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LOCAL EXISTENCE OF BOUNDED SOLUTIONS TO THE DEGENERATE STEFAN PROBLEM WITH JOULE'S HEATING *

Yuan Guangwei
(Institute of Applied Physics and Computational Mathematics,
P.O. Box 8009, Beijing 100088, China)
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Abstract This paper deals with the degenerate Stefan problem with Joule's heating, which describes the combined effects of heat and electrical current flows in a metal. The local existence of a bounded weak solution for the problem in proved. Also a degenerate thermistor problem with discontinuous conductivity is considered.
Key Words Degenerate; bounded solutions; compactness; Stefan problem.
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1. Introduction

A multidimensional, two-phase problem of Stefan type that describes the processes of electric heating in a conducting material is considered. When an electrical current flows across the conductor, Joule heating is generated by the resistance of the conductor to the electrical current, which brings about the increase of the temperature. Once the melting temperature is crossed, latent heat will be absorbed and the phase change phenomena occurs.

Let Ω be a smooth bounded domain in $\mathcal{R}^N$, $N \geq 1$. The electrical potential and the temperature distribution inside $\Omega_T \equiv \Omega \times [0, T]$ are denoted by $\varphi = \varphi(x, t)$ and $u = u(x, t)$, respectively. Let u_*, be the melting temperature, which is a positive constant, $h = h(x, t)$ be the enthalpy density. Then the mathematical model under consideration is the follows:

Find a triplet $\{h, u, \varphi\}$, such that

$$\frac{\partial h}{\partial t} - \Delta u = \sigma(u)|\nabla \varphi|^2 \quad \text{in } \Omega_T \tag{1.1}$$

$$\nabla \cdot (\sigma(u) \nabla \varphi) = 0 \quad \text{in } \Omega_T \tag{1.2}$$

$$h \subset \alpha(u) \quad \text{in } \Omega_T \tag{1.3}$$

$$u = u_0(x) \quad \text{on } \Omega \times \{0\} \tag{1.4}$$

$$u = 0 \quad \text{on } \partial\Omega \times [0, T] \tag{1.5}$$

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$$\varphi = \varphi_0 \quad \text{on } \partial\Omega \times [0,T] \tag{1.6}$$

where $\sigma(u)$ is the temperature depending on electrical conductivity, $\alpha(u)$ is the maximal monotone graph modelling the phase change process,

$$\alpha(s) = \begin{cases} s-\lambda & \text{if } s < u_* \\ |u_* - \lambda, u_* + \lambda| & \text{if } s = u_* \\ s+\lambda & \text{if } s > u_* \end{cases} \tag{1.7}$$

For simplicity we have assumed that the temperature is equal to zero on $\partial\Omega$, and will assume $\lambda = 1$.

When $\lambda = 0$ (i.e., $h \equiv u$, no phase change occurs), the problem (1.1)–(1.6) is called the thermistor problem and has been studied by several authors; see, *e.g.*, [1–6]. For the physical background and the known results for the problem (1.1)–(1.6) with $\lambda = 1$, we refer to [7] for more details and the references therein. In [8] we have proved the existence of the C^0-solution in two-space dimension. The results mentioned above are proved by assuming that the conductivity is continuous and uniformly positive. When the conductivity $\sigma(s)$ has limit zero as $|s| \to \infty$, Xu [9] introduced a notion of capacity solution to avoid the difficulty which is caused by the fact that the boundedness of solution u has not been proved. In this paper we shall estimate the uniform bound of approximated temperature u in local time by using heat potential analysis and comparison principle, and then obtain a sequence of approximate solution converging in L^2 space to the local weak solution of (1.1)–(1.6) by using a generalized compactness lemma, which is the improvement of the well-known Lions-Aubin compactness lemma and is crucial to the proof of existence results for the Stefan-like problem. As a corollary the global existence of bounded weak solution for the problem (1.1)–(1.6) with uniformly positive conductivity is obtained. We note that the boundedness of weak solution has not been discussed in [7], and it seems that the proof of (4.21) in [7] is false.

The plan of the paper is as follows. In Section 2, we state the definition of the weak solution and main results, and prove two auxiliary lemmas. The one is a generalized compactness lemma, and the other one is a maximum principle for parabolic equation with the source in Morrey space $L^\infty(0,T;L^{1,N-2+2\alpha}(\Omega))$ (See Section 2 below). In Section 3 we introduce a family of regularized problems, to whom the existence and uniform estimate are proved. Next in Section 4 we will conclude that there exists a sequence of approximating solutions converging to the weak solution of (1.1)–(1.6). Finally in Section 5 a time-dependent thermistor problem with degenerate and discontinuous conductivity is considered and the local existence of bounded weak solution for the problem is obtained

2. Formulation of the Problem and Auxiliary Lemmas

Let us assume

$$\sigma(s) \in \mathrm{Lip}\,(\mathcal{R}^1),\ 0 < \sigma(s) \le \sigma_0 < +\infty, \quad \forall s \in \mathcal{R}^1,\ \lim_{|s|\to\infty} \sigma(s) = 0 \tag{2.1}$$

$$u_0(x) \in H_0^1(\Omega) \cap C(\overline{\Omega}),\ u_0(x) \neq u_*,\ a.e.\ x \in \Omega \tag{2.2}$$

$$\varphi_0(x,t) \in C(0,T;C^{1+\alpha}(\overline{\Omega}))\quad (0<\alpha<1) \tag{2.3}$$

We assume $\lambda = 1$ for simplicity.

Definition 2.1 We say that a triplet $\{h, u, \varphi\}$ is a bounded weak solution of (1.1)–(1.6) if

$$\begin{aligned} &h \in L^\infty(\Omega_T),\ h \subset \alpha(u),\ h(x,0) = \alpha(u_0(x)),\ a.e.\ \text{in } \Omega \\ &u \in L^\infty(\Omega_T) \cap C(0,T;L^2(\Omega)) \cap L^2(0,T;H_0^1(\Omega)) \\ &\varphi - \varphi_0 \in L^\infty(0,T;H_0^1(\Omega)) \end{aligned} \tag{2.4}$$

and for all $\xi \in L^2(0,T;H_0^1(\Omega)) \cap C^1(\overline{\Omega}_T)$ with $\xi = 0$ on $\Omega \times \{T\}$, there holds

$$\int_{\Omega_T} \Big\{ -h\frac{\partial \xi}{\partial t} + \nabla u \cdot \nabla \xi \Big\} dxdt = \int_{\Omega_T} \sigma(u)|\nabla \varphi|^2 \xi dxdt + \int_\Omega \xi(x,0)\alpha(u_0(x))dx \tag{2.5}$$

and for all $\eta \in H_0^1(\Omega)$, a.e. $t \in [0,T]$, there holds

$$\int_\Omega \sigma(u) \nabla \varphi \cdot \nabla \eta dx = 0 \tag{2.6}$$

In this paper the following existence results will be proved.

Theorem 2.2 *Under the assumption (2.1)–(2.3), there exists a bounded weak solution to the problem (1.1)–(1.6) for some $T > 0$.*

Corollary 2.3 Assume (2.1)–(2.3) holds, and $\sigma(s) \geq \sigma_1 > 0$ for all $s \in \mathcal{R}^1$, where σ_1 is a positive constant. Then there exists a global bounded weak solution to problem (1.1)–(1.6).

Remark If we replace (1.1) with the equation

$$\frac{\partial h}{\partial t} - \nabla(k(u)\nabla u) = \sigma(u)|\nabla \varphi|^2 \tag{1.1$'$}$$

where $k(\cdot)$ is a continuous function on R^1 and satisfies

$$\lim_{s\to\infty} k(s) = 0, \quad k(s) > 0,\ \forall s \in R^1$$

then for the problem (1.1)$'$ (1.2)–(1.6) the same results as Theorem 2.2 and Corollary 2.3 hold.

For the presence of the singular term $\dfrac{\partial h}{\partial t}$ we shall need the following generalized compactness lemma:

Lemma 2.4 *Assume $f_n(s) \in C^1(\mathcal{R}^1)$, $f_n'(s) \geq 1$, $|f_n(s)| \leq 1 + |s|$, for all $s \in \mathcal{R}^1$, $n = 1, 2, \cdots$. Let $u_n(x,t) \in C(0,T;H_0^1(\Omega))$ such that*

$$\|u_n\|_{L^2(0,T;H_0^1(\Omega))} + \Big\|\frac{\partial f_n(u_n)}{\partial t}\Big\|_{L^2(0,T;H^{-1}(\Omega))} \leq C \tag{2.7}$$

where C is a constant independent of n. Then $\{u_n\}$ *is precompact in* $L^2(\Omega_T)$.

Remark Note that Lemma 2.4 is not the corollary of Lions-Aubin compactness Lemma (see [10, Chapt. I, Thm 5.1] or [11]). The upper bound of $f_n'(s)$ is not assumed, which makes Lemma 2.4 applicable to Stefan-like problem.

Proof of Lemma 2.4 Denote by e_i the unit coordinate vector in the x_i direction. Extend $u_n(\cdot,t)$ by zero into $\mathcal{R}^N\backslash\Omega$ for each $t\in[0,T]$. By Lemma 7.23 in [12] we know, for each $1\le i\le N$,

$$\iint_{\Omega_T}|u_n(x+he_i,t)-u_n(x,t)|^2dxdt\le|h|\iint_{\Omega_T}|\nabla u_n|^2,\quad\forall h\in\mathcal{R}^1 \tag{2.8}$$

Assume that for any $\eta>0$ the following inequality holds,

$$\begin{aligned}\|u(\cdot,t+h)-u(\cdot,t)\|^2_{L^2(\Omega)}\le&\varepsilon\|u(\cdot,t+h)-u(\cdot,t)\|^2_{H^1(\Omega)}\\&+\frac{1}{\varepsilon}\|f_n(u(\cdot,t+h))-f_n(u(\cdot,t))\|^2_{H^{-1}(\Omega)}\end{aligned} \tag{2.9}$$

for all $u\in C(0,T;H_0^1(\Omega))$ and all $t,t+h\in[0,T]$. Then, for any $0<h<h_1$,

$$\begin{aligned}&\iint_{\Omega_{T-h_1}}|u_n(x,t+h)-u_n(x,t)|^2dxdt\le\varepsilon\int_0^{T-h}\|u_n(\cdot,t+h)-u_n(\cdot,t)\|^2_{H^1(\Omega)}dt\\&\quad+\frac{1}{\varepsilon}\int_0^{T-h}\|f_n(u_n(\cdot,t+h))-f_n(u_n(\cdot,t))\|^2_{H^{-1}(\Omega)}dt\\&\le4\varepsilon\int_0^T\|u_n(\cdot,t)\|^2_{H^1(\Omega)}+\frac{h}{\varepsilon}\int_0^T\Big\|\frac{\partial f_n(u_n(\cdot,t))}{\partial t}\Big\|^2_{H^{-1}(\Omega)}dt\\&\le C(\varepsilon+h\varepsilon^{-1})\end{aligned} \tag{2.10}$$

By combining (2.8) with (2.10) and using Theorem 2.21 in [13] one obtains the conclusion of Lemma 3.2. It remains to show (2.9). In fact, setting

$$\begin{aligned}w&=w(x,t)=u(x,t+h)-u(x,t)\\\widetilde{w}&=\widetilde{w}(x,t)\\&=\begin{cases}\dfrac{f_n(u(x,t+h))-f_n(u(x,t))}{u(x,t+h)-u(x,t)}&\text{if } u(x,t+h)-u(x,t)\ne0\\1&\text{if } u(x,t+h)-u(x,t)=0\end{cases}\end{aligned}$$

we get $\widetilde{w}\ge1$ and

$$\begin{aligned}\|w\cdot\widetilde{w}\|_{H^{-1}(\Omega)}=&\sup_{\|\xi\|_{H_0^1(\Omega)}=1}\langle w\cdot\widetilde{w},\xi\rangle=\sup_{\|\xi\|_{H_0^1(\Omega)}=1}\int_\Omega w\cdot\widetilde{w}\cdot\xi dx\\&\ge\int_\Omega w\cdot\widetilde{w}\cdot\frac{w}{\|w\|_{H^1(\Omega)}}dx\ge\frac{\|w\|^2_{L^2(\Omega)}}{\|w\|_{H^1(\Omega)}}dx\end{aligned}$$

and then (2.9) follows.

For the proof of Theorem 2.2, we need the following maximum principle, which is a direct consequence of Lemma 2.2 in [6].

Lemma 2.5 *Let $f = f(x,t) \in L^\infty(0,T;L^1(\Omega))$ such that $f \geq 0$ in Ω_T and*

$$\operatorname*{ess\,sup}_{0\le t\le T} \int_{\Omega\cap B(x_0,\rho)} f(x,t)dx \leq C_0\rho^{N-2+2\alpha}, \quad \forall x_0 \in \overline{\Omega},\ \forall \rho > 0 \tag{2.11}$$

where $0 < \alpha < 1$, $B(x_0,\rho) = \{x \in \mathcal{R}^N; |x - x_0| < \rho\}$. Let $w = w(x,t)$ be the weak solution of the following problem:

$$\frac{\partial w}{\partial t} - \Delta w = f \quad \text{in } \Omega_T \tag{2.12}$$

$$w(x,0) = 1 + u_* + \|u_0\|_{L^\infty(\Omega)} \quad \text{on } \Omega \tag{2.13}$$

$$w(x,t) = 1 + u_* \quad \text{on } \partial\Omega \times [0,T] \tag{2.14}$$

Then we have

$$1 + u_* \leq w(x,t) \leq Ct^\beta + 2(1 + u_* + \|u_0\|_{L^\infty(\Omega)}) \quad \text{in } \Omega_T \tag{2.15}$$

where $\beta \in (0,1)$ depends only on α and N, and C depends only on the constant C_0 in (2.11).

Proof Decompose w into the sum of w_1 and w_2 which are, respectively, the solution of the following two problems:

$$\frac{\partial w_1}{\partial t} - \Delta w_1 = f \quad \text{in } \Omega_T \tag{2.16}$$

$$w_1(x,0) = 0 \quad \text{on } \Omega \tag{2.17}$$

$$w_1(x,t) = 0 \quad \text{on } \partial\Omega \times [0,T] \tag{2.18}$$

and

$$\frac{\partial w_2}{\partial t} - \Delta w_2 = 0 \quad \text{in } \Omega_T \tag{2.19}$$

$$w_2(x,0) = w(x,0) \quad \text{on } \Omega \tag{2.20}$$

$$w_2(x,t) = w(x,t) \quad \text{on } \partial\Omega \times [0,T] \tag{2.21}$$

Introduce the fundamental solution of the heat equation:

$$\Gamma(x-\xi,t-\tau) = \begin{cases} \dfrac{1}{|4\pi(t-\tau)|^{N/2}} \exp\Big\{-\dfrac{|x-\xi|^2}{4(t-\tau)}\Big\} & \text{for } t > \tau \\ 0 & \text{for } t \leq \tau \end{cases}$$

Then

$$w_1 = \int_0^t \int_{\mathcal{R}^N} \Gamma(x-\xi,t-\tau) f(\xi,\tau) d\xi d\tau + v \equiv w_f + v \tag{2.22}$$

where v is the solution of the problem

$$\frac{\partial v}{\partial t} - \Delta v = 0 \quad \text{in } \Omega_T \tag{2.23}$$

$$v=-w_f \quad \text{on } \partial\Omega\times[0,T] \tag{2.24}$$

$$v=0 \quad \text{on } \Omega\times\{0\} \tag{2.25}$$

Note that

$$\Gamma(x-\xi,t-\tau)\le C|t-\tau|^{-\beta}|x-\xi|^{-N+2\beta} \quad (t>\tau, 0<\beta<1)$$

We can estimate w_f by

$$\begin{aligned}|w_f(x,t)|\le& C\sum_{i=0}^{\infty}\int_0^t\int_{B(x,2^{i+1})\setminus B(x,2^i)}\frac{|f(\xi,\tau)|}{|t-\tau|^{\beta_1}|x-\xi|^{N-2\beta_1}}d\xi d\tau\\ &+C\sum_{i=0}^{\infty}\int_0^t\int_{B(x,2^{-i})\setminus B(x,2^{-i-1})}\frac{|f(\xi,\tau)|}{|t-\tau|^{\beta_2}|x-\xi|^{N-2\beta_2}}d\xi d\tau\\ \le& CC_0\sum_{i=0}^{\infty}\Big(\frac{1}{2^i}\Big)^{N-2\beta_1}2^{(i+1)(N-2+2\alpha)}\int_0^t\frac{d\tau}{(t-\tau)^{\beta_1}}\\ &+CC_0\sum_{i=0}^{\infty}2^{(i+1)(N-2\beta_2)}\Big(\frac{1}{2^i}\Big)^{N-2+2\alpha}\int_0^t\frac{d\tau}{(t-\tau)^{\beta_2}}\\ \le& Ct^{1-\beta_2}\end{aligned}$$

where $0<\beta_1<1-\alpha<\beta_2<1$, and C depends on C_0. By the standard maximum principle and $w=w_f+v+w_2$ we obtain (2.12).

Remark If one replace (2.12) with

$$\frac{\partial w}{\partial t}-\nabla(a(x,t)\nabla w)=f \quad \text{in } \Omega_T \tag{2.12$'$}$$

where $a(x,t)$ is a bounded function with positive lower bound, then the conclusion of Lemma 2.5 holds still. The proof is the same as above provided that the estimate for the fundamental solution in [14] is used.

Lemma 2.6 *If f satisfies (2.11), then $f\in L^\infty(0,T;H^{-1}(\Omega))$. Moreover $\|f\|_{L^\infty(0,T;H^{-1}(\Omega))}\le C(C_0)$.* (see [15])

3. Approximate Problems

Set $\alpha_n(s)=s+H_n(s-u_*)$ $(n=1,2,\cdots)$, and $H_n(s)$ satisfies the following conditions:

$$H_n(s)\in C^1(\mathcal{R}^1),\ H_n(0)=0,\quad -1\le H_n(s)\le 1$$

$$0\le H_n'(s)\le 4n,\quad \forall s\in\mathcal{R}^1,\ n=1,2,\cdots \tag{3.1}$$

$$H_n'(s-u_*)=0,\quad \forall s\in\mathcal{R}^1\Big\backslash\Big[u_*-\frac{1}{n},u_*+\frac{1}{n}\Big],\ n=1,2,\cdots \tag{3.2}$$

$$\alpha_n(s)\to\alpha(s) \text{ in } C^1[a,b] \text{ for all } [a,b] \text{ such that } u_*\notin[a,b] \tag{3.3}$$

Let $u_{0n}(x)$ $(n = 1, 2, \cdots)$ satisfy

$$u_{0n}(x) \in C^1(\overline{\Omega}),\ u_{0n}(x) = 0 \quad \text{on } \partial\Omega$$
$$\|u_{0n} - u_0\|_{C(\overline{\Omega})} + \|u_{0n} - u_0\|_{H_0^1(\Omega)} \to 0 \text{ (as } n \to \infty) \tag{3.4}$$

Denote

$$\overline{\sigma}(s) = \begin{cases} \sigma(s) & \text{for } |s| < \overline{M} \\ \sigma(\overline{M}) & \text{for } |s| \geq \overline{M} \end{cases} \tag{3.5}$$

where $\overline{M}$ is a positive constant to be determined late.

Denote $[a]_n = \min(a, n)$. For each n we consider the following approximating problem (P_n): Find a pair $\{u_n, \varphi_n\}$, such that

$$\frac{\partial \alpha_n(u_n)}{\partial t} - \Delta u_n = \overline{\sigma}(u_n)[|\nabla \varphi_n|^2]_n \quad \text{in } \Omega_T \tag{3.6}$$
$$\nabla \cdot (\overline{\sigma}(u_n) \nabla \varphi_n) = 0 \quad \text{in } \Omega_T, \quad \forall t \in [0, T] \tag{3.7}$$
$$u_n(x, 0) = u_{0n}(x) \quad \text{in } \Omega \tag{3.8}$$
$$u_n = 0 \quad \text{on } \partial\Omega \times [0, T] \tag{3.9}$$
$$\varphi_n = \varphi_0 \quad \text{on } \partial\Omega \times [0, T] \tag{3.10}$$

Lemma 3.1 *For each $n = 1, 2, \cdots$, the problem (P_n) has a weak solution satisfying*

$$u_n \in L^p(0, T; W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega))$$
$$\frac{\partial u_n}{\partial t} \in L^p(0, T; L^p(\Omega)) \ for\ any\ p \in (1, \infty) \tag{3.11}$$
$$\varphi_n \in L^\infty(0, T; C^{1+\alpha}(\overline{\Omega})) \cap C(0, T; H^1(\Omega)) \cap C(\overline{\Omega}_T) \tag{3.12}$$

and for all $\xi \in L^2(0, T; H_0^1(\Omega)) \cap H^1(0, T; L^2(\Omega))$ with $\xi = 0$ on $\Omega \times \{T\}$

$$\int_{\Omega_T} \Big\{ - \alpha_n(u_n)\frac{\partial \xi}{\partial t} + \nabla u_n \cdot \nabla \xi \Big\} dx dt$$
$$= \int_{\Omega_T} \overline{\sigma}(u_n)[|\nabla \varphi_n|^2]_n \xi dx dt + \int_\Omega \alpha_n(u_{0n}(x))\xi(x, 0) dx \tag{3.13}$$

and for all $\xi \in H_0^1(\Omega)$, all $t \in [0, T]$,

$$\int_\Omega \overline{\sigma}(u_n) \nabla \varphi_n \cdot \nabla \eta dx = 0 \tag{3.14}$$

The proof is much the same as that of Lemma 3.1 in [8]. We omit the proof.

Taking $\overline{M} = 3 + 2(u_* + \|u_0\|_{L^\infty(\Omega)})$, we have

Lemma 3.2 $\|u_n\|_{L^\infty(\Omega \times (0,t))} \leq \overline{M}$, *for $t \leq t_0 = t_0(\overline{M})$.*

Proof Since the right hand side of (3.6) is nonnegative, we get

$$u_n \geq -1 - \|u_0\|_{L^\infty(\Omega)} \quad \text{in } \Omega_T \tag{3.15}$$

To obtain the upper bound of u_n, we compare u_n with the solution w of the following problem:

$$\frac{\partial w}{\partial t} - \Delta w = \overline{\sigma}(u_n)[|\nabla \varphi_n|^2]_n \quad \text{in } \Omega_T \tag{3.16}$$

$$w(x,0) = 1 + u_* + \|u_0\|_{L^\infty(\Omega)} \quad \text{on } \Omega \tag{3.17}$$

$$w(x,t) = 1 + u_* + \|u_0\|_{L^\infty(\Omega)} \quad \text{on } \partial\Omega \times [0,T] \tag{3.18}$$

By using the classical Hölder estimates for (3.7) (3.10), we obtain

$$\|\varphi_n\|_{L^\infty(0,T;C^\alpha(\overline{\Omega}))} \le C_1 \tag{3.19}$$

where $0 < \alpha < 1$, and C_1 depends only on $\overline{M}$, σ_0, $\|\varphi_0\|_{L^\infty(0,T;C^\alpha(\overline{\Omega}))}$ and smoothness of $\partial\Omega$. Let $\xi \in C_0^\infty(\mathcal{R}^N)$ be the cut-off function satisfying $\xi = 0$ in $\mathcal{R}^N \backslash B(x_0, 2\rho)$, $\xi = 1$ in $B(x_0,\rho)$, $0 \le \xi \le 1$, $|\nabla \xi| \le \frac{2}{\rho}$ in $\mathcal{R}^N$. Obviously, for $x_0 \in \Omega$, either dist $\{x_0, \partial\Omega\} > 2\rho$ or dist $\{x_0, \partial\Omega\} \le 2\rho$. In the first case, we can take $\eta = (\varphi_n - \varphi_n(x_0,t))\xi^2$ as a test function in (3.14) and obtain

$$\begin{aligned} 0 &= \int_\Omega \overline{\sigma}(u_n) \nabla \varphi_n \cdot \nabla((\varphi_n - \varphi_n(x_0,t))\xi^2) \\ &= \int_\Omega \overline{\sigma}(u_n)|\nabla \varphi_n|^2\xi^2 + 2\int_\Omega \overline{\sigma}(u_n)(\varphi_n - \varphi_n(x_0,t))\xi \nabla \varphi_n \nabla \xi \\ &\ge \frac{1}{2}\int_\Omega \overline{\sigma}(u_n)|\nabla \varphi_n|^2\xi^2 - C\rho^{N-2+2\alpha}\|\varphi_n(\cdot,t)\|^2_{C^\alpha(\overline{\Omega})} \end{aligned}$$

In the second case, taking $\eta = (\varphi_n - \varphi_0(x,t))\xi^2 \in H_0^1(\Omega)$, we get

$$\begin{aligned} 0 &= \int_\Omega \overline{\sigma}(u_n) \nabla \varphi_n \cdot \nabla((\varphi_n - \varphi_0)\xi^2) \\ &\ge \frac{1}{2}\int_\Omega \overline{\sigma}(u_n)|\nabla \varphi_n|^2\xi^2 - \int_\Omega \overline{\sigma}(u_n)|\nabla \varphi_0|^2\xi^2 \\ &\quad - 4\int_\Omega \overline{\sigma}(u_n)(\varphi_n - \varphi_0)^2|\nabla \xi|^2 \\ &\ge \frac{1}{2}\int_{\Omega\cap B(x_0,\rho)} \overline{\sigma}(u_n)|\nabla \varphi_n|^2 - C\rho^{N-2+2\alpha}(1 + \|\varphi_n(\cdot,t)\|^2_{C^\alpha(\overline{\Omega})}) \end{aligned}$$

Thus, for the function $f \equiv \overline{\sigma}(u_n)[|\nabla \varphi_n|^2]_n$, the inequality (2.11) holds. By Lemma 2.5, we obtain

$$1 + u_* \le w = w(x,t) \le C(\overline{M})t^\beta + 2(1 + u_* + \|u_0\|_{L^\infty(\Omega)}) \quad \text{in } \Omega_T$$

Now, noticing the definition of $\alpha_n(s)$, we have $\alpha_n(w) = w$ and then

$$\frac{\partial(\alpha_n(u_n) - \alpha_n(w))}{\partial t} - \Delta(u_n - w) = 0 \quad \text{in } \Omega_T \tag{3.20}$$

$$u_n(x,0) - w(x,0) \le 0 \quad \text{on } \Omega \tag{3.21}$$

$$u_n(x,t)-w(x,t)\le 0 \quad \text{on } \partial\Omega\times[0,T] \tag{3.22}$$

Denote

$$S_\delta(r)=\begin{cases}\sqrt{r^2+\delta^2}-\delta & \text{for } r>0\\ 0 & \text{for } r\le 0\end{cases}$$

Multiplying (3.24) by $S_\delta(u_n-w)$ and then integrating over $\Omega\times(0,t)$, we get

$$\begin{aligned}0&=\iint_{\Omega_t}\frac{\partial}{\partial t}(\alpha_n(u_n)-\alpha_n(w))S_\delta(u_n-w)+\iint_{\Omega_t}\nabla(u_n-w)\cdot\nabla S_\delta(u_n-w)\\ &\equiv I_1^\delta+I_2^\delta\end{aligned}$$

Obviously, $S'_\delta(r)\ge 0$ and

$$S'_\delta(r)\to \operatorname{sign}^+(r)=\begin{cases}1 & \text{if } r>0\\ 0 & \text{if } r\le 0\end{cases}$$

and $\operatorname{sign}^+(\alpha_n(u_n)-\alpha_n(w))=\operatorname{sign}^+(u_n-w)$, it follows that

$$\lim_{\delta\to 0} I_1^\delta=\int_0^t\int_\Omega\frac{\partial}{\partial t}(\alpha_n(u_n)-\alpha_n(w))^+=\int_{\Omega\times\{t\}}(\alpha_n(u_n)-\alpha_n(w))^+$$

and

$$I_2^\delta=\int_0^t\int_\Omega S'_\delta(u_n-w)|\nabla(u_n-w)|^2\ge 0$$

Thus, $\int_{\Omega\times\{t\}}(\alpha_n(u_n)-\alpha_n(w))^+\le 0$, *i.e.*, $u_n\le w$ in Ω_T, and then $u_n\le\overline{M}$ in $\Omega\times(0,t)$, for $t\le t_0=t_0(\overline{M})$.

Remark By Lemma 3.3 and the definition of $\overline{\sigma}(s)$ we know that $\sigma(u_n)=\overline{\sigma}(u_n)$ for $t\le t_0=t_0(\overline{M})$, $n=1,2,\cdots$.

Lemma 3.3

(i) $\|\varphi_n\|_{L^\infty(\Omega_T)}\le C,\ \sup_{0\le t\le T}\int_\Omega|\nabla\varphi_n(\cdot,t)|^{p_0}dx\le C(\overline{M})$ *for some* $p_0>2$ (3.23)

(ii) $\sup_{0\le t\le T}\int_\Omega u_n^2(\cdot,t)+\iint_{\Omega_T}|\nabla u_n|^2\le C(\overline{M})$ (3.24)

(iii) $\left\|\frac{\partial\alpha_n(u_n)}{\partial t}\right\|_{L^2(0,T;H^{-1}(\Omega))}\le C(\overline{M})$ (3.25)

where C *and* $C(\overline{M})$ *are constants independent of* n.

Proof (i) follows from the maximum principle and Meyers' estimate (see [16]). Multiply (3.6) by u_n and integrate over $\Omega_t\equiv\Omega\times(0,t)$. We proceed to evaluate separately the three terms: By using (3.1) (3.2) we have

$$\begin{aligned}\iint_{\Omega_t}u_n\frac{\partial\alpha_n(u_n)}{\partial t}&=\iint_{\Omega_t}\Big(\frac{\partial}{\partial t}\int_0^{\alpha_n(u_n)}\alpha_n^{-1}(s)ds\Big)\\ &=\int_\Omega\Big(\int_0^{u_n}r\alpha'_n(r)dr-\int_0^{u_0}r\alpha'_n(r)dr\Big)\ge\frac12\int_\Omega u_n^2(\cdot,t)-C\end{aligned}$$

and

$$\iint_{\Omega_t} u_n(-\Delta u_n) = \iint_{\Omega_t} |\nabla u_n|^2$$

and

$$\left|\iint_{\Omega_t} \overline{\sigma}_n(u_n)[|\nabla \varphi_n|^2]_n u_n\right| \le C(\overline{M}) \iint_{\Omega_t} |\nabla \varphi_n|^2 \le C(\overline{M})$$

Combining these estimates we easily conclude that (ii) holds. Note that $\overline{\sigma}(U_n)[|\nabla\varphi_n|^2]_n$ satisfy (2.11). Then the estimate (3.25) follows from (3.6) (3.24) and Lemma 2.6.

4. The Limit as $n \to \infty$

Lemma 4.1 (i) *The family* $\{u_n\}$ *is precompact in* $L^2(\Omega_T)$, (ii) *There exists a subsequence out of* $\{\varphi_n\}$ *(still denoted by* $\{\varphi_n\}$*) such that, as* $n \to \infty$,

$$\varphi_n \to \varphi \quad in\ L^2(0,T;H_0^1(\Omega))$$
$$\nabla \varphi_n(\cdot,t) \to \nabla\varphi(\cdot,t) \quad in\ L^2(\Omega)\ for\ a.e.\ t \in [0,T]$$

Proof By Lemmas 2.4 and 3.2 we obtain (i) holds. Now we prove (ii).
Using (i) we get, for a subsequence (still denote $\{u_n\}$),

$$u_n \to u \quad a.e.\ \text{in } \Omega_T,\ \text{as } n \to \infty \tag{4.1}$$
$$u_n(\cdot,t) \to u(\cdot,t) \quad a.e.\ \text{in } \Omega,\ \text{for } a.e. t \in [0,T] \tag{4.2}$$

For $\forall t \in [0,T]$, $\forall \eta \in H_0^1(\Omega)$ and $\forall n, m$, there holds

$$\int_\Omega \sigma(u_n)\nabla(\varphi_n - \varphi_m)\cdot\nabla\eta = \int_\Omega (\sigma(u_n) - \sigma(u_m))\nabla\varphi_m \cdot \nabla\eta$$

Taking $\eta(\cdot) = (\varphi_n - \varphi_m)(\cdot,t) \in H_0^1(\Omega)$, we have

$$\begin{aligned}\int_\Omega |\nabla(\varphi_n - \varphi_m)|^2 &\le C\int_\Omega |\sigma(u_n) - \sigma(u_m)|^2|\nabla\varphi_m|^2 \\ &\le C\|\nabla\varphi_m\|^2_{L^{p_0}}\|\sigma(u_n) - \sigma(u_m)\|^2_{L^{\frac{2p_0}{p_0-2}}(\Omega)} \\ &\le C\|\sigma(u_n) - \sigma(u_m)\|^2_{L^{\frac{2p_0}{p_0-2}}(\Omega)}\end{aligned}$$

Then, $\|\nabla(\varphi_n - \varphi_m)\|_{L^2(\Omega)} \to 0$ for $a.e.\ t \in [0,T]$, as $n\to\infty$, $m\to\infty$,

$$\|(\varphi_n - \varphi_m)\|_{L^2(0,T;H_0^1(\Omega))} \to 0 \quad \text{as } n, m \to \infty$$

Let $\varphi_n(\cdot,t) \to \varphi(\cdot,t)$ in $H^1(\Omega)$ for $a.e.\ t \in [0,T]$

$$\varphi_n \to \widetilde{\varphi} \quad \text{in } L^2(\Omega_T),\ \nabla\varphi_n \to \nabla\widetilde{\varphi} \quad \text{in } L^2(\Omega_T)$$

It is easily seen that $\varphi = \widetilde{\varphi}$ $a.e.$ in Ω_T and the conclusion (ii) follows.

Now, take $T = T(\overline{M})$ small enough, and note that

$$\alpha_n(u_n) \to h \subset \alpha(u) \quad \text{weakly in } L^2(\Omega_T)$$
$$\alpha_n(u_{0n}) \to \alpha(u_0) \quad \text{weakly in } L^2(\Omega)$$

and let $n \to \infty$ in (3.13) for the subsequence chosen as in Lemma 4.1. We have, for $\forall \xi \in L^2(0,T;H_0^1(\Omega)) \cap C^1(\overline{\Omega}_T)$ with $\xi = 0$ on $\Omega \times \{T\}$,

$$\int_{\Omega_T} \left\{ -h\frac{\partial \xi}{\partial t} + \nabla u \cdot \nabla \xi \right\} = \int_{\Omega_T} \sigma(u)| \nabla \varphi|^2 \xi + \int_{\Omega} \alpha(u_0(x))\xi(x,0)$$

i.e., (2.5) holds. Finally (2.6) is true by Lemma 4.1 and the proof of Theorem 2.2 is completed.

Remark From the proof of Theorem 2.2 it is easy to see that Corollary 2.3 holds.

5. A Degenerate Thermistor Problem with Discontinuous Conductivity

Now we consider the time-dependent thermistor problem with the degenerate and discontinuous conductivity, which can be formulated as follows:

Find a triplet $\{h, u, \varphi\}$, such that

$$\frac{\partial u}{\partial t} - \Delta u = f| \nabla \varphi|^2 \quad \text{in } \Omega_T \tag{5.1}$$
$$u(x,0) = u_0(x) \quad \text{on } \Omega \tag{5.2}$$
$$u = 0 \quad \text{on } \partial\Omega \times [0,T] \tag{5.3}$$
$$f = f(x,t) \in \sigma(u(x,t)) \quad \text{in } \Omega_T \tag{5.4}$$
$$\nabla \cdot (f \nabla \varphi) = 0 \quad \text{in } \Omega_T \tag{5.5}$$
$$\varphi = \varphi_0 \quad \text{on } \partial\Omega \times [0,T] \tag{5.6}$$

where

$$\sigma(s) = \begin{cases} \sigma_1 & \text{if } s < \lambda \\ [\sigma_2(\lambda), \sigma_1(\lambda)] & \text{if } s = \lambda \\ \sigma_2(s) & \text{if } s > \lambda \end{cases}$$

It represents the electric conductivity, and we have assumed $\sigma_2(\lambda) < \sigma_1(\lambda)$ for simplicity.

In [5] by using divergence-curl lemma the existence of weak solution to the problem (5.1)–(5.6) was obtained under the assumption that the conductivity is uniformly positive, which is essential to the proof of existence. Also the boundedness of u has not been discussed in [5]. We can relax the assumption of [5] allowing the limit of the conductivity $\sigma(s)$ being zero as $|s| \to \infty$ and prove the local existence of bounded solution. Since the method of proof is much the same as that in Sections 2–4 above and that used in [5], we omit it and only state the existence results as follows.

Assume Ω is a bounded smooth domain and

$$\sigma_1(s) \in C(-\infty, \lambda], \ \sigma_2(s) \in C[\lambda, +\infty), \quad \lambda > 0 \text{ is a constant}$$
$$0 < \sigma_i(s) \le \sigma_0 < +\infty, \quad i = 1, 2. \ \forall s \in \mathrm{dom}(\sigma_i), \ i = 1, 2$$
$$\lim_{s \to -\infty} \sigma_1(s) = 0, \ \lim_{s \to +\infty} \sigma_2(s) = 0 \tag{5.7}$$
$$u_0 \in W_0^{1,2+r}(\Omega) \cap C^\alpha(\overline{\Omega}), \text{ for some } r > 0 \tag{5.8}$$
$$\varphi_0 \in C(0, T; C^\alpha(\overline{\Omega}) \cap H^1(\Omega)), \quad (0 < \alpha < 1) \tag{5.9}$$

Definition 5.1 We say that $\{f, u, \varphi\}$ is a weak solution to the problem (5.1)–(5.6), if

$$f \in L^\infty(\Omega_T), \quad u \in C^\alpha(\overline{\Omega}_T) \cap L^2(0, T; H_0^1(\Omega))$$
$$\varphi - \varphi_0 \in L^\infty(0, T; W_0^{1,2p}(\Omega))$$
$$u_t \in L^p(\Omega_T), \ \Delta u \in L^p(\Omega_T) \text{ for some } p > 1$$

and (5.1) is satisfied almost everywhere and (5.2)–(5.4) hold, and for all $\xi \in H^1(\Omega)$ and *a.e.* $t \in (0, T)$, there holds

$$\int_\Omega f(\cdot, t) \nabla \varphi \cdot \nabla(\xi - \varphi_0) dx = 0$$

Theorem 5.2 *Assume* (5.7)–(5.9) *hold. Then for some* $T > 0$ *there exists a weak solution* $\{f, u, \varphi\}$ *to the problem* (5.1)–(5.6).

Note that the Hölder continuity of u can be proved as in [6]. We don't repeat it here.

Remark For the case of other boundary data such as the third boundary data and mixed boundary data, the analogue results can be obtained similarly.

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