|_{C^{4+\alpha,\frac{\alpha}{2}}(\mathcal{M}\times[0,T_{\varepsilon}])} < \frac{L}{8} \tag{5.9}$$

For any $T \in [T_{\varepsilon}, T_0]$ (T_0 is defined in Lemma 3.1), we define a subset of A_T by

$$B_{\varepsilon,T} \equiv \left\{\Lambda \middle| \Lambda \equiv \Lambda^{\varepsilon} \text{ for } t\in[0,T_{\varepsilon}] \text{ and } \|\Lambda\|_{C^{4+\alpha,\frac{\alpha}{2}}(\mathcal{M}\times[0,T_{\varepsilon}])} \le \frac{L}{2}\right\}$$

Since $\mathcal{F}_{\varepsilon}\Lambda^{\varepsilon} = \Lambda^{\varepsilon}$ for $t \in [0,T_{\varepsilon}]$, where $\mathcal{F}_{\varepsilon}$ is the operator defined in Section 4, then $\mathcal{F}_{\varepsilon}\Lambda = \Lambda^{\varepsilon}$ in $\mathcal{M}\times[0,T_{\varepsilon}]$ for any $\Lambda \in B_{\varepsilon,T}$.

It is obvious that the solution of (2.6) and (2.7) provided by Lemma 2.1 can exist upto T_m in the way that, either $T_m = T$, or

$$\|\Lambda\|_{C^{1,0}(\mathcal{M}\times[0,T_m])} = \min\left\{\frac{L}{2}, 1\right\}$$

(See [1].) If we assume that $\mathcal{F}_\varepsilon\Lambda$ is well-defined in $\mathcal{M}\times[0,T]$, then we have from Lemma 2.1 and Lemma 3.1 that

$$\begin{aligned}
\|\mathcal{F}_\varepsilon\Lambda\|_{C^{4+\alpha,\frac{\alpha}{2}}(\mathcal{M}\times[0,T])} &\le \|\mathcal{F}_\varepsilon\Lambda\|_{C^{4+\alpha,\frac{\alpha}{2}}(\mathcal{M}\times[T_\varepsilon,T])} + \|\mathcal{F}_\varepsilon\Lambda\|_{C^{4+\alpha,\frac{\alpha}{2}}(\mathcal{M}\times[0,T_\varepsilon])} \\
&\le \|\Lambda^\varepsilon\|_{C^{4+\alpha,\frac{\alpha}{2}}(\mathcal{M}\times[0,T_\varepsilon])} + \|\Lambda^\varepsilon(s',T_\varepsilon)\|_{C^{4+\alpha,\frac{\alpha}{2}}(\mathcal{M}\times[T_\varepsilon,T])} \\
&\quad + \|\mathcal{F}_\varepsilon\Lambda - \Lambda^\varepsilon(s',T_\varepsilon)\|_{C^{4+\alpha,\frac{\alpha}{2}}(\mathcal{M}\times[T_\varepsilon,T])} \\
&\le \frac{L}{4} + C(\varepsilon)|T-T_\varepsilon|^{\frac{\beta-\alpha}{2}}
\end{aligned}$$

Set $\Delta T_\varepsilon \equiv \left(\dfrac{L}{4C(\varepsilon)}\right)^{\frac{2}{\beta-\alpha}} > 0$ and $T = \min\{T^0, T_\varepsilon + \Delta T_\varepsilon\}$. Then we yield

$$\|\mathcal{F}_\varepsilon\Lambda\|_{C^{4+\alpha,\frac{\alpha}{2}}(\mathcal{M}\times[0,T])} \le \frac{L}{2} < 1 \quad \text{for } L<1$$

Hence $\mathcal{F}_\varepsilon\Lambda$ is indeed well-defined in $\mathcal{M}\times[0,T]$ with $T=\min\{T^0, T_\varepsilon+\Delta T_\varepsilon\}$ and $\mathcal{F}_\varepsilon$ is an operator from $B_{\varepsilon,T}$ into $B_{\varepsilon,T}$. Thus from the argument in Section 4, we see that $\mathcal{F}_\varepsilon$ has a unique fixed point in $B_{\varepsilon,T}$, that is, we have extended the existence interval of $\{\Lambda^\varepsilon, u^\varepsilon_{\Lambda^\varepsilon}\}$ from $[0,T_\varepsilon]$ to $[0,T]$. Repeating this process finite times, we can extend the domain in which $\{\Lambda^\varepsilon, u^\varepsilon_{\Lambda^\varepsilon}\}$ exists classically from $\mathcal{M}\times[0,T_\varepsilon]$ to $\mathcal{M}\times[0,T^*_\varepsilon]$, such that, either $T^*_\varepsilon = T_0$ or the inequality (5.7) holds for $T_\varepsilon = T^*_\varepsilon$. So if we take $\lambda = \min\{\lambda_1, T_0\}$, then (5.4) holds.

It is easy to see from Lemma 5.1 and Lemma 5.2 that there is a sequence $\{\varepsilon_j\}$ with $\varepsilon_j \to 0$ as $j\to+\infty$, such that the solution $\{\Lambda^{\varepsilon_j}, u^{\varepsilon_j}_{\Lambda^{\varepsilon_j}}\}$ of the system (2.6), (2.7) and (3.2)–(3.5) for $\varepsilon=\varepsilon_j$ tends to a local classical solution $\{u,\Lambda\}$ of (1.1)–(1.5) with $\Gamma \equiv \bigcup_{0\le t\le T}(\Gamma_t\times\{t\})$ defined by (2.4), which has the regularities desired by Theorem 1.1.

Remark 5.3 If $\Gamma^0 \in C^{k+\alpha}$ and $\partial\Omega \in C^{k-2+\alpha}$ $(k\ge 4, 0<\alpha<1)$, the arguments in Sections 2–5 give a local classical solution $\{u,\Gamma\}$ of (1.1)–(1.5), which satisfies

$$\begin{aligned}
&u(\cdot,t)\in C^{k-2+\alpha}(\overline{\Omega^1(t)})\cap C^{k-2+\alpha}(\overline{\Omega^2(t)}), \quad 0\le t\le T\\
&u\in C^{k-3+\alpha,\frac{\alpha}{2}}(\overline{Q^1_T})\cap C^{k-3+\alpha,\frac{\alpha}{2}}(\overline{Q^2_T})\\
&\Gamma\in C^{k-1+\alpha,\frac{2+\alpha}{2}}_{x,t}
\end{aligned}$$

6. The Uniqueness of Classical Solution of (1.1)–(1.5)

Suppose that $\{u_i,\Lambda_i\}$ $(i=1,2)$ are two classical solutions of (1.1)–(1.5), which satisfy

$$\|\Lambda_i\|_{C^{4+\alpha,\frac{\alpha}{2}}(\mathcal{M}\times[0,T])} \le C \tag{6.1}$$

$$\|u_i\|_{C^{2+\alpha,\frac{\alpha}{2}}(\overline{Q^1_{iT}})} \cap \|u_i\|_{C^{2+\alpha,\frac{\alpha}{2}}(\overline{Q^2_{iT}})} \le C, \quad i=1,2 \tag{6.2}$$

$$\sup_{0\le t\le T} \|u_i\|_{C^{3+\alpha}(\overline{\Omega^1_i}(t))} + \sup_{0\le t\le T} \|u_i\|_{C^{3+\alpha}(\overline{\Omega^2_i}(t))} \le C, \quad i=1,2 \tag{6.3}$$

where $\Omega^1_i(t)$ is the domain bounded by $\Gamma_i(t) \equiv \{X(s',s_n)|s_n = \Lambda_i(s',t), s' \in \mathcal{M}\}$, $\Omega^2_i(t) \equiv \Omega\backslash(\overline{\Omega^1_i}(t))$ and $Q^k_{iT} \equiv \bigcup_{0\le t\le T} (\Omega^k_i(t)\times\{t\})$ $(i=1,2, k=1,2)$.

Set $\Phi(s',t) = \Lambda_1(s',t) - \Lambda_2(s',t)$. Due to (2.6) and (2.7), we have

$$\begin{cases} \sum_{i,j=1}^{n-1} \widetilde{a}_{ij}\dfrac{\partial^2\Phi}{\partial s_i\partial s_j} + \sum_{i=1}^{n-1}\widetilde{b}_i\dfrac{\partial\Phi}{\partial s_i} + \widetilde{c}\Phi - \dfrac{\partial\Phi}{\partial t} = \widetilde{e}(v_1 - v_2) \quad \text{in } \mathcal{M}\times[0,T] & (6.4)\\ \Phi(s',0) = 0, \ s' \in \mathcal{M} & (6.5)\end{cases}$$

where

$$v_i \equiv u_i(X(s',\Lambda_i(s',t)),t), \quad i=1,2$$

$$\widetilde{a}_{ij} \equiv \widetilde{a}_{ij}(s',t) = a_{ij}(s',\Lambda_1,\nabla_{s'}\Lambda_1), \quad i,j=1,\cdots,N-1$$

$$\widetilde{e} \equiv \widetilde{e}(s',t) = c(s',\Lambda_1,\nabla_{s'}\Lambda_1) \ge 1$$

and

$$\|\widetilde{a}_{ij},\widetilde{b}_i,\widetilde{c},\widetilde{e}, \ i,j=1,\cdots,N-1\|_{C^{2+\alpha,\frac{\alpha}{2}}(\mathcal{M}\times[0,T])} \le C \tag{6.6}$$

Without loss of generality, we can assume $T\in(0,1]$. Thus the constant C in (6.1)–(6.3) and (6.6) is independent of T.

Using the same argument as in the proof of Lemma 2.1, we can show

$$\begin{aligned}\|\Lambda_1-\Lambda_2\|_{C^{3+\beta,\frac{\beta}{2}}(\mathcal{M}\times[0,T])} &= \|\Phi\|_{C^{3+\beta,\frac{\beta}{2}}(\mathcal{M}\times[0,T])}\\ &\le C\Big\{\|v_1-v_2\|_{C^{1+\alpha,\frac{\alpha}{2}}(\mathcal{M}\times[0,T])} + \sup_{0\le t\le T}\|v_1(\cdot,t)-v_2(\cdot,t)\|_{C^2(\mathcal{M})}\Big\},\\ &\qquad \alpha<\beta<1\end{aligned} \tag{6.7}$$

here and below, the capital letter C denotes constants independent of T.

We may assume T so small that

$$\|\Lambda_i\|_{C(\mathcal{M}\times[0,T])} \le \frac{L}{2}, \quad i=1,2 \tag{6.8}$$

and let $Y_i \equiv Y_{\Lambda_i}$ $(i=1,2)$ be the transformations introduced in the proof of Lemma 3.1. After simple computations, we find

$$\begin{cases}\|Y_1-Y_2\|_{C^{3+\alpha,\frac{\alpha}{2}}(\overline{\Omega}_T)} \le C\|\Lambda_1-\Lambda_2\|_{C^{3+\alpha,\frac{\alpha}{2}}(\mathcal{M}\times[0,T])} & (6.9)\\ \|Y_1^{-1}-Y_2^{-1}\|_{C^{3+\alpha,\frac{\alpha}{2}}(\overline{\Omega}_T)} \le \|\Lambda_1-\Lambda_2\|_{C^{3+\alpha,\frac{\alpha}{2}}(\mathcal{M}\times[0,T])} & (6.10)\end{cases}$$

The function $Z_i \equiv u_i(Y^{-1}(y,t))$ is the solution of (3.11)–(3.14) with $\varepsilon = 0$, $\Lambda = \Lambda_i$ for $i = 1$ and 2 respectively. Denote $\phi(y,t) \equiv Z_1(y,t) - Z_2(y,t)$. Then we can easily verify that, for any $t \in [0,T]$,

$$\begin{cases} \sum_{i,j=1}^{n-1} \widehat{a}_{ij} \frac{\partial^2 \phi^k}{\partial y_i \partial y_j} + \sum_{i=1}^{n-1} \widehat{b}_i \frac{\partial \phi^k}{\partial y_i} = \overline{f}^k \quad \text{in } \Omega_0^k, \quad k = 1,2 & (6.11) \\ \frac{\partial \phi^1}{\partial n} = 0 \quad \text{on } \partial\Omega & (6.12) \\ \widehat{\nu} \cdot [\nabla_y \phi^1 - \nabla_y \phi^2] = \overline{g} \quad \text{on } \Gamma^0 & (6.13) \\ \phi^1 = \phi^2 \quad \text{on } \Gamma^0 & (6.14) \end{cases}$$

where $\phi \equiv \phi^k$ in Ω_0^k $(k = 1,2)$

$$\|\overline{f}^k\|_{C^{\alpha,\frac{\alpha}{2}}(\overline{\Omega}_0^k \times [0,T])} \le C\|\Lambda_1 - \Lambda_2\|_{C^{3+\alpha,\frac{\alpha}{2}}(\mathcal{M}\times[0,T])}, \quad k = 1,2 \tag{6.15}$$

$$\|\overline{g}\|_{C^{1+\alpha,\frac{\alpha}{2}}(\overline{\Omega}_T)} \le C\|\Lambda_1 - \Lambda_2\|_{C^{3+\alpha,\frac{\alpha}{2}}(\mathcal{M}\times[0,T])} \tag{6.16}$$

We can check that the conditions of Lemma 3.2 for $l = 0$ with respect to (6.11)–(6.14) are satisfied if T is small enough. So (3.30) and (3.31) with $l = 0$ give that, for small T,

$$\sum_{i=1}^{2} \Big\{ \|Z_1 - Z_2\|_{C^{1+\alpha,\frac{\alpha}{2}}(\overline{\Omega}_0^i \times [0,T])} + \sup_{0 \le t \le T} \sum_{i=1}^{2} \|Z_1(\cdot,t) - Z_2(\cdot,t)\|_{C^{2+\alpha}(\overline{\Omega}_0^i)} \Big\}$$
$$\le C\{\|\Lambda_1 - \Lambda_2\|_{C^{3+\alpha,\frac{\alpha}{2}}(\mathcal{M}\times[0,T])} + \|Z_1 - Z_2\|_{C(\overline{\Omega}_T)}\} \tag{6.17}$$

Since $u_i(x,t) = Z_i(Y_i(x,t),t)$ and $v_i(s',t) = u_i(X(s',\Lambda_i(s',t)),t)$ $(i = 1,2)$, it follows from (6.9) and (6.17) that

$$\|v_1 - v_2\|_{C^{1+\alpha,\frac{\alpha}{2}}(\mathcal{M}\times[0,T])} + \sup_{0 \le t \le T} \|v_1(\cdot,t) - v_2(\cdot,t)\|_{C(\mathcal{M})}$$
$$\le C\{\|\Lambda_1 - \Lambda_2\|_{C^{3+\alpha,\frac{\alpha}{2}}(\mathcal{M}\times[0,T])} + \|Z_1 - Z_2\|_{C(\overline{\Omega}_T)}\}$$
$$\text{for sufficiently small } T \tag{6.18}$$

Applying Maximum principle to (6.11)–(6.14), we obtain

$$\|Z_1 - Z_2\|_{C(\overline{\Omega}_T)} \le C\{\|\Lambda_1 - \Lambda_2\|_{C^{3+\alpha,\frac{\alpha}{2}}(\mathcal{M}\times[0,T])} + \|Z_1 - Z_2\|_{C(\Gamma^0\times[0,T])}\}$$
$$\equiv C\{\|\Lambda_1 - \Lambda_2\|_{C^{3+\alpha,\frac{\alpha}{2}}(\mathcal{M}\times[0,T])} + \|v_1 - v_2\|_{C(\mathcal{M}\times[0,T])}\} \tag{6.19}$$

On the other hand, by using an argument similar to that in the proof of (5.3), we can show

$$\|v_1 - v_2\|_{C(\mathcal{M}\times[0,T])}$$
$$\le \rho \sum_{i=1}^{2} \|Z_1 - Z_2\|_{C^{1+\alpha,\frac{\alpha}{2}}(\overline{\Omega}_0^i\times[0,T])} + \rho\|\Lambda_1 - \Lambda_2\|_{C^{3+\alpha,\frac{\alpha}{2}}(\mathcal{M}\times[0,T])} + C_\rho$$

which, together with (6.17)–(6.19), gives

$$\|Z_1 - Z_1\|_{C(\overline{\Omega}_T)} \le C\|\Lambda_1 - \Lambda_2\|_{C^{3+\alpha,\frac{\alpha}{2}}(\mathcal{M}\times[0,T])} \tag{6.20}$$

Combining (6.7), (6.18) and (6.19), we yield

$$\|\Lambda_1 - \Lambda_2\|_{C^{3+\beta,\frac{\beta}{2}}(\mathcal{M}\times[0,T])} \le C\|\Lambda_1 - \Lambda_2\|_{C^{3+\alpha,\frac{\alpha}{2}}(\mathcal{M}\times[0,T])}$$
$$\text{for } \alpha < \beta < 1,\ T \text{ small enough} \tag{6.21}$$

Recalling $\Lambda_1(s',0) = \Lambda_2(s',0)$ in $\mathcal{M}$, we obtain from (6.20) that

$$\|\Lambda_1 - \Lambda_2\|_{C^{3+\beta,\frac{\beta}{2}}(\mathcal{M}\times[0,T])} \le C\|\Lambda_1 - \Lambda_2\|_{C^{3+\beta,\frac{\beta}{2}}(\mathcal{M}\times[0,T])} \cdot T^{\frac{\beta-\alpha}{2}}$$

which implies

$$\Lambda_1 \equiv \Lambda_2 \text{ in } (\mathcal{M}\times[0,T]) \text{ for } T \text{ small enough}$$

Therefore, $u_1 \equiv u_2$ in $(\mathcal{M}\times[0,T])$ for T small enough.

Thus the solutions of (1.1)–(1.5) provided by Theorem 1.1 is unique.

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