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GLOBAL SOLUTIONS AND THEIR LARGE-TIME BEHAVIOR OF CAUCHY PROBLEM FOR EQUATIONS OF DEEP WATER TYPE

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Abstract We consider the large time behavior of the global solution of the Cauchy problem for the equation of Benjamin-Ono type. A series of large-time global estimates for Cauchy problem are constructed. By means of the obtained global estimates uniformly for $0 \le t < \infty$, the attractors of the Cauchy problem for the mentioned nonlinear equations are considered. And also the dimensions of the global attractor are estimated.
Key Words Large time behavior; Benjamin-Ono type equations; global attractor.

Classification 35Q20.

1. Introduction

The equation, which describes the propagation of the internal waves in the stratified fluid with great depth, can be expressed in the form [1–7]

$$u_t + 2uu_x + Hu_{xx} = 0 \tag{1}$$

where H is the Hilbert transform

$$Hu(x,t) = \frac{1}{\pi} P \int_{-\infty}^{\infty} \frac{u(y,t)}{y-x} dy \tag{2}$$

and P denotes the principle value of the integral. The equation (1) of deep water is also called Benjamin-Ono equation. If the effect of the amplitude of the internal waves is taken into account in the deep fluid, the equation (1) has an additional linear term as follows

$$u_t + c_0 u_x + 2uu_x + Hu_{xx} = 0 \tag{3}$$

There are many investigations of the physical purpose for the nonlinear partial differential equation (1) with singular integral term of deep water. The Backlund transformations, the conservation laws, various soliton solutions and their interactions for the Benjamin-Ono equation (1) are studied in [8–12].

In [13] the initial value problem for the nonlinear singular integral-differential equation

$$u_t + 2uu_x + Hu_{xx} + b(x,t)u_x + c(x,t)u = f(x,t) \tag{4}$$

is studied by the method of fixed point and integral *a priori* estimates. The generalized and classical global solutions are obtained. The classical solution of Cauchy problem for the simple Benjamin-Ono equation (1) is also derived in [14] by the method of semigroups. In [15] the global generalized and classical solution are considered again for the original Benjamin-Ono equation (1) in the Hilbert spaces with half order derivatives.

The purpose of this work is to establish the solutions for the Cauchy problem of the general equation

$$u_t + 2uu_x + \alpha Hu_{xx} - \beta Hu_x + \gamma(x,t)Hu + b(x,t)u_x + c(x,t)u = f(x,t) \tag{5}$$

of Benjamin-Ono type, where $\alpha > 0$ and $\beta \geq 0$ are constants. The term $-\beta Hu_x$ of the equation (5) has special character. The change of the coefficient β shows the interesting behavior of solution of the equation (5). This is a nonlinear partial differential equation with singular integral operators. The solutions of the problem for the above equation are approximated by the solutions of the Cauchy problem for the nonlinear parabolic equation

$$u_t - \epsilon u_{xx} + 2uu_x + \alpha Hu_{xx} - \beta Hu_x + \gamma(x,t)Hu + b(x,t)u_x + c(x,t)u = f(x,t) \tag{6}$$

with Hilbert transforms terms, which is obtained by the addition of a diffusion term ϵu_{xx} with small coefficient $\epsilon > 0$ in the equation (5). The solutions of the Cauchy problem (7) for the nonlinear equation (5) are established by the limiting process of the vanishing of diffusion coefficient $\epsilon \to 0$. The convergence speed is estimated in order of $\epsilon > 0$. And then in later part of this work, we are going to consider the large time behavior of the global solutions of the Cauchy problem for the equation of Benjamin-Ono type. A series of large-time global estimates for the solutions of the problems for the nonlinear parabolic equations with Hilbert operators and the corresponding nonlinear equations of Benjamin-Ono type are constructed. By means of these obtained global estimates, the attractors of the Cauchy problems for the mentioned nonlinear equations are considered. And also the dimensions of the global attractor are estimated.

Let us state some fundamental properties of Hilbert transform [8–12], which are used repeatedly in the further investigation as follows.

Lemma 1 *For any $f(x)$ and $g(x) \in L_2(R)$, there are*

⟨1⟩ $H^2 f = -f$,

⟨2⟩ $H(fg) = H(HfHg) + fHg + gHf$,

⟨3⟩ $\int_{-\infty}^{\infty} f(x)Hg(x)dx = -\int_{-\infty}^{\infty} g(x)Hf(x)dx$,

hence

$$\int_{-\infty}^{\infty} g(x)Hg(x)dx = 0$$

Lemma 2 *For any differentiable function $f(x) \in H^s(R)$ $(s = 1, 2)$, there are*

$\langle 1\rangle$ $(Hf)_x = Hf_x$,

$\langle 2\rangle$ $\int_{-\infty}^{\infty} f_x(x)Hf(x)dx \geq 0$,

$\langle 3\rangle$ $\|f_x\|^2_{L_2(R)} \leq \left(\int_{-\infty}^{\infty} f_x(x)Hf(x)dx\right)^{\frac{1}{2}}\left(\int_{-\infty}^{\infty} f_{xx}(x)Hf_x(x)dx\right)^{\frac{1}{2}}$.

Proof Denote by $\widehat{f}(\xi)$ the Fourier transform of $f(x)$. Since the Fourier transform of $Hf(x)$ is $\widehat{Hf}(\xi) = \mathrm{i}\,\mathrm{sgn}\xi\widehat{f}(\xi)$ and $\widehat{f_x}(\xi) = 2\pi\xi\widehat{f}(\xi)$, then

$$\widehat{Hf_x}(\xi) = \mathrm{i}\,\mathrm{sgn}\xi\widehat{f_x}(\xi) = -2\pi\xi\mathrm{sgn}\,\xi\widehat{f}(\xi) = 2\pi i\xi\widehat{Hf}(\xi) = (\widehat{Hf})_x(\xi)$$

This shows $Hf_x = (Hf)_x$ by means of the inverse Fourier transformation. Hence $\langle 1\rangle$ is proved.

For $\langle 2\rangle$, we have

$$\int_{-\infty}^{\infty} f_x(x)Hf(x)dx = \int_{-\infty}^{\infty} \widehat{f}_x\overline{\widehat{Hf}(\xi)}d\xi = \int_{-\infty}^{\infty} 2\pi i\xi\widehat{f}(\xi)\overline{\mathrm{i}\,\mathrm{sgn}\xi\widehat{f}(\xi)}d\xi$$
$$= 2\pi\int_{-\infty}^{\infty}|\xi||\widehat{f}|^2d\xi \geq 0$$

For $\langle 3\rangle$, we have

$$\|f_x\|^2_{L_2(R)} = 4\pi^2\int_{-\infty}^{\infty}|\xi|^2|\widehat{f}(\xi)|^2d\xi \leq \left(2\pi\int_{-\infty}^{\infty}|\xi||\widehat{f}(\xi)|^2d\xi\right)^{\frac{1}{2}}\left(2\pi\int_{-\infty}^{\infty}|\xi|^3|\widehat{f}(\xi)|^2d\xi\right)^{\frac{1}{2}}$$
$$= \left(\int_{-\infty}^{\infty} f_x(x)Hf(x)dx\right)^{\frac{1}{2}}\left(\int_{-\infty}^{\infty} f_{xx}(x)Hf_x(x)dx\right)^{\frac{1}{2}}$$

Hence the lemma is proved.

2. Equations with Diffusion Term

1. In this section we are going to consider the solution of the problem for the nonlinear parabolic equation (6) in the domain $Q_T = \{x \in R, 0 \leq t \leq T\}$ with the initial value condition

$$u(x,0) = \phi(x) \tag{7}$$

where $\phi(x)$ is a given initial function for $x \in R$ and $0 < T < \infty$ is a given constant.

We are finding the solutions of mentioned problems in the space of functions with the derivatives of any order tending to zero as $|x| \to \infty$.

Let us begin with the initial value problems for linear parabolic equations.

Denote by $W_2^{(k,[\frac{k}{2}])}(Q_T)$ the functional spaces of functions $f(x,t)$, which have derivatives $f_{x^rt^s}(x,t) \in L_2(Q_T)$ for $2r+s \leq k$ where $k = 0,1,\cdots$. Also denote by $W_\infty^{(k,[\frac{k}{2}])}(Q_T)$ the functional spaces of functions $f(x,t)$, which have derivatives $f_{x^rt^s}(x,t) \in L\infty(Q_T)$ for $2r+s \leq k$ where $k = 0,1,\cdots$. For $k=0$, $W_2^{(0,0)}(Q_T) \equiv L_2(Q_T)$ and $W_\infty^{(0,0)}(Q_T) \equiv L_\infty(Q_T)$.

We state the existence theorem of the initial value problem for the linear parabolic equations as follows

Theorem 1 *Suppose that* $b(x,t)$ *and* $c(x,t) \in W_\infty^{(k,[\frac{k}{2}])}(Q_T)$ *and* $f(x,t) \in W_2^{(k,[\frac{k}{2}])}(Q_T)$ *and suppose that* $\phi(x) \in H^{k+1}(R)$. *The initial value problem* (7) *in the domain* Q_T *for the linear parabolic equation*

$$Lu \equiv u_t - \epsilon u_{xx} + b(x,t)u_x + c(x,t)u = f(x,t) \tag{8}$$

has a unique global solution $u(x,t) \in W_2^{(k+2,[\frac{k}{2}]+1)}(Q_T)$.

2. Let us now investigate the initial value problem (7) for the linear parabolic equation with singular integral operator

$$\begin{aligned} L_\lambda u \equiv & Lu + \lambda\widetilde{L}u = u_t - \epsilon u_{xx} + \lambda\alpha H u_{xx} - \lambda\beta H u_x + \lambda\gamma(x,t)Hu \\ & + b(x,t)u_x + c(x,t)u = f(x,t) \end{aligned} \tag{9}$$

where $0 \le \lambda \le 1$ is a parameter, α and β are constants and

$$\widetilde{L}u \equiv \alpha H u_{xx} - \beta H u_x + \gamma(x,t)Hu \tag{10}$$

It is evident that for $\lambda = 0$, the initial value problem (7) of linear parabolic equation $L_0 u = f(x,t)$ has a unique solution $u(x,t) \in W_2^{(2,1)}(Q_T)$, under the assumptions that $b(x,t), c(x,t) \in L_\infty(Q_T)$; $f(x,t) \in L_2(Q_T)$ and $\phi(x) \in H^1(R)$.

Let E be the set of values of $\lambda \in [0,1]$, for which the initial value problem (7) of the linear parabolic equation $L_\lambda u = f(x,t)$ containing singular integral operator $\lambda\widetilde{L}u$ has a unique global solution $u_\lambda(x,t) \in W_2^{(2,1)}(Q_T^*)$, then the set E is nonempty in the segment $[0,1]$.

For the solutions $u_\lambda(x,t)$ of the initial value problem (7) of the linear parabolic equation (9), we can make some *a priori* estimations by the usual energy method.

Taking the scalar product of the solution $u_\lambda(x,t)$ and the equation (9), we have

$$\begin{aligned} & \frac{d}{dt}\|u(\cdot,t)\|_{L_2(R)}^2 + \epsilon\|u_x(\cdot,t)\|_{L^2(R)}^2 \\ \le & \Big\{\frac{1}{\epsilon}\|b\|_{L_\infty(Q_T)}^2 + 2\|c\|_{L_\infty(Q_T)} + 2\|\gamma\|_{L_\infty(Q_T)} + 1\Big\}\|u(\cdot,t)\|_{L_2(R)}^2 + \|f(\cdot,t)\|_{L_2(R)}^2 \end{aligned}$$

There is the estimation

$$\sup_{0\le t\le T}\|u(\cdot,t)\|_{L_2(R)} + \|u_x\|_{L_2(Q_T)} \le C_1\{\|\phi\|_{L_2(R)} + \|f\|_{L_2(Q_T)}\}$$

where C_1 depends on the norms $\|b\|_{L_\infty(Q_T)}$, $\|c\|_{L_\infty(Q_T)}$, $\|\gamma\|_{L_\infty(Q_T)}$, the diffusion coefficient $\epsilon > 0$ and the constant β is independent of the parameter $0 \le \lambda \le 1$.

Then taking the scalar product of $u_{xx}(x,t)$ and the equation (9), we get

$$\frac{d}{dt}\|u_x(\cdot,t)\|_{L_2(R)}^2 + \epsilon\|u_{xx}(\cdot,t)\|_{L_2(R)}^2 \le \Big(\frac{4}{\epsilon}\|b\|_{L_\infty(Q_T)}^2 + \frac{4\beta^2}{\epsilon}\Big)\|u_x(\cdot,t)\|_{L_2(R)}^2$$

$$+\frac{4}{\epsilon}\|c\|^2_{L_\infty(Q_T)}\|u(\cdot,t)\|^2_{L_2(R)}+\frac{4}{\epsilon}\|\gamma\|^2_{L_\infty(Q_T)}\|u(\cdot,t)\|^2_{L_2(R)}+\frac{4}{\epsilon}\|f(\cdot,t)\|^2_{L_2(R)}$$

This implies

$$\sup_{0\le t\le T}\|u_x(\cdot,t)\|_{L_2(R)}+\|u_{xx}\|_{L_2(Q_T)}\le C_2\{\|\phi\|_{H^1(R)}+\|f\|_{L_2(Q_T)}\} \tag{11}$$

where C_2 is a constant dependent on the norms $\|b\|_{L_\infty(Q_T)}$, $\|c\|_{L_\infty(Q_T)}$, $\|\gamma\|_{L_\infty(Q_T)}$, the constant β and the diffusion coefficient $\epsilon>0$ are independent of the parameter $0\le\lambda\le 1$.

Hence under given conditions, the solutions $u_\lambda(x,t)$ for the initial value problem (7) of the linear parabolic equation (9) with singular integral operator have the estimations

$$\sup_{0\le t\le T}\|u(\cdot,t)\|_{L_2(R)}+\|u_{xx}\|_{L_2(Q_T)}+\|u_t\|_{L_2(Q_T)}\le C_3\{\|\phi\|_{H^1(R)}+\|f\|_{L_2(Q_T)}\} \tag{12}$$

in other words

$$\|u\|_{W_2^{(2,1)}(Q_T)}\le C_3\{\|\phi\|_{H^1(R)}+\|f\|_{L_2(Q_T)}\} \tag{13}$$

where C_3 depends on the norms $\|b\|_{L_\infty(Q_T)}$, $\|c\|_{L_\infty(Q_T)}$, the constant β and the diffusion coefficient $\epsilon>0$ are independent of the parameter $0\le\lambda\le 1$.

By means of these estimations, we can prove that the subset E is closed in $[0,1]$. Suppose that $\widetilde{\lambda}\in[0,1]$ is a limiting point of E, i.e., there is a sequence $\{\lambda_k\}\subset E$ such that $\lim\limits_{k\to\infty}\lambda_k=\widetilde{\lambda}$. Let $u_k(x,t)$ be the unique generalized global solution of the initial value problem (7) (9): $L_{\lambda_k}u=f(x,t)$ corresponding to the parameter $\lambda=\lambda_k$ $(k=0,1,\cdots)$. The set $\{u_k(x,t)\}$ is uniformly bounded in the space $W_2^{(2,1)}(Q_T)$. Then there exist the subsequences of $\{\lambda_k\}$ and $\{u_k(x,t)\}$, still denoted by $\{\lambda_k\}$ and $\{u_k(x,t)\}$, such that as $k\to\infty$, then $\lambda_k\to\widetilde{\lambda}$ and $\{u_k(x,t)\}$ converges to $\widetilde{u}(x,t)\in W_2^{(2,1)}(Q_T)$ in the sense that $\{u_k(x,t)\}$ converges uniformly to $\widetilde{u}(x,t)$ in any compact domain of Q_T, $\{u_{kx}(x,t)\}$ converges strongly to $\widetilde{u}_x(x,t)$ in $L_p(0,T;L_2(R))$ for $2\le p<\infty$ and the sequences $\{u_{kxx}(x,t)\}$ and $\{u_{kt}(x,t)\}$ converge weakly to $\widetilde{u}_{xx}(x,t)$ and $\widetilde{u}_t(x,t)$ respectively in $L_2(Q_T)$. Thus obtained function $\widetilde{u}(x,t)\in W_2^{(2,1)}(Q_T)$ is a unique generalized global solution of the initial value problem (7) (9): $L_{\widetilde{\lambda}}u=f(x,t)$ with $\lambda=\widetilde{\lambda}$. This means that E is closed in $[0,1]$.

Let us now prove that E is open in $[0,1]$. Define A_λ a mapping of $W_2^{(2,1)}(Q_T)$ into itself as follows: for any $v\in W_2^{(2,1)}(Q_T)\subset W_2^{(2,0)}(Q_T)$, let $u=A_\lambda v$ be the unique generalized global solution of the initial value problem (7) for the linear parabolic equation

$$\begin{aligned}&u_t-\epsilon u_{xx}+\lambda_0(\alpha Hu_{xx}-\beta Hu_x+\gamma(x,t)Hu)+b(x,t)u_x+c(x,t)u\\=&f(x,t)+(\lambda_0-\lambda)(\alpha Hv_{xx}-\beta Hv_x+\gamma(x,t)Hv)\end{aligned}$$

with singular integral operator, where $\lambda_0\in E$. For $v_1,v_2\in W_2^{(2,0)}(Q_T)$, there are $u_1=A_\lambda v_1$ and $u_2=A_\lambda v_2$ belonging to $W_2^{(2,1)}(Q_T)$. Then u_1-u_2 satisfies the linear equation

$$(u_1-u_2)_t-\epsilon(u_1-u_2)_{xx}+\lambda_0(\alpha H(u_1-u_2)_{xx}-\beta H(u_1-u_2)_x$$

$$+\gamma(x,t)H(u_1-u_2))+b(x,t)(u_1-u_2)_x+c(x,t)(u_1-u_2)$$
$$=(\lambda_0-\lambda)(\alpha H(v_1-v_2)_{xx}-\beta H(v_1-v_2)_x+\gamma(x,t)H(v_1-v_2))$$

and the homogeneous initial condition

$$u_1(x,0)-u_2(x,0)=0$$

From the estimation formula (12), we have

$$\begin{aligned}\|u_1-u_2\|_{W_2^{(2,1)}(Q_T^*)}&\le C_3|\lambda_0-\lambda|\|v_1-v_2\|_{W_2^{(2,0)}(Q_T)}\\&\le C_3|\lambda_0-\lambda|\|v_1-v_2\|_{W_2^{(2,1)}(Q_T^*)}\end{aligned}$$

For the sufficiently small $|\lambda_0-\lambda|$, the defined $A_\lambda: W_2^{(2,1)}(Q_T)\to W_2^{(2,1)}(Q_T)$ is a contraction mapping. Hence there is a unique solution $u(x,t)\in W_2^{(2,1)}(Q_T)$, such that $u=A_\lambda u$. This means $\lambda\in E$. Then E is open in $[0,1]$.

Therefore $E\equiv[0,1]$, i.e., for any $\lambda\in[0,1]$, then for $\lambda=1$, the problem (7) and (9) has a unique generalized global solution.

Theorem 2 *Suppose that $\epsilon>0$, β and α are constants and suppose that $b(x,t)$, $c(x,t),\gamma(x,t)\in L_\infty(Q_T)$; $f(x,t)\in L_2(Q_T)$ and $\phi(x)\in H^1(R)$. The initial value problem (7) of the linear parabolic equation (9) has a unique generalized global solution $u(x,t)\in W_2^{(2,1)}(Q_T)$.*

Corollary *Under the conditions of Theorem 2, the generalized global solution $u(x,t)$ of the initial value problem (7) and (9) has the estimation*

$$\|u\|_{W_2^{(2,1)}(Q_T)}\le K_1\{\|\phi\|_{H^1(R)}+\|f\|_{L_2(Q_T)}\} \tag{13$_0$}$$

where K_1 is a constant dependent on the constants α and β and the norms $\|b\|_{L_\infty(Q_T)}$, $\|c\|_{L_\infty(Q_T)}$, $\|\gamma\|_{L_\infty(Q_T)}$ and the diffusion coefficient $\epsilon>0$.

Corollary *Suppose that $b(x,t),c(x,t),\gamma(x,t)\in W_\infty^{(k,[\frac{k}{2}])}(Q_T)$, $f(x,t)\in W_2^{(k,[\frac{k}{2}])}(Q_T)$ and $\phi(x)\in H^{k+1}(R)$ for $k\ge1$ integer. Then the unique global solution $u(x,t)$ of the initial value problem (7) and (9) belongs to the space $W_2^{(k+2,[\frac{k}{2}]+1)}(Q_T)$.*

3. Now we turn to prove the existence of the generalized global solution for the initial value problem (7) of the nonlinear parabolic equation (6) with singular integral operator.

Theorem 3 *Suppose that β and α are constants and assume that $b(x,t),c(x,t),\gamma(x,t)\in L_\infty(Q_T)$, $f(x,t)\in L_2(Q_T)$ and $\phi(x)\in H^1(R)$. The initial value problem (7) of the nonlinear parabolic equation (6) with the Hilbert operators has a generalized global solution $u(x,t)\in W_2^{(2,1)}(Q_T)$, which satisfies the equation (6) in generalized sence and the initial condition (7) in classical sence. Furthermore there is the estimation*

$$\sup_{0\le t\le T}\|u(\cdot,t)\|_{H^1(R)}+\|u_{xx}\|_{L_2(Q_T)}+\|u_t\|_{L_2(Q_T)}\le K_2\{\|\phi\|_{H^1(R)}+\|f\|_{L_2(Q_T)}\} \tag{14}$$

where K_2 is a constant dependent on the constants α, β and the norms of the coefficients $b(x,t), c(x,t)$ and $\gamma(x,t)$, the diffusion coefficient $\epsilon > 0$ and $T > 0$.

Proof We want to prove the existence of the generalized global solution for the present problem by the fixed-point technique.

We define a mapping $T_\lambda : B \to B$ of the functional space $B = L_\infty(Q_T)$ into itself with a parameter $0 \le \lambda \le 1$ as follows: For any $v(x,t) \in B$, let $u(x,t)$ be the unique generalized global solution of the linear parabolic equation

$$u_t - \epsilon u_{xx} + \alpha H u_{xx} - \beta H u_x + \gamma H u + 2v u_x + b u_x + c u = f \tag{15}$$

with the initial condition

$$u(x,0) = \lambda \phi(x) \tag{16}$$

When $v(x,t) \in B$, the obtained function $u(x,t) \in W_2^{(2,1)}(Q_T)$. We can prove that the mapping $T_\lambda : B \to B$ defined by $u = T_\lambda v$ for $v \in B$ is completely continuous for any $0 \le \lambda \le 1$.

As $\lambda = 0$, $T_0(B) = 0$.

In order to justify the existence of the generalized global solution of the original problem (6) and (7), it is sufficient to prove the uniform boundedness in the base space B of all possible fixed point of the mapping $T_\lambda : B \to B$ with respect to the parameter $0 \le \lambda \le 1$, i.e., it needs to give *a priori* estimations of the solutions $u_\lambda(x,t)$ for the initial problem (16) for the nonlinear parabolic equation

$$u_t - \epsilon u_{xx} + \alpha H u_{xx} - \beta H u_x + \gamma H u + 2u u_x + b u_x + c u = \lambda f \tag{17}$$

with respect to the parameter $0 \le \lambda \le 1$.

Taking the scalar product of the function $u(x,t)$ and the equation (17) in Hilbert space, we get

$$\int_{-\infty}^{\infty} u(u_t - \epsilon u_{xx} + \alpha H u_{xx} - \beta H u_x + \gamma H u + 2u u_x + b u_x + c u - \lambda f) dx = 0$$

By simple calculations as before, this can be replaced by the inequality

$$\frac{d}{dt}\|u(\cdot,t)\|^2_{L_2(R)} + \|u_x(\cdot,t)\|^2_{L_2(R)} \le C_4\{\|u(\cdot,t)\|^2_{L_2(R)} + \|f(\cdot,t)\|^2_{L_2(R)}\}$$

Hence we have the estimation

$$\sup_{0\le t\le T} \|u(\cdot,t)\|_{L_2(R)} + \|u_x\|_{L_2(Q_T)} \le C_5\{\|\phi\|_{L_2(R)} + \|f\|_{L_2(Q_T)}\}$$

where C_4 and C_5 are dependent on $\|b\|_{L_\infty(Q_T)}$, $\|c\|_{L_\infty(Q_T)}$, $\|\gamma\|_{L_\infty(Q_T)}$ and $\epsilon > 0$, but are independent of $0 \le \lambda \le 1$.

Again multiplying the equation (17) by u_{xx} and then integrating the resulting product with respect to $x \in R$, we obtain

$$\int_{-\infty}^{\infty} u_{xx}(u_t - \epsilon u_{xx} + \alpha H u_{xx} - \beta H u_x + \gamma H u + 2u u_x + b u_x + c u - \lambda f) dx = 0$$

Similarly, we get the estimation

$$\sup_{0\le t\le T}\|u_x(\cdot,t)\|_{L_2(R)}+\|u_{xx}\|_{L_2(Q_T)}\le C_6\{\|\phi\|_{H^1(R)}+\|f\|_{L_2(Q_T)}\}$$

where C_6 is independent of $0\le\lambda\le 1$.

This shows that all possible solutions of initial value problem (17) and (16) are uniformly bounded in space $L_\infty(0,T;H^1(R))$ hence in space B with respect to $0\le\lambda\le 1$.

Therefore the existence of generalized global solution $u(x,t)\in W_2^{(2,1)}(Q_T)$ for the initial value problem (7) for the nonlinear parabolic equation (6) is proved. Hence the theorem is proved.

Suppose that there are two generalized global solutions $u(x,t)$ and $v(x,t)$ in $L_\infty(0,T;H^2(R))$ for the initial value problem (7) for the nonlinear parabolic equation (6). The difference function $w(x,t)=u(x,t)-v(x,t)$ satisfies the homogeneous linear equation

$$w_t-\epsilon w_{xx}+\alpha Hw_{xx}-\beta Hw_x+\gamma Hw+(b+u+v)w_x+(c+u_x+v_x)w=0$$

in generalized sense and the homogeneous initial condition

$$w(x,0)=0$$

From the estimation formulas of the generalized global solution, it gives $w(x,t)=0$, where for the coefficients of w_x and w terms are bounded, since $u,v\in L_\infty(0,T;H^2(R))$.

Hence the generalized global solution $u(x,t)\in L_\infty(0,T;H^2(R))$ of the problem (6) and (7) is unique.

3. *A Priori* Estimations

1. In order to obtain the global solution of the initial value problem (7) for the nonlinear singular integral-differential equation (5) by the limiting process as the coefficient $\epsilon>0$ of the additional diffusion term tends to zero, we must derive a series of *a priori* uniform estimations for the solutions of the initial value problem (7) of the nonlinear parabolic equation (6) containing the Hilbert operator with respect to the coefficient $\epsilon>0$.

Lemma 3 *Suppose that $\epsilon>0$, $\beta\ge 0$ and α are constants and suppose that $b(x,t),b_x(x,t),c(x,t),\gamma(x,t)\in L_\infty(Q_T)$; $f(x,t)\in L_2(Q_T)$ and $\phi(x)\in L_2(R)$. The generalized global solutions $u_\epsilon\in W_2^{(2,1)}(Q_T)$ of the initial value problem (7) for the nonlinear parabolic equation (6) with Hilbert operators have the estimation*

$$\sup_{0\le t\le T}\|u_\epsilon(\cdot,t)\|^2_{L_2(R)}+\beta\|u_{\epsilon x}Hu_\epsilon\|_{L_1(Q_T)}\le K_3\{\|\phi\|^2_{L_2(R)}+\|f\|^2_{L_2(Q_T)}\}\qquad(18)$$

where K_2 is a constant independent of $\epsilon>0$, but dependent on the norms $\|b_x\|_{L_\infty(Q_T)}$, $\|c\|_{L_\infty(Q_T)}$ and $\|\gamma\|_{L_\infty(Q_T)}$.

Proof In the integral equality

$$\int_{-\infty}^{\infty} u(u_t + 2uu_x + \alpha H u_{xx} - \beta H u_x + \gamma H u - \epsilon u_{xx} + bu_x + cu - f)dx = 0$$

Regarding

$$\int_{-\infty}^{\infty} buu_x dx = -\frac{1}{2}\int_{-\infty}^{\infty} b_x u^2 dx$$

we have

$$\frac{d}{dt}\|u(\cdot,t)\|^2_{L_2(R)} + 2\epsilon\|u_x(\cdot,t)\|^2_{L_2(R)} + 2\beta\int_{-\infty}^{\infty} u_x H u dx$$
$$\leq \{\|b_x\|^2_{L_\infty(Q_T)} + 2\|c\|_{L^\infty(Q_T)} + 2\|\gamma\|_{L_\infty(Q_T)} + 1\}\|u(\cdot,t)\|^2_{L_2(R)} + \|f(\cdot,t)\|^2_{L_2(R)}$$

Hence the lemma is proved.

2. Now we turn to estimate $\|u_{\epsilon x}(\cdot,t)\|_{L_2(R)}$.

By the direct calculations and by use of the behaviors of the Hilbert operator, we can obtain a series of identities as follows.

At first,

$$\frac{d}{dt}\int_{-\infty}^{\infty} u_x^2 dx = -2\int_{-\infty}^{\infty} u_{xx}u_t dx$$
$$= 2\int_{-\infty}^{\infty} u_{xx}[2uu_x + \alpha H u_{xx} - \beta H u_x + \gamma H u - \epsilon u_{xx} + bu_x + cu - f]dx = 0$$

Here we have

$$\int_{-\infty}^{\infty} u_{xx} H u_{xx} dx = 0, \quad \int_{-\infty}^{\infty} u_{xx} H u_x dx \geq 0$$
$$\int_{-\infty}^{\infty} bu_x u_{xx} dx = -\frac{1}{2}\int_{-\infty}^{\infty} b_x u_x^2 dx$$
$$\int_{-\infty}^{\infty} \gamma u_{xx} H u dx = -\int_{-\infty}^{\infty} (\gamma u_x H u_x + \gamma_x u_x H u)dx$$
$$\int_{-\infty}^{\infty} cuu_{xx} dx = -\int_{-\infty}^{\infty} (cu_x^2 + c_x uu_x)dx$$
$$\int_{-\infty}^{\infty} u_{xx} f dx = -\int_{-\infty}^{\infty} u_x f_x dx$$

Then we obtain the identity

$$\frac{d}{dt}\int_{-\infty}^{\infty} u_x^2 dx = 4\int_{-\infty}^{\infty} uu_x u_{xx} dx - 2\epsilon\|u_{xx}\|_{L_2(R)} - 2\beta\int_{-\infty}^{\infty} u_{xx} H u_x dx$$
$$- \int_{-\infty}^{\infty} (b_x + 2c)u_x^2 dx - 2\int_{-\infty}^{\infty} c_x uu_x dx + 2\int_{-\infty}^{\infty} u_x f_x dx$$
$$- 2\int_{-\infty}^{\infty} \gamma u_x H u_x dx - 2\int_{-\infty}^{\infty} \gamma_x u_x H u dx \quad (19)$$

Secondly let us calculate

$$\frac{d}{dt}\int_{-\infty}^{\infty}u^2Hu_xdx=\int_{-\infty}^{\infty}(2uu_tHu_x+u^2Hu_{xt})dx$$
$$=2\int_{-\infty}^{\infty}[uHu_x+H(uu_x)]u_tdx$$
$$=-2\int_{-\infty}^{\infty}[uHu_x+H(uu_x)][2uu_x+\alpha Hu_{xx}-\beta Hu_x+\gamma Hu-\epsilon u_{xx}+bu_x+cu-f]dx$$

From the fundamental property ⟨3⟩ of the Hilbert transform, we have evidently

$$\int_{-\infty}^{\infty}uu_xH(uu_x)dx=0$$

and from Lemma 1, we have

$$\int_{-\infty}^{\infty}H(uu_x)Hu_{xx}dx=\int_{-\infty}^{\infty}uu_xu_{xx}dx$$

Here we also have

$$\int_{-\infty}^{\infty}uHu_xHu_{xx}dx=\frac{1}{2}\int_{-\infty}^{\infty}u[(Hu_x)^2]_xdx=-\frac{1}{2}\int_{-\infty}^{\infty}u_x(Hu_x)^2dx$$
$$=\frac{1}{2}\int_{-\infty}^{\infty}u_xH(u_xHu_x)dx$$

From the property ⟨2⟩ of the Hilbert operator,

$$H(fHf)=H(HfH^2f)+(Hf)^2+fH^2f$$

This shows

$$H(fHf)=\frac{1}{2}(Hf)^2-\frac{1}{2}f^2$$

Then

$$\int_{-\infty}^{\infty}u_xH(u_xHu_x)dx=\frac{1}{2}\int_{-\infty}^{\infty}u_x(Hu_x)^2dx-\frac{1}{2}\int_{-\infty}^{\infty}u_x^3dx$$
$$=-\int_{-\infty}^{\infty}uHu_xHu_{xx}dx+\int_{-\infty}^{\infty}uu_xu_{xx}dx$$

Hence we get

$$\int_{-\infty}^{\infty}uHu_xHu_{xx}dx=\frac{1}{3}\int_{-\infty}^{\infty}uu_xu_{xx}dx$$

Therefore we obtain the identity

$$\frac{d}{dt}\int_{-\infty}^{\infty}u^2Hu_xdx=-4\int_{-\infty}^{\infty}u^2u_xHu_xdx-\frac{8}{3}\alpha\int_{-\infty}^{\infty}uu_xu_{xx}dx-2\int_{-\infty}^{\infty}buu_xHu_xdx$$
$$-2\int_{-\infty}^{\infty}bu_xH(uu_x)dx+2\beta\int_{-\infty}^{\infty}u(Hu_x)^2dx+2\epsilon\int_{-\infty}^{\infty}(uHu_x+H(uu_x))u_{xx}dx$$

$$+2\beta\int_{-\infty}^{\infty} uu_x^2dx - 2\int_{-\infty}^{\infty}(uHu_x + H(uu_x))(\gamma Hu + cu - f)dx \tag{20}$$

Thirdly, we have

$$\begin{aligned}\frac{d}{dt}\int_{-\infty}^{\infty} buHu_xdx &= \int_{-\infty}^{\infty} b_tuHu_xdx + \int_{-\infty}^{\infty}(bu_tHu_x + buHu_{xt})dx\\ &= \int_{-\infty}^{\infty} b_tuHu_xdx - \int_{-\infty}^{\infty}(bHu_x + H(bu)_x)[2uu_x\\ &\quad + \alpha Hu_{xx} - \beta Hu_x + \gamma Hu - \epsilon u_{xx} + bu_x + cu - f]dx\end{aligned}$$

Here we have

$$\int_{-\infty}^{\infty} uu_xH(bu)_xdx = -\int_{-\infty}^{\infty}(b_xu + bu_x)H(uu_x)dx$$

$$\int_{-\infty}^{\infty} bHu_xHu_{xx} = -\frac{1}{2}\int_{-\infty}^{\infty} b_x(Hu_x)^2dx$$

$$\int_{-\infty}^{\infty} H(bu)_xHu_{xx}dx = \int_{-\infty}^{\infty}(bu)_xu_{xx}dx = -\int_{-\infty}^{\infty}\left(b_{xx}uu_x + \frac{3}{2}b_xu_x^2\right)dx$$

Then we get

$$\begin{aligned}\frac{d}{dt}\int_{-\infty}^{\infty} buHu_xdx &= -2\int_{-\infty}^{\infty} buu_xHu_xdx + 2\int_{-\infty}^{\infty} bu_xH(uu_x)dx\\ &+ \int_{-\infty}^{\infty} b_tuHu_xdx + 2\int_{-\infty}^{\infty} b_xuH(uu_x)dx\\ &+ \frac{\alpha}{2}\int_{-\infty}^{\infty} b_x(Hu_x)^2dx + \alpha\int_{-\infty}^{\infty} b_{xx}uu_xdx + \frac{3}{2}\alpha\int_{-\infty}^{\infty} b_xu_x^2dx\\ &+ \epsilon\int_{-\infty}^{\infty}(bHu_x + H(bu)_x)u_{xx}dx\\ &- \int_{-\infty}^{\infty}(bHu_x + H(bu)_x)(-\beta Hu_x + \gamma Hu + bu_x + cu - f)dx\end{aligned} \tag{21}$$

Similarly we can derive other two more identities as follows:

$$\begin{aligned}\frac{d}{dt}\int_{-\infty}^{\infty} u^4dx =& 12\alpha\int_{-\infty}^{\infty} u^2u_xHu_xdx + 4\epsilon\int_{-\infty}^{\infty} u^3u_{xx}dx\\ &- 4\int_{-\infty}^{\infty} u^3(-\beta Hu_x + \gamma Hu + bu_x + cu - f)dx\end{aligned} \tag{22}$$

and

$$\begin{aligned}\frac{d}{dt}\int_{-\infty}^{\infty} bu^3dx &= 6\alpha\int_{-\infty}^{\infty} buu_xHu_xdx\\ &+ \int_{-\infty}^{\infty} b_tu^3dx - 6\int_{-\infty}^{\infty} bu^3u_xdx + 3\alpha\int_{-\infty}^{\infty} b_xu^2Hu_xdx\\ &+ 3\epsilon\int_{-\infty}^{\infty} bu^2u_{xx}dx - 3\int_{-\infty}^{\infty} bu^2(-\beta Hu_x + \gamma Hu + bu_x + cu - f)dx\end{aligned} \tag{23}$$

In the derivations of these identities, we must notice that all the derivatives of the global solution $u_\epsilon(x,t)$ tend to zero as $|x| \to \infty$ and we have to use the fundamental properties of Hilbert transform repeatedly in the calculations.

Now making the linear combination of the five identities (19)–(23), in order to eliminate the four terms

$$\int_{-\infty}^{\infty} uu_x u_{xx} dx, \quad \int_{-\infty}^{\infty} u^2 u_x Hu_x dx, \quad \int_{-\infty}^{\infty} buu_x Hu_x dx, \quad \int_{-\infty}^{\infty} bu_x H(uu_x) dx$$

in the right parts of identities (19)–(23), we obtain the final equality:

$$\begin{aligned}
&\frac{d}{dt}\int_{-\infty}^{\infty}\Big[2\alpha u_x^2 + 3u^2 Hu_x + 3buHu_x + \frac{1}{\alpha}u^4 + \frac{2}{\alpha}bu^3\Big]dx + 4\alpha\epsilon\|u_{xx}(\cdot,t)\|^2_{L_2(R)}\\
&= -4\alpha\beta\int_{-\infty}^{\infty} u_{xx}Hu_x dx + 6\beta\int_{-\infty}^{\infty} u[(Hu_x)^2 + u_x^2]dx\\
&+\epsilon\int_{-\infty}^{\infty}\Big[b(uHu_x + H(uu_x)) + 3(bHu_x + H(bu)_x) + \frac{4}{\alpha}u^3 + \frac{6}{\alpha}bu^2\Big]u_{xx}dx\\
&+\int_{-\infty}^{\infty}\Big[\Big(\frac{5}{2}b_x + 3\beta b - 4\alpha c\Big)u_x^2 + \Big(\frac{3}{2}\alpha b_x + 3\beta b\Big)(Hu_x)^2 - (3b^2 + 4\alpha\gamma)u_x Hu_x\Big]dx\\
&+\int_{-\infty}^{\infty}\Big[-16bu^3u_x + \frac{4\beta}{\alpha}u^3 Hu_x\Big]dx + 6\int_{-\infty}^{\infty}\Big[\Big(b_x + \frac{\beta}{\alpha}b - c\Big)u^2 Hu_x\\
&+ (b_x - c)uH(uu_x) - \frac{b^2}{\alpha}u^2u_x - \gamma uHuu_x - \gamma HuH(uu_x)\Big]dx\\
&+3\int_{-\infty}^{\infty}\Big[\Big(\alpha b_{xx} + \beta b_x - \frac{4\alpha}{3}c_x\Big)uu_x + (-bc + b_t)uHu_x - bu_xH(b_xu)\\
&- cuH(bu_x) - b\gamma HuHu_x - \frac{4}{3}\gamma_x Huu_x - \gamma HuH(bu_x)\Big]dx\\
&-\frac{4}{\alpha}\int_{-\infty}^{\infty}(cu^4 + \gamma u^3 Hu)dx + \int_{-\infty}^{\infty}\Big[\frac{1}{\alpha}(2b_t - 6bc)u^3 - \frac{6b\gamma}{\alpha}u^2 Hu\Big]dx\\
&+3\int_{-\infty}^{\infty}[cuH(b_xu) + \gamma HuH(b_xu)]dx + 6\int_{-\infty}^{\infty} f[uHu_x + H(uu_x)]dx\\
&+\int_{-\infty}^{\infty}[f(3bHu_x + 3H(bu_x)) + 4f_x\alpha u_x]dx + \frac{4}{\alpha}\int_{-\infty}^{\infty} fu^3 dx\\
&+6\int_{-\infty}^{\infty} fbu^2 dx + 3\int_{-\infty}^{\infty} fH(b_xu)dx \qquad (24)
\end{aligned}$$

Denote by J_k $(k = 1, 2, \cdots, 15)$ the k-th integral term on the right hand side of the above equality (24).

3. Let us now suppose that the coefficients $b(x,t)$, $c(x,t)$ and $\gamma(x,t)$, the free term $f(x,t)$ and the initial function $\phi(x)$ satisfy the following assumptions:

(1) $f(x,t) \in W_2^{(1,0)}(Q_T)$,

(2) $c(x,t), \gamma(x,t) \in W_\infty^{(1,0)}(Q_T)$,

(3) $b(x,t) \in W_\infty^{(2,1)}(Q_T)$,

(4) $\phi \in H^1(R)$.

And also assume that $\alpha > 0$ and $\beta \geq 0$.

Now we turn to simplify the equality (24).

At first let us consider the integral J_3 of the right hand side of (24) with coefficient $\epsilon > 0$. Denote by $\widetilde{J}(x,t)$ the expression in the curved parenthesis of this integral. Then

$$\Big|\int_{-\infty}^{\infty} \widetilde{J} u_{xx} dx\Big| \leq \frac{\alpha}{2}\|u_{xx}(\cdot,t)\|^2_{L_2(R)} + \frac{1}{2\alpha}\|\widetilde{J}(\cdot,t)\|^2_{L_2(R)} \tag{25}$$

where

$$\begin{aligned}\|\widetilde{J}(\cdot,t)\|^2_{L_2(R)} \leq & C_7\int_{-\infty}^{\infty}\{u^2(Hu_x)^2 + (H(uu_x))^2 + b^2(Hu_x)^2 \\ & + (H(bu_x))^2 + u^6 + b^2u^4\}dx\end{aligned} \tag{26}$$

Here we have

$$\begin{aligned}&\int_{-\infty}^{\infty} u^2(Hu_x)^2 dx \leq \|u(\cdot,t)\|^2_{L_\infty(R)}\|Hu_x(\cdot,t)\|^2_{L_2(R)} \\ &\quad = \|u(\cdot,t)\|^2_{L_\infty(R)}\|u_x(\cdot,t)\|^2_{L_2(R)} \leq \eta\|u_{xx}(\cdot,t)\|^2_{L_2(R)} + C_8(\eta)\|u(\cdot,t)\|^{10}_{L_2(R)} \\ &\int_{-\infty}^{\infty}(H(uu_x))^2 dx = \int_{-\infty}^{\infty} u^2u_x^2 dx \leq \eta\|u_{xx}(\cdot,t)\|^2_{L_2(R)} + C_8(\eta)\|u(\cdot,t)\|^{10}_{L_2(R)} \\ &\int_{-\infty}^{\infty} b^2(Hu_x)^2 dx \leq \|b\|^2_{L_\infty(Q_T)}\|u_x(\cdot,t)\|^2_{L_2(R)} \\ &\quad \leq \eta\|u_{xx}(\cdot,t)\|^2_{L_2(R)} + C_9(\eta)\|b\|^4_{L_\infty(Q_T)}\|u(\cdot,t)\|^2_{L_2(R)} \\ &\int_{-\infty}^{\infty}(H(bu)_x)^2 dx \leq \eta\|u_{xx}(\cdot,t)\|^2_{L_2(R)} \\ &\quad + C_{10}(\eta)(\|b_x\|^2_{L_\infty(Q_T)} + \|b\|^4_{L_\infty(Q_T^*)})\|u(\cdot,t)\|^2_{L_2(R)} \\ &\int_{-\infty}^{\infty} u^6 dx = \|u(\cdot,t)\|^6_{L_6(R)} \leq \eta\|u_{xx}(\cdot,t)\|^2_{L_2(R)} + C_{11}(\eta)\|u(\cdot,t)\|^{10}_{L_2(R)} \\ &\int_{-\infty}^{\infty} b^2u^4 dx = \eta\|u_{xx}(\cdot,t)\|^2_{L_2(R)} + C_{12}(\eta)\|b\|^{\frac{8}{3}}_{L_\infty(Q_T^*)}\|u(\cdot,t)\|^{\frac{14}{3}}_{L_2(R)}\end{aligned}$$

Substituting these estimations into (26), we get

$$\|\widetilde{J}(\cdot,t)\|^2_{L_2(R)} \leq bC_7\eta\|u_{xx}(\cdot,t)\|^2_{L_2(R)} + C_{13}(\eta)\|u(\cdot,t)\|^2_{L_2(R)}$$

where $C_{13}(\eta)$ depends on the norms $\|b\|_{W^{(1,0)}_\infty(Q_T)}$ and $\sup\limits_{0\leq t\leq T}\|u(\cdot,t)\|_{L_2(R)}$. Let us take η so small that $6C_7\eta = \alpha$. Hence

$$\|\widetilde{J}(\cdot,t)\|^2_{L_2(R)} \leq \alpha^2\|u_{xx}(\cdot,t)\|^2_{L_2(R)} + C_{14}\|u(\cdot,t)\|^2_{L_2(R)}$$

and that

$$\Big|\int_{-\infty}^{\infty} \widetilde{J} u_{xx} dx\Big| \leq \alpha\|u_{xx}(\cdot,t)\|^2_{L_2(R)} + \frac{1}{2\alpha}C_{14}\|u(\cdot,t)\|^2_{L_2(R)} \tag{27}$$

where C_{14} is a constant dependent on the norms $\|b\|_{W^{(1,0)}_\infty(Q_T^*)}$ and $\sup\limits_{0\leq t\leq T}\|u(\cdot,t)\|_{L_2(R)}$.

Using the interpolation formulas, we can estimate some of the remaining terms of (24) as follows:

$$
\begin{aligned}
|J_4| \le & \Big(\frac{1}{2}(5+3\alpha)\|b_x\|_{L_\infty(Q_T)} + 6\beta\|b\|_{L_\infty(Q_T)} + 4\alpha\|c\|_{L_\infty(Q_T)} \\
& + 4\alpha\|\gamma\|_{L_\infty(Q_T)} + 3\|b\|^2_{L_\infty(Q_T)}\Big)\|u_x(\cdot,t)\|^2_{L_2(R)} \\
|J_5| \le & C_{15}\Big(\|b\|_{L_\infty(Q_T)} + \frac{\beta}{\alpha}\Big)\|u(\cdot,t)\|^2_{L_2(R)}\|u_x(\cdot,t)\|^2_{L_2(R)} \\
|J_6| \le & C_{16}\Big(\|b_x\|_{L_\infty(Q_T)} + \frac{\beta}{\alpha}\|b\|_{L_\infty(Q_T)} + \|c\|_{L_\infty(Q_T)} + \|\gamma\|_{L_\infty(Q_T)} \\
& + \frac{1}{\alpha}\|b\|^2_{L_\infty(Q_T)}\Big)(\|u_x(\cdot,t)\|^2_{L_2(R)} + \|u(\cdot,t)\|^6_{L_2(R)}) \\
|J_7| \le & 3\Big[(\alpha+\beta+1)\|b\|_{W^{(2,1)}_\infty(Q_T)} + \|b\|_{L_\infty(Q_T)}(2\|c\|_{L_\infty(Q_T)} \\
& + 2\|\gamma\|_{L_\infty(Q_T)} + \|b_x\|_{L_\infty(Q_T)}) + \frac{4\alpha}{3}\|c_x\|_{L_\infty(Q_T)} + \|\gamma_x\|_{L_\infty(Q_T)}\Big](\|u_x(\cdot,t)\|^2_{L_2(R)} \\
& + \|u(\cdot,t)\|^2_{L_2(R)}) \\
|J_8| \le & \frac{4}{\alpha}(\|c\|_{L_\infty(Q_T)} + \|\gamma\|_{L_\infty(Q_T)})(\|u_x(\cdot,t)\|^2_{L_2(R)} + C_{17}\|u(\cdot,t)\|^6_{L_2(R)}) \\
|J_9| \le & \frac{C_{18}}{\alpha}(\|b_t\|_{L_\infty(Q_T)} + \|b\|_{L_\infty(Q_T)}\|c\|_{L_\infty(Q_T)} + \|b\|_{L_\infty(Q_T)}\|\gamma\|_{L_\infty(Q_T)}) \\
& \cdot(\|u_x(\cdot,t)\|^2_{L_2(R)} + \|u(\cdot,t)\|^{\frac{10}{3}}_{L_2(R)}) \\
|J_{10}| \le & 3\|b_x\|_{L_\infty(Q_T)}(\|c\|_{L_\infty(Q_T)} + \|\gamma\|_{L_\infty(Q_T)})\|u(\cdot,t)\|^2_{L^2(R)}
\end{aligned}
$$

where C's are independent of $\epsilon > 0$. For the terms containing the factors $f(x,t)$ or $f_x(x,t)$, we have

$$
\begin{aligned}
|J_{12}| &\le \Big(\frac{9}{2}\|b\|_{L_\infty(Q_T)} + 2\Big)\|u_x(\cdot,t)\|^2_{L_2(R)} + 2\|f(\cdot,t)\|^2_{H^1(R)} \\
|J_{13}| &\le \Big(\frac{C_{19}}{\alpha}\|u(\cdot,t)\|^4_{L_2(R)}\|u_x(\cdot,t)\|^2_{L_2(R)} + \frac{1}{2}\|f(\cdot,t)\|^2_{L_2(R)} \\
|J_{14}| &\le 3\|b\|^2_{L_\infty(Q_T)}(\|u_x(\cdot,t)\|^2_{L_2(R)} + \|u(\cdot,t)\|^6_{L_2(R)}) + 3\|f(\cdot,t)\|^2_{L_2(R)} \\
|J_{15}| &\le \frac{3}{2}\|b_x\|^2_{L_\infty(Q_T)}\|u(\cdot,t)\|^2_{L_2(R)} + \frac{3}{2}\|f(\cdot,t)\|^2_{L_2(R)}
\end{aligned}
$$

and furthermore for J_{11} we have

$$
\begin{aligned}
\Big|\int_{-\infty}^{\infty} fuHu_x dx\Big| &\le \|u(\cdot,t)\|_{L_2(R)}\|f(\cdot,t)\|_{L_\infty(Q_T)}\|Hu_x(\cdot,t)\|_{L_2(R)} \\
&\le \frac{1}{2}\|u_x(\cdot,t)\|^2_{L_2(R)} + \frac{1}{2}\|f(\cdot,t)\|^2_{L_\infty(Q_T)}\|u(\cdot,t)\|^2_{L_2(R)} \\
&\le \frac{1}{2}\|u_x(\cdot,t)\|^2_{L_2(R)} + C_{20}\|f(\cdot,t)\|^2_{H^1(R)}
\end{aligned}
$$

and

$$
\Big|\int_{-\infty}^{\infty} fH(uu_x)dx\Big| \le \|Hf(\cdot,t)\|_{L_\infty(Q_T)}\|u(\cdot,t)\|_{L_2(R)}\|u_x(\cdot,t)\|_{L_2(R)}
$$

$$\leq \frac{1}{2}\|u_x(\cdot,t)\|^2_{L_2(R)} + C_{20}\|u(\cdot,t)\|^2_{L_2(R)}\|Hf(\cdot,t)\|_{L_2(R)}\|Hf_x(\cdot,t)\|_{L_2(R)}$$
$$\leq \frac{1}{2}\|u_x(\cdot,t)\|^2_{L_2(R)} + C_{21}\|f(\cdot,t)\|^2_{H^1(R)}$$

where C_{21} depends on $\sup\limits_{0\leq t\leq T}\|u(\cdot,t)\|_{L_2(R)}$.

For J_2, by using Lemma 2, we have

$$|J_2| \leq 12\beta\|u(\cdot,t)\|_{L_\infty(Q_T)}\|u_x(\cdot,t)\|^2_{L_2(R)}$$
$$\leq 12\beta\|u(\cdot,t)\|_{L_\infty(Q_T)}\|u_{xx}(\cdot,t)Hu_x(\cdot,t)\|^{\frac{1}{2}}_{L_1(R)}\|u_x(\cdot,t)Hu(\cdot,t)\|^{\frac{1}{2}}_{L_1(R)}$$
$$\leq 2\alpha\beta\|u_{xx}(\cdot,t)Hu_x(\cdot,t)\|_{L_1(R)} + 18\frac{\beta}{\alpha}\|u(\cdot,t)\|^2_{L_\infty(Q_T)}\|u_x(\cdot,t)Hu(\cdot,t)\|_{L_1(R)}$$

Hence

$$J_1 + J_2 \leq -2\alpha\beta\|u_{xx}(\cdot,t)Hu_x(\cdot,t)\|_{L_1(R)} + C_{22}\frac{\beta}{\alpha}\|u(\cdot,t)\|^2_{L_2(R)}\|u_x(\cdot,t)\|^2_{L_2(R)}$$

Let us substitute all these estimations into right hand side of the inequality (24). Then the equality (24) is simplified and can be replaced by the inequality

$$\frac{d}{dt}\int_{-\infty}^{\infty}\Big(\frac{1}{\alpha}u^4 + \frac{2b}{\alpha}u^3 + 3buHu_x + 3u^2Hu_x + 2\alpha u_x^2\Big)dx + 3\alpha\epsilon\|u_{xx}\|^2_{L_2(R)}$$
$$+ 4\alpha\beta\|u_{xx}(\cdot,t)Hu_x(\cdot,t)\|_{L_1(R)} \leq C_{23}(\|u(\cdot,t)\|^2_{H^1(R)} + \|f(\cdot,t)\|^2_{H^1(R)}) \tag{28}$$

where C_{23} is a constant dependent on $\alpha > 0$, $\beta \geq 0$ and the norms $\|b\|_{W^{(2,1)}_\infty(Q_T)}$, $\|c\|_{W^{(1,0)}_\infty(Q_T)}$, $\|\gamma\|_{W^{(1,0)}_\infty(Q_T)}$ and $\sup\limits_{0\leq t\leq T}\|u(\cdot,t)\|_{L_2(R)}$, but independent of the diffusion coefficient $\epsilon > 0$.

Integrating the both sides of the inequality (28) with respect to the time variable in the interval $[0,t]$, then we have

$$\int_{-\infty}^{\infty}\Big(\frac{1}{\alpha}u^4 + \frac{2b}{\alpha}u^3 + 3buHu_x + 3u^2Hu_x + 2\alpha u_x^2\Big)dx$$
$$\leq C_{23}\int_0^t\|u_x(\cdot,\tau)\|^2_{L_2(R)}d\tau + C_{24}\{\|\phi\|^2_{L_2(R)} + \|f\|^2_{W_2^{(1,0)}(Q_T)}\}$$
$$+ \int_{-\infty}^{\infty}\Big(\frac{1}{\alpha}\phi^4 + \frac{2}{\alpha}b(x,0)\phi^3 + 3b(x,0)\phi H\phi_x + 3\phi^2H\phi_x + 2\alpha\phi_x^2\Big)dx \tag{29}$$

where C_{24} is independent of $\epsilon > 0$.

For the last integral, we have

$$\Big|\int_{-\infty}^{\infty}\Big(\frac{1}{\alpha}\phi^4 + \frac{2}{\alpha}b(x,0)\phi^3 + 3b(x,0)\phi H\phi_x + 3\phi^2H\phi_x + 2\alpha\phi_x^2\Big)dx\Big|$$
$$\leq C_{25}(\|b(x,0)\|_{L_\infty(Q_T)} + 1)\|\phi_x\|^2_{L_2(R)} + C_{25}(\|\phi\|^4_{L_2(R)} + \|\phi\|^{\frac{4}{3}}_{L_4(R)} + 1)\|\phi\|^2_{L_2(R)}$$

or simply that the last integral can be dominated by $C_{26}\|\phi\|^2_{H^1(R)}$, where C_{26} depends on α, $\|b(\cdot,0)\|_{L_\infty(R)}$ and $\|\phi\|_{L_2(R)}$.

By similar method of estimation, the first integral of the inequality (29) can be dominated below by

$$\alpha\|u_x(\cdot,t)\|^2_{L_2(R)} - C_{27}\|u(\cdot,t)\|^2_{L_2(R)}$$

where C_{27} is a constant dependent on α and the norms $\|b\|_{L_\infty(Q_T)}$ and $\sup_{0\le t\le T}\|u(\cdot,t)\|_{L_2(R)}$ and independent of $\epsilon > 0$. Hence the inequality (29) becomes

$$\|u_x(\cdot,t)\|^2_{L_2(R)} \le C_{23}\int_0^t \|u_x(\cdot,\tau)\|^2_{L_2(R)}d\tau + C_{28}\{\|\phi\|^2_{H^1(R)} + \|f\|^2_{W^{(1,0)}_\infty(Q_T)}\} \tag{30}$$

where the constants C_{23} and C_{28} are dependent on $\alpha > 0$, $\beta \ge 0$ and the norms $\|b\|_{W^{(2,1)}_\infty(Q_T)}$, $\|c\|_{W^{(1,0)}_\infty(Q_T)}$, $\|\gamma\|_{W^{(1,0)}_\infty(Q_T)}$ and $\sup_{0\le t\le T}\|u(\cdot,t)\|_{L_2(R)}$ and independent of $\epsilon > 0$. From the inequality (30), we get the estimation

$$\|u_x(\cdot,t)\|^2_{L_2(R)} \le C_{29}\{\|\phi\|^2_{H^1(R)} + \|f\|^2_{W^{(1,0)}_\infty(Q_T)}\}$$

for any $0 \le t \le T$, where C_{29} is independent of $\epsilon > 0$.

Lemma 4 *Suppose that $\alpha > 0$, $\beta \ge 0$ and assume that $b(x,t) \in W^{(2,1)}_\infty(Q_T)$, $c(x,t) \in W^{(1,0)}_\infty(Q_T)$, $\gamma(x,t) \in W^{(1,0)}_\infty(Q_T)$, $f(x,t) \in W^{(1,0)}_2(Q_T)$ and $\phi(x) \in H^1(R)$. The generalized global solutions $u_\epsilon(x,t) \in W^{(2,1)}_2(Q_T)$ of the initial value problem (7) for the nonlinear parabolic equation (6) with Hilbert operator have the estimation*

$$\sup_{0\le t\le T}\|u_{\epsilon x}(\cdot,t)\|^2_{L_2(R)} + \beta\|u_{\epsilon xx}Hu_{\epsilon x}\|_{L_1(Q_T)} \le K_4\{\|\phi\|^2_{H^1(R)} + \|f\|^2_{W^{(1,0)}_2(Q_T)}\} \tag{31}$$

where K_4 is a constant independent of $\epsilon > 0$.

Lemma 5 *Under the conditions of Lemma 4, the solutions of the initial value problem (6) and (7) $u_\epsilon(x,t) \in W^{(2,1)}_2(Q^*_T)$ have the estimation*

$$\|u_\epsilon\|_{L_\infty(Q_T)} \le K_5\{\|\phi\|_{H^1(R)} + \|f\|_{W^{(1,0)}_2(Q_T)}\} \tag{32}$$

where K_5 is a constant independent of $\epsilon > 0$.

4. In order to estimate $\|u_{\epsilon xx}(\cdot,t)\|_{L_2(R)}$, we must to verify some necessary identities as before. We obtain the following identity

$$\begin{aligned}
&\frac{d}{dt}\int_{-\infty}^{\infty}(2\alpha u^2_{xx} + 10uu_xHu_{xx} + 5u_x^2Hu_x)dx \\
&\quad + 4\alpha\epsilon\|u_{xxx}(\cdot,t)\|^2_{L_2(R)} + 4\alpha\beta\|u_{xxx}(\cdot,t)Hu_{xx}(\cdot,t)\|_{L_1(R)} \\
&= 20\epsilon\int_{-\infty}^{\infty}(u_{xx}u_{xxx}Hu - uu_{xx}Hu_{xxx})dx - 20\int_{-\infty}^{\infty}u^2u_{xx}Hu_{xx}dx \\
&\quad + 10\int_{-\infty}^{\infty}[\beta u(Hu_{xx})^2 + \beta uu^2_{xx} - buu_{xx}Hu_{xx} - bu_{xx}H(uu_{xx})]dx
\end{aligned}$$

$$
\begin{aligned}
&-\alpha\int_{-\infty}^{\infty}[(6b_x+4c)u_{xx}^2-4\gamma u_{xx}Hu_{xx}]dx\\
&+20\int_{-\infty}^{\infty}[-2uu_x^2u_{xx}+uu_xHu_xu_{xx}+uu_x^2Hu_{xx}+u_xH(uu_x)u_{xx}]dx\\
&+10\int_{-\infty}^{\infty}[\beta(Hu_x)^2u_{xx}+2\beta u_x^2u_{xx}-bH(u_x^2)u_{xx}-bu_xHu_xu_{xx}-bu_xH(u_xu_{xx})]dx\\
&-10\int_{-\infty}^{\infty}[(b_x+c)uu_xHu_{xx}+\gamma Hu_xH(uu_{xx})+(b_x+c)u_xH(uu_{xx})\\
&-\gamma uHu_xHu_{xx}+\gamma HuH(u_xu_{xx})+cuH(u_xu_{xx})-\gamma HuHu_xu_{xx}-cuHu_xu_{xx}]dx\\
&-4\alpha\int_{-\infty}^{\infty}[(b_{xx}+2c_x)u_xu_{xx}+2\gamma_xHu_xu_{xx}]dx\\
&-10\int_{-\infty}^{\infty}[c_xu^2Hu_{xx}+c_xuH(uu_{xx})+\gamma_xuHuHu_{xx}+\gamma_xHuH(uu_{xx})]dx\\
&-4\alpha\int_{-\infty}^{\infty}(c_{xx}uu_{xx}+\gamma_{xx}Huu_{xx})dx\\
&-10\int_{-\infty}^{\infty}[(b_x+c)u_xH(u_x^2)+\gamma Hu_xH(u_x^2)]dx\\
&-10\int_{-\infty}^{\infty}[c_xuH(u_x^2)+\gamma_xHuH(u_x^2)]dx-10\int_{-\infty}^{\infty}f[Hu_xu_{xx}+H(u_xu_{xx})]dx\\
&+10\int_{-\infty}^{\infty}f_x[uHu_{xx}+H(uu_{xx})]dx+4\alpha\int_{-\infty}^{\infty}f_{xx}u_{xx}dx+10\int_{-\infty}^{\infty}f_xH(u_x^2)dx \quad (33)
\end{aligned}
$$

Denote by $\overline{J}_k$ ($k=1,2,\cdots,16$) the k-th integral term on the right hand side of identity (33).

For $\overline{J}_1$, we have

$$\overline{J}_1=20\epsilon\int_{-\infty}^{\infty}(u_{xx}u_{xxx}Hu-uu_{xx}Hu_{xxx})dx$$

Then

$$
\begin{aligned}
|\overline{J}_1|\le&20\epsilon\{\|Hu\|_{L_\infty(Q_T)}\|u_{xx}(\cdot,t)\|_{L_2(R)}\|u_{xxx}(\cdot,t)\|_{L_2(R)}\\
&+\|u\|_{L_\infty(Q_T)}\|u_{xx}(\cdot,t)\|_{L_2(R)}\|Hu_{xxx}(\cdot,t)\|_{L_2(R)}\}\\
\le&4\alpha\epsilon\|u_{xxx}(\cdot,t)\|_{L_2(R)}^2+50\frac{\epsilon}{\alpha}(\|u\|_{L_\infty(Q_T)}+\|Hu\|_{L_\infty(Q_T)}\|u_{xx}(\cdot,t)\|_{L_2(R)}^2
\end{aligned}
$$

where

$$
\begin{aligned}
\|Hu\|_{L_\infty(Q_T)}^2=&\sup_{0\le t\le T}\|Hu(\cdot,t)\|_{L_\infty(R)}^2\\
\le&C_{30}\sup_{0\le t\le T}\|Hu(\cdot,t)\|_{L_2(R)}\|Hu_x(\cdot,t)\|_{L_2(R)}\\
\le&C_{30}\sup_{0\le t\le T}\|u(\cdot,t)\|_{L_2(R)}\sup_{0\le t\le T}\|u_x(\cdot,t)\|_{L_2(R)}
\end{aligned}
$$

is bounded with respect to $\epsilon>0$.

Similarly using the interpolation formulas, we can estimate the other integral terms of (33). We obtain the following inequality in stead of the equality (33)

$$\frac{d}{dt}\int_{-\infty}^{\infty}(2(u_{xx}^2+5u_x^2Hu_x+10uu_xHu_{xx})dx$$
$$\leq C_{35}(\|u_{xx}(\cdot,t)\|_{L_2(R)}^2+\|u_x(\cdot,t)\|_{L_2(R)}^2+\|u(\cdot,t)\|_{L_2(R)}^2+\|f(\cdot,t)\|_{L_2(R)}^2)$$

where C_{36} is a constant and independent of $\epsilon>0$. Integrating the both sides of this inequality with respect to the variable t in the interval $[0,t]$ and regarding the estimations given in Lemmas 3 and 4, we have

$$\|u_{xx}(\cdot,t)\|_{L_2(R)}^2\leq C_{36}\int_0^t\|u_{xx}(\cdot,\tau)\|_{L_2(R)}^2d\tau+C_{40}\{\|\phi\|_{H^2(R)}^2+\|f\|_{W_2^{(2,0)}(Q_T)}^2\}$$

where C_{40} is independent of $\epsilon>0$. This gives the following lemma.

Lemma 6 *Suppose that $\alpha>0$, $\beta\geq 0$, $b(x,t)\in W_\infty^{(2,1)}(Q_T)$, $c(x,t),\gamma(x,t)\in W_\infty^{(2,0)}(Q_T)$ and $f(x,t)\in W_2^{(2,0)}(Q_T)$ and assume that $\phi(x)\in H^2(R)$. Then the generalized global solutions $u_\epsilon(x,t)\in W_2^{(2,1)}(Q_T)$ of the initial value problem (7) for the nonlinear parabolic equation (6) with Hilbert operators have the estimate*

$$\sup_{0\leq t\leq T}\|u_{\epsilon xx}(\cdot,t)\|_{L_2(R)}\leq K_6\{\|\phi\|_{H^2(R)}+\|f\|_{W_\infty^{(2,0)}(Q_T)}\} \tag{34}$$

where the constant K_6 depends on the norms $\|b\|_{W_\infty^{(2,1)}(Q_T)}$, $\|c\|_{W_\infty^{(2,0)}(Q_T)}$, $\|\gamma\|_{W_\infty^{(2,0)}(Q_T)}$, $\|\phi\|_{H^1(R)}$ and is independent of $\epsilon>0$.

5. By means of the equation (6) and the interpolation relations, we have the following lemmas as the immediate consequences of the previous lemmas.

Lemma 7 *Under the conditions of Lemma 6, the generalized global solutions $u_\epsilon(x,t)\in W_2^{(2,1)}(Q_T)$ of the initial value problem (7) for the nonlinear parabolic equation (6) with Hilbert operator have the estimate*

$$\sup_{0\leq t\leq T}\|u_{\epsilon t}(\cdot,t)\|_{L_2(R)}\leq K_7\{\|\phi\|_{H^2(R)}+\|f\|_{W_2^{(2,0)}(Q_T)}\} \tag{35}$$

where K_7 is a constant dependent on the norms $\|b\|_{W_\infty^{(2,1)}(Q_T)}$, $\|c\|_{W_\infty^{(2,0)}(Q_T)}$, $\|\gamma\|_{W_\infty^{(2,0)}(Q_T)}$ and $\|\phi\|_{H^1(R)}$ but independent of $\epsilon>0$.

Lemma 8 *Under the conditions of Lemma 6, the generalized global solutions $u_\epsilon(x,t)\in W_2^{(2,1)}(Q_T)$ of the initial value problem (6) and (7) have the estimate*

$$\|u_{\epsilon x}(\cdot,t)\|_{L_\infty(Q_T)}\leq K_8\{\|\phi\|_{H^2(R)}+\|f\|_{W_2^{(2,0)}(Q_T)}\} \tag{36}$$

where K_8 is independent of $\epsilon>0$.

4. Generalized Solutions

In the present section we are going to establish the generalized global solution of the initial value problem (7) (5) under the conditions of Lemma 6.

From Lemmas 3–8, the set of functions $\{u_\epsilon(x,t)\}$ is uniformly bounded in the functional space $Z = L_\infty(0,T;H^2(R)) \cap W_\infty^{(1)}(0,T;L_2(R))$ with respect to the diffusion coefficient $\epsilon > 0$ of the nonlinear parabolic equation (6) with Hilbert operators. By means of interpolation formulas for the functional spaces, we have the following lemma of the uniform estimations with respect to $\epsilon > 0$.

Lemma 9 *Under the conditions of Lemma 6, the set $\{u_\epsilon(x,t)\}$ of the generalized global solutions of initial value problem (6) and (7) has the following estimates:*

$$|u_\epsilon(\overline{x},t) - u_\epsilon(x,t)| \le K_9|\overline{x} - x| \tag{37}$$

$$|u_\epsilon(x,\overline{t}) - u_\epsilon(x,t)| \le K_{10}|\overline{t} - t|^{\frac{3}{4}} \tag{38}$$

$$|u_{\epsilon x}(\overline{x},t) - u_{\epsilon x}(x,t)| \le K_{11}|\overline{x} - x|^{\frac{1}{2}} \tag{39}$$

and

$$|u_{\epsilon x}(x,\overline{t}) - u_{\epsilon x}(x,t)| \le K_{12}|\overline{t} - t|^{\frac{1}{4}} \tag{40}$$

where $\overline{x}, x \in R$, $\overline{t}, t \in [0,T]$ and the constants K's are independent of $\epsilon > 0$.

Proof The estimation (37) is an immediate consequence of (36).

In the interpolation formula

$$|u(x,\overline{t}) - u(x,t)| \le C_1\|u(\cdot,\overline{t}) - u(\cdot,t)\|_{L_2(R)}^{\frac{3}{4}}\|u_{xx}(\cdot,\overline{t}) - u_{xx}(\cdot,t)\|_{L_2(R)}^{\frac{1}{4}}$$

we have

$$\|u(\cdot,\overline{t}) - u(\cdot,t)\|_{L_2(R)}^2 = \int_{-\infty}^{\infty}|u(x,\overline{t}) - u(x,t)|^2dx = \int_{-\infty}^{\infty}\left|\int_t^{\overline{t}} u_t(x,\tau)d\tau\right|^2 dx$$
$$\le |\overline{t} - t|^2 \sup_{0\le t\le T}\|u_t(\cdot,t)\|_{L_2(R)}|^2$$

and

$$\|u_{xx}(\cdot,\overline{t}) - u_{xx}(\cdot,t)\|_{L_2(R)} \le 2\sup_{0\le t\le T}\|u_{xx}(\cdot,t)\|_{L_2(R)}$$

Thus the relation (38) is valid.

The formula (39) follows directly from

$$|u_x(\overline{x},t) - u_x(x,t)| \le \left|\int_x^{\overline{x}} u_{xx}(\xi,t)d\xi\right| \le |\overline{x} - x|^{\frac{1}{2}}\|u_{xx}(\cdot,t)\|_{L_2(R)}$$

The last formula (40) can be obtained from

$$|u_x(x,\overline{t}) - u_x(x,t)| \le C_2\|u(\cdot,\overline{t}) - u(\cdot,t)\|_{L_2(R)}^{\frac{1}{4}}\|u_{xx}(\cdot,\overline{t}) - u_{xx}(\cdot,t)\|_{L_2(R)}^{\frac{3}{4}}$$
$$\le 2C_2|\overline{t} - t|^{\frac{1}{4}}\sup_{0\le t\le T}\|u_t(\cdot,t)\|_{L_2(R)}^{\frac{1}{4}}\sup_{0\le t\le T}\|u_{xx}(\cdot,t)\|_{L_2(R)}^{\frac{3}{4}}$$

The lemma is proved.

Hence the constructed limiting function $u(x,t)$ belongs to the functional space Z.

We can prove that the limiting function $u(x,t)$ satisfies the nonlinear singular integral-differential equation (5) with Hilbert operator in generalized sense.

Theorem 4 *Suppose that $\alpha > 0$, $\beta \geq 0$, $b(x,t) \in W_\infty^{(2,1)}(Q_T)$, $c(x,t), \gamma(x,t) \in W_\infty^{(2,0)}(Q_T)$ and $f(x,t) \in W_2^{(2,0)}(Q_T)$ and assume that $\phi(x) \in H^2(R)$. For the initial value problem with the initial condition (7) for the nonlinear singular integral-differential equation (5) with Hilbert operator, there exists at least one generalized global solution $u(x,t) \in Z$, which satisfies the equation (5) in generalized sense and satisfies the initial condition (7) in classical sense.*

For the uniqueness of the generalized global solution of the mentioned problem, we have the following theorem.

Theorem 5 *Suppose that $b(x,t) \in W_\infty^{(1,0)}(Q_T)$ and $c(x,t), \gamma(x,t) \in L_\infty(Q_T)$. The generalized global solution $u(x,t) \in Z$ for the initial problem (7) of the nonlinear singular integral-differential equation (5) is unique.*

5. Rate of Convergence

Theorem 6 *Under the conditions of Theorem 4, as $\epsilon \to 0$, the generalized global solution $u_\epsilon(x,t) \in W_2^{(2,1)}(Q_T)$ of the initial problem (7) for the nonlinear parabolic equation (6) with Hilbert operator converges to the unique generalized global solution $u(x,t) \in Z$ of the initial problem (7) for the nonlinear singular integral-differential equation (5) in the sense that $\{u_\epsilon(x,t)\}$ and $\{u_{\epsilon x}(x,t)\}$ are uniformly convergent to $u(x,t)$ and $u_x(x,t)$ respectively in any compact set of Q_T and $\{u_{\epsilon xx}(x,t)\}$ and $\{u_{\epsilon t}(x,t)\}$ are weakly convergent to $u_{xx}(x,t)$ and $u_t(x,t)$ respectively in $L_p(0,T;L_2(R))$ for $2 \leq p < \infty$.*

The following theorem is concerned to the estimation of the rate of convergence.

Theorem 7 *Under the conditions of Theorem 4, for the generalized global solutions $u_\epsilon(x,t) \in W_2^{(2,1)}(Q_T)$ and $u(x,t) \in Z$ of the initial problems (9) for the nonlinear parabolic equation (8) and the nonlinear integral-differential equation (7) respectively, there are the estimations for the rate of convergence in term of the power of the diffusion coefficient $\epsilon > 0$ as follows:*

$$\sup_{0\leq t\leq T} \|u_\epsilon(\cdot,t) - u(\cdot,t)\|_{L_2(R)} \leq K_{13}\epsilon \tag{41}$$

$$\|u_\epsilon - u\|_{L_\infty(Q_T)} \leq K_{14}\epsilon^{\frac{3}{4}} \tag{42}$$

$$\sup_{0\leq t\leq T} \|u_{\epsilon x}(\cdot,t) - u_x(\cdot,t)\|_{L_2(R)} \leq K_{15}\epsilon^{\frac{1}{2}} \tag{43}$$

and

$$\|u_{\epsilon x} - u_x\|_{L_\infty(Q_T)} \leq K_{16}\epsilon^{\frac{1}{2}} \tag{44}$$

where K's are the constants independent of $\epsilon > 0$.

Proof The difference $z(x,t) = u_\epsilon(x,t) - u(x,t)$ between these two generalized global solutions satisfies the linear equation

$$z_t + \alpha H z_{xx} - \beta H z_x + \gamma H z + (u_\epsilon + u + b) z_x + (u_{\epsilon x} + u_x + c) z = \epsilon u_{\epsilon xx}$$

in generalized sense and the homogeneous initial condition

$$z(x,0) = 0$$

Since $u_\epsilon(x,t) \in Z$, the coefficients of z_x and z are all bounded. Multiplying the equation by z and then integrating over the real axis with respect to variable x, we have

$$\frac{d}{dt}\|z(\cdot,t)\|^2_{L_2(R)} + 2\beta \int_{-\infty}^{\infty} z_x H z dx + 2\gamma \int_{-\infty}^{\infty} z H z dx + \int_{-\infty}^{\infty} (u_{\epsilon x} + u_x + 2c - b_x) z^2 dx$$
$$= 2\epsilon \int_{-\infty}^{\infty} z u_{\epsilon xx} dx$$

Thus we have

$$\frac{d}{dt}\|z(\cdot,t)\|^2_{L_2(R)} \le C_1 \|z(\cdot,t)\|^2_{L_2(R)} + \epsilon^2 \|u_{\epsilon xx}(\cdot,t)\|^2_{L_2(R)}$$

This implies

$$\sup_{0\le t\le T} \|z(\cdot,t)\|_{L_2(R)} \le C_2 \epsilon$$

This is the estimation (41). Then the estimations (42), (43) and (44) follow immediately from the following interpolation formulas

$$\|z(\cdot,t)\|_{L_\infty(Q_T)} \le C_3 \|z(\cdot,t)\|^{\frac{3}{4}}_{L_2(R)} \|z_{xx}(\cdot,t)\|^{\frac{1}{4}}_{L_2(R)}$$
$$\|z_x(\cdot,t)\|_{L_2(R)} \le C_4 \|z(\cdot,t)\|^{\frac{1}{2}}_{L_2(R)} \|z_{xx}(\cdot,t)\|^{\frac{1}{2}}_{L_2(R)}$$

and

$$\|z_x(\cdot,t)\|_{L_\infty(Q_T)} \le C_5 \|z(\cdot,t)\|^{\frac{1}{4}}_{L_2(R)} \|z_{xx}(\cdot,t)\|^{\frac{3}{4}}_{L_2(R)}$$

Hence the theorem is proved.

6. Remark on $T = \infty$

In the previous discussions, the value T is the arbitrary given constant. Hence the results obtained may be extended to the case of domain with infinite time interval $R_+ = [0,\infty)$. We have the theorem.

Theorem 8 *Suppose that the conditions of Theorem 4 are satisfied for any value of $T > 0$. The initial value problem (7) for the nonlinear integral-differential equation (5) has a unique generalized global solution $u(x,t) \in L_{\infty\,\mathrm{loc}}(R_+; H^2(R)) \cap W^{(1)}_{\infty\,\mathrm{loc}}(R_+; L_2(R))$ in the infinite domain $Q_\infty = \{x \in R; t \in R_+\}$.*

7. Remark on Regularity

Lemma 10 *Suppose that $\alpha > 0$, $\beta \geq 0$, $b(x,t) \in W_\infty^{(M,1)}(Q_T)$, $c(x,t), \gamma(x,t) \in W_\infty^{(M,0)}(Q_T)$ and $f(x,t) \in W_2^{(M,0)}(Q_T)$ and suppose that $\phi(x) \in H^M(R)$, for $M \geq 2$. Then the generalized global solution $u_\epsilon(x,t) \in W_2^{(M,1)}(Q_T)$ of the initial value problem (7) for the nonlinear parabolic equation (6) with Hilbert operators has the estimate*

$$\sup_{0\leq t\leq T} \|u_{\epsilon x^M}(\cdot,t)\|_{L_2(R)} \leq K_{16}\{\|\phi\|_{H^M(R)} + \|f\|_{W_2^{(M,0)}(Q_T)}\} \tag{45}$$

where K_{16} is a constant dependent on the norms $\|b\|_{W_\infty^{(M,1)}(Q_T)}$, $\|c\|_{W_\infty^{(M,0)}(Q_T)}$, $\|\gamma\|_{W_\infty^{(M,0)}(Q_T)}$ and $\|\phi\|_{H^{M-1}(R)}$ but independent of $\epsilon > 0$.

Lemma 11 *Under the conditions of above lemma, the generalized global solutions $u_\epsilon(x,t) \in W_2^{(M,1)}(Q_T)$ of the initial value problem (6), (7) have the estimates*

$$\sup_{0\leq t\leq T} \|u_{\epsilon x^{M-2}t}(\cdot,t)\|_{L_2(R)}, \|u_{\epsilon x^{M-1}}\|_{L_\infty(Q_T)} \leq K_{17}\{\|\phi\|_{H^M(R)} + \|f\|_{W_2^{(M,0)}(Q_T)}\} \tag{46}$$

where K_{17} is independent of $\epsilon > 0$.

Thus we can state the theorem of regularity of the solutions for the initial problem (7) of the homogeneous equation

$$u_t - 2uu_x + \alpha H u_{xx} - \beta H u_x + \gamma H u + b u_x + cu = 0 \tag{5$_{co}$}$$

of Benjamin-Ono type with the constant coefficients $\alpha, \beta, \gamma, b, c$ as follows, where $\alpha > 0$, $\beta \geq 0$.

Theorem 9 *Suppose that $\phi(x) \in H^M(R)$ for $M \geq 2$. The initial value problem (7) for the homogeneous equation $(5)_{co}$ of Benjamin-Ono type with constant coefficients $\alpha > 0$, $\beta \geq 0$, γ, b and c has a unique global solution $u(x,t) \in Z_M \equiv \bigcap_{k=0}^{[M/2]} W_{\infty,\text{loc}}^{(k)}(R_+, H^{M-2k}(R))$, which has the derivatives $u_{x^r t^s}(x,t) \in L_{\infty,\text{loc}}(R_+; L_2(R))$ for $0 \leq 2s + r \leq M$.*

Then as an immediate consequence, we have a theorem for the original Benjamin-Ono equation as follows:

Theorem 10 *Suppose that $\phi(x) \in H^M(R)$ for $M \geq 2$. The initial value problem (7) for the original Benjamin-Ono equation (1) has a unique global solution $u(x,t) \in Z_M$.*

For the nonhomogeneous equation $(5)_{co}$ with constant coefficients, we have the following theorem for the regularity of the global solutions.

Theorem 11 *Suppose that $\phi(x) \in H^M(R)$ and $f(x,t) \in W_2^{(M,0)}(Q_T)$ for $M \geq 2$. The initial value problem (7) for the nonhomogeneous equation $(5)_{co}$ of Benjamin-Ono type with constant coefficients $\alpha > 0$, $\beta \geq 0$, γ, b and c has a unique global solution*

$$u(x,t) \in L_{\infty\,\text{loc}}(R_+; H^M(R)) \cap W_{\infty,\text{loc}}^{(1)}(R_+; H^{M-2}(R))$$

which has the derivatives $u_{x^k}(x,t)$ $(k = 1, 2, \cdots, M)$ and $u_{x^k t}(x,t)$ $(k = 0, 1, \cdots, M-2)$.

8. Large-Time Global Estimate

1.Let us now consider the equation

$$u_t + 2uu_x + \alpha H u_{xx} - \beta H u_x + c(x,t)u = f(x,t) \tag{5$_0$}$$

of Benjamin-Ono type with the coefficients $b = 0$ and $\gamma = 0$, and the corresponding nonlinear parabolic equation

$$u_t + 2uu_x - \epsilon u_{xx} + \alpha H u_{xx} - \beta H u_x + c(x,t)u = f(x,t) \tag{6$_0$}$$

For the global solutions $u_\epsilon(x,t)$ of the Cauchy problem (7) $(6)_0$, there is the estimate as follows:

Lemma 12 *Suppose that $\alpha > 0$, $\beta \geq 0$, $c(x,t) \in W_\infty^{(1,0)}(Q_\infty)$, $f(x,t) \in L_\infty(R_+; H^1(R))$ and $\phi(x) \in H^1(R)$ and again suppose that $c(x,t) \geq c_0 > 0$. Then the generalized global solutions $u_\epsilon(x,t) \in W_2^{(2,1)}(Q_\infty)$ of the initial value problem (7) for the nonlinear parabolic equation $(6)_0$ with Hilbert operators have the estimates*

$$\|u_{\epsilon x}(\cdot,t)\|_{L_2(R)} \leq K_{22}\Big\{e^{-c_0 t}\|\phi\|_{H^1(R)} + \sup_{t\in R_+}\|u_\epsilon(\cdot,t)\|_{L_2(R)} + \sup_{t\in R_+}\|f(\cdot,t)\|_{H^1(R)}\Big\} \tag{47}$$

where K_{22} is a constant independent of $\epsilon > 0$ and dependent on the norm of $c(x,t)$ and $\sup_{t\in R_+}\|u_\epsilon(\cdot,t)\|_{L_2(R)}$.

Proof From the equality (24), we have in the present case as

$$\begin{aligned}
&\frac{d}{dt}\int_{-\infty}^{\infty}\Big[2\alpha u_x^2 + 3u^2 H u_x + \frac{1}{\alpha}u^4\Big]dx + 4\alpha\epsilon\|u_{xx}(\cdot,t)\|_{L_2(R)}^2\\
&= -4\alpha\beta\int_{-\infty}^{\infty} u_{xx} H u_x dx + 6\beta\int_{-\infty}^{\infty} u[(Hu_x)^2 + u_x^2]dx\\
&\quad + \epsilon\int_{-\infty}^{\infty}\Big[6(uHu_x + H(uu_x)) + \frac{4}{\alpha}u^3\Big]u_{xx}dx - 4\alpha\int_{-\infty}^{\infty} c u_x^2 dx + \frac{4\beta}{\alpha}\int_{-\infty}^{\infty} u^3 H u_x dx\\
&\quad - 6\int_{-\infty}^{\infty}(cu^2 H u_x - cuH(uu_x))dx - 4\alpha\int_{-\infty}^{\infty} c_x u u_x dx - \frac{\alpha}{4}\int_{-\infty}^{\infty} c u^4 dx\\
&\quad + 6\int_{-\infty}^{\infty} f[uHu_x + H(uu_x)]dx + 4\int_{-\infty}^{\infty} f_x \alpha u_x dx + \frac{4}{\alpha}\int_{-\infty}^{\infty} f u^2 dx
\end{aligned} \tag{24$_0$}$$

Denote by J_k^0 $(k = 1, 2, \cdots, 11)$, the k-th integral term on the right hand part of the above equality.

Using the interpolation formulas, we estimate the integral terms as follows.

$$\begin{aligned}
&|J_3^0| \leq 2\alpha\epsilon\|u_{xx}(\cdot,t)\|_{L_2(R)}^2 + \epsilon\overline{C}_1\|u(\cdot,t)\|_{L_2(R)}^2\\
&|J_2^0| + |J_5^0| \leq 2\alpha\beta\|u_{xx}(\cdot,t)Hu_\epsilon(\cdot,t)\|_{L_1(R)} + \beta\overline{C}_2\|u(\cdot,t)\|_{L_2(R)}^2\\
&|J_6^0| + |J_7^0| + |J_8^0| \leq \eta\|u_x(\cdot,t)\|_{L_2(R)}^2 + C(\eta)\|u(\cdot,t)\|_{L_2(R)}^2
\end{aligned}$$

where $\eta > 0$ and $C(\eta)$ depends on η and $\sup_{t\in R_+} \|u_\epsilon(\cdot,t)\|_{L_2(R)}$.

$$|J_9^0| + |J_{10}^0| + |J_{11}^0| \le \eta |u_x(\cdot,t)\|^2_{L_2(R)} + \overline{C}(\eta)\|f(\cdot,t)\|^2_{L_2(R)}$$

where $\eta > 0$ and $\overline{C}(\eta)$ depends on η and $\sup_{t\in R_+} \|u_\epsilon(\cdot,t)\|_{L_2(R)}$.

Taking $2\eta = c_0\alpha$, we have

$$\frac{d}{dt}\int_{-\infty}^{\infty}\Big[2\alpha u_x^2 + 3u^2 H u_x + \frac{1}{\alpha}u^4\Big]dx + 2\alpha\beta\|u_{xxx}(\cdot,t)Hu_{xx}(\cdot,t)\|_{L_1(R)}$$
$$+ 2\alpha\epsilon\|u_{xx}(\cdot,t)\|^2_{L_2(R)} \le 2\alpha c_0\|u_x(\cdot,t)\|^2_{L_2(R)} + C_{16}\|u(\cdot,t)\|^2_{L_2(R)} + \|f(\cdot,t)\|_{H^1(R)}$$

where C_{16} depends on $\sup_{t\in R_+} \|u_\epsilon(\cdot,t)\|_{L_2(R)}$.

Hence the lemma is proved.

2. Lemma 13 *Suppose $\alpha > 0$, $\beta \ge 0$, $c(x,t) \in W_\infty^{(2,0)}(Q_\infty)$, $f(x,t) \in L_\infty(R_+; H^2(R))$ for the equation $(6)_0$ and $\phi(x) \in H^2(R)$. And further assume that $c(x,t) \ge c_0 > 0$. Then the solutions $u_\epsilon(x,t) \in W_2^{(2,1)}(Q_\infty)$ for the Cauchy problem (7), $(6)_0$ have estimate*

$$\|u_{\epsilon xx}(\cdot,t)\|_{L_2(R)}$$
$$\le K_{23}\Big\{e^{-c_0 t}\|\phi\|_{H^2(R)} + \sup_{t\in R_+}\|u_\epsilon(\cdot,t)\|_{L_2(R)} + \sup_{t\in R_+}\|f(\cdot,t)\|_{H^2(R)}\Big\} \tag{48}$$

where K_{23} depends on $\sup_{t\in R_+}\|u_\epsilon(\cdot,t)\|_{L_2(R)}$ and is independent of $\epsilon > 0$.

Proof For the equation $(6)_0$, we have the following identities:

$$\frac{d}{dt}\int_{-\infty}^{\infty} u_{xx}^2 dx + 2\epsilon\|u_{xxx}(\cdot,t)\|^2_{L_2(R)} + 2\beta\|u_{xxx}(\cdot,t)Hu_{xx}(\cdot,t)\|_{L_1(R)}$$
$$= -10\int_{-\infty}^{\infty} u_x u_{xx}^2 dx - 2\int_{-\infty}^{\infty}(cu-f)_{xx}u_{xx}dx \tag{$33)_0$}$$

$$\frac{d}{dt}\int_{-\infty}^{\infty} u u_x H u_{xx} dx = -\frac{1}{2}\alpha\int_{-\infty}^{\infty} u_x u_{xx}^2 dx + \frac{5}{2}\alpha\int_{-\infty}^{\infty} u_x (Hu_{xx})^2 dx$$
$$-2\int_{-\infty}^{\infty} u^2 u_{xx} H u_{xx} dx - \epsilon\int_{-\infty}^{\infty}(uHu_{xxx} + H(uu_x)_{xx})u_{xx}dx$$
$$+\beta\int_{-\infty}^{\infty}[u(Hu_{xx})^2 + Hu_{xx}H(uu_x)_x]dx$$
$$-\int_{-\infty}^{\infty}(cu-f)_x(uHu_{xx} + H(uu_x)_x)dx - 4\int_{-\infty}^{\infty} u u_x^2 H u_{xx} dx \tag{$34)_0$}$$

$$\frac{d}{dt}\int_{-\infty}^{\infty} u_x^2 H u_x dx = \alpha\int_{-\infty}^{\infty} u_x (Hu_{xx})^2 dx - \alpha\int_{-\infty}^{\infty} u_x u_{xx}^2 dx$$
$$+2\int_{-\infty}^{\infty}[(u_x Hu_x)_x - H(u_x u_{xx})](-2\epsilon u_{xx} + \beta H u_x + cu - f)dx$$
$$+4\int_{-\infty}^{\infty}[uu_x(u_x H u_x)_x + u_x H(uu_x)u_{xx}]dx \tag{$35)_0$}$$

$$\frac{d}{dt}\int_{-\infty}^{\infty} u^2u_x^2dx = 2\alpha\int_{-\infty}^{\infty} u^2u_{xx}Hu_{xx}dx + 2\alpha\int_{-\infty}^{\infty} uu_x^2Hu_{xx}dx$$
$$+2\int_{-\infty}^{\infty}[uu_x^2+u^2u_{xx}](2uu_x-\epsilon u_{xx}-\beta Hu_x+cu-f)dx \tag{49}$$

We can eliminate the integral terms

$$\int_{-\infty}^{\infty} u_xu_{xx}^2dx,\ \int_{-\infty}^{\infty} u_x(Hu_{xx})^2dx,\ \text{and}\ \int_{-\infty}^{\infty} u^2u_{xx}Hu_{xx}dx$$

from the four identities $(33)_0$, $(34)_0$, $(35)_0$ and (49). Then we obtain the following equality

$$\frac{d}{dt}\int_{-\infty}^{\infty}(2\alpha^2u_{xx}^2+10\alpha uu_xHu_{xx}+5\alpha u_x^2Hu_x+10u^2u_x^2)dx$$
$$+4\alpha^2\epsilon\|u_{xxx}(\cdot,t)\|_{L_2(R)}^2+4\alpha^2\beta\|u_{xxx}(\cdot,t)Hu_{xx}(\cdot,t)\|_{L_1(R)}$$
$$=20\epsilon\int_{-\infty}^{\infty}(\alpha u_{xx}u_{xxx}Hu-\alpha uu_{xx}Hu_{xx}-10uu_x^2u_{xx}+10u^2u_xu_{xxx})dx$$
$$+10\alpha\beta\int_{-\infty}^{\infty}[u(Hu_{xx})^2-uu_{xx}^2]dx-20\beta\int_{-\infty}^{\infty}(uu_x^2+u^2u_{xx})dx$$
$$+10\alpha\beta\int_{-\infty}^{\infty}[(Hu_x)^2+2u_x^2]dx$$

$$-4\alpha\int_{-\infty}^{\infty}cu_{xx}^2dx+20\alpha\int_{-\infty}^{\infty}[-2uu_x^2u_{xx}+2uu_x^2Hu_{xx}+uu_xHu_xu_{xx}$$
$$+u_xH(uu_x)u_{xx}]dx+40\int_{-\infty}^{\infty}u^3u_xu_{xx}dx-10\alpha\int_{-\infty}^{\infty}[cuu_xHu_{xx}+cu_xH(uu_{xx}$$
$$+cuH(u_xu_{xx})-cuHu_xu_{xx}]dx-8\alpha^2\int_{-\infty}^{\infty}c_xu_xu_{xx}dx+20\int_{-\infty}^{\infty}cu^3u_{xx}dx$$
$$-10\alpha\int_{-\infty}^{\infty}[c_xu^2Hu_{xx}+c_xuH(uu_{xx})]dx-4\alpha^2\int_{-\infty}^{\infty}c_{xx}uu_{xx}dx$$
$$+40\int_{-\infty}^{\infty}u^2u_x^3dx-10\alpha\int_{-\infty}^{\infty}cu_xH(u_x^2)dx+20\int_{-\infty}^{\infty}cu^2u_x^2dx$$
$$-10\alpha\int_{-\infty}^{\infty}c_xuH(u_x^2)dx-10\alpha\int_{-\infty}^{\infty}f[Hu_xu_{xx}+H(u_xu_{xx})]dx$$
$$-20\int_{-\infty}^{\infty}fu^2u_{xx}dx+10\alpha\int_{-\infty}^{\infty}f_x[uHu_{xx}+H(uu_{xx})]dx+4\alpha^2\int_{-\infty}^{\infty}f_{xx}u_{xx}dx$$
$$-20\int_{-\infty}^{\infty}fuu_x^2dx+10\alpha\int_{-\infty}^{\infty}f_xH(u_x^2)dx \tag{50}$$

Similarly let us denote the k-th integral term of the right hand part of the above equality by J_k^*, where $k=1,2,\cdots,22$.

For J_1^*, we have

$$|J_1^*|\le 2\alpha^2\epsilon\|u_{xxx}(\cdot,t)\|_{L_2(R)}^2+\epsilon C_{17}\|u(\cdot,t)\|_{L_2(R)}^2$$

For J_2^*, J_3^* and J_4^*, we have

$$|J_2^*|+|J_3^*|+|J_4^*|\le 2\alpha^2\beta\|u_{xxx}(\cdot,t)Hu_{xx}(\cdot,t)\|_{L_1(R)}+\beta C_{18}\|u(\cdot,t)\|^2_{L_2(R)}$$

By means of interpolation formulas, we can estimate some of the integral terms as follows:

$$|J_6^*|\le\eta\|u_{xx}(\cdot,t)\|^2_{L_2(R)}+\widetilde{C}_6\eta^{-7}\|u(\cdot,t)\|^{\frac{21}{4}}_{L_2(R)}\|u_x(\cdot,t)\|^{\frac{21}{2}}_{L_2(R)}$$
$$|J_7^*|\le\eta\|u_{xx}(\cdot,t)\|^2_{L_2(R)}+\widetilde{C}_7\eta^{-7}\|u(\cdot,t)\|^{\frac{63}{4}}_{L_2(R)}\|u_x(\cdot,t)\|^{7}_{L_2(R)}$$
$$|J_8^*|\le\eta\|u_{xx}(\cdot,t)\|^2_{L_2(R)}+\widetilde{C}_8\eta^{-\frac{8}{3}}\|u(\cdot,t)\|^{2}_{L_2(R)}\|u_x(\cdot,t)\|^{\frac{8}{3}}_{L_2(R)}$$
$$|J_9^*|\le\eta\|u_{xx}(\cdot,t)\|^2_{L_2(R)}+\widetilde{C}_9\eta^{-1}\|u_x(\cdot,t)\|^{2}_{L_2(R)}$$
$$|J_{10}^*|\le\eta\|u_{xx}(\cdot,t)\|^2_{L_2(R)}+\widetilde{C}_{10}\eta^{-4}\|u(\cdot,t)\|^{10}_{L_2(R)}$$
$$|J_{11}^*|\le\eta\|u_{xx}(\cdot,t)\|^2_{L_2(R)}+\widetilde{C}_{11}\eta^{-\frac{8}{3}}\|u(\cdot,t)\|^{\frac{14}{3}}_{L_2(R)}$$
$$|J_{12}^*|\le\eta\|u_{xx}(\cdot,t)\|^2_{L_2(R)}+\widetilde{C}_{12}\eta^{-1}\|u(\cdot,t)\|^{2}_{L_2(R)}$$
$$|J_{13}^*|\le\eta\|u_{xx}(\cdot,t)\|^2_{L_2(R)}+\widetilde{C}_{13}\eta^{-4}\|u(\cdot,t)\|^{6}_{L_2(R)}\|u_x(\cdot,t)\|^{8}_{L_2(R)}$$
$$|J_{14}^*|\le\eta\|u_{xx}(\cdot,t)\|^2_{L_2(R)}+\widetilde{C}_{14}\eta^{-\frac{4}{3}}\|u_x(\cdot,t)\|^{\frac{10}{3}}_{L_2(R)}$$
$$|J_{15}^*|\le\eta\|u_{xx}(\cdot,t)\|^2_{L_2(R)}+\widetilde{C}_{15}\eta^{-\frac{8}{5}}\|u(\cdot,t)\|^{\frac{14}{5}}_{L_2(R)}\|u_x(\cdot,t)\|^{\frac{12}{5}}_{L_2(R)}$$
$$|J_{16}^*|\le\eta\|u_{xx}(\cdot,t)\|^2_{L_2(R)}+\widetilde{C}_{16}\eta^{-\frac{4}{3}}\|u(\cdot,t)\|^{\frac{4}{3}}_{L_2(R)}\|u_x(\cdot,t)\|^{2}_{L_2(R)}$$

For the last six integral terms we have the estimates as follows:

$$|J_{17}^*|\le\eta\|u_{xx}(\cdot,t)\|^2_{L_2(R)}+\widetilde{C}_{17}\eta^{-1}\|u_x(\cdot,t)\|^2_{L_2(R)}\|f(\cdot,t)\|^2_{H^1(R)}$$
$$|J_{18}^*|\le\eta\|u_{xx}(\cdot,t)\|^2_{L_2(R)}+\widetilde{C}_{18}\eta^{-1}\|u(\cdot,t)\|^{\frac{3}{2}}_{L_2(R)}\|u_x(\cdot,t)\|^{\frac{1}{2}}_{L_2(R)}\|f(\cdot,t)\|^2_{H^1(R)}$$
$$|J_{19}^*|\le\eta\|u_{xx}(\cdot,t)\|^2_{L_2(R)}+\widetilde{C}_{19}\eta^{-1}\|u(\cdot,t)\|^2_{L_2(R)}\|f(\cdot,t)\|^2_{H^2(R)}$$
$$|J_{20}^*|\le\eta\|u_{xx}(\cdot,t)\|^2_{L_2(R)}+\widetilde{C}_{20}\eta^{-1}\|f(\cdot,t)\|^2_{H^2(R)}$$
$$|J_{21}^*|\le\eta\|u_{xx}(\cdot,t)\|^2_{L_2(R)}+\widetilde{C}_{21}\eta^{-1}\|u_x(\cdot,t)\|^6_{L_2(R)}+\widetilde{C}_{21}\|u(\cdot,t)\|^2_{L_2(R)}\|f(\cdot,t)\|^2_{H^1(R)}$$
$$|J_{22}^*|\le\eta\|u_{xx}(\cdot,t)\|^2_{L_2(R)}+\widetilde{C}_{22}\eta^{-1}\|u_x(\cdot,t)\|^6_{L_2(R)}+\widetilde{C}_{22}\|f(\cdot,t)\|^2_{H^1(R)}$$

Taking sufficiently small $\eta>0$ that $2\alpha^2c_0=17\eta$, we can replace the equality (50) by the following inequality

$$\begin{aligned}&\frac{d}{dt}\int_{-\infty}^{\infty}(2\alpha^2u_{xx}^2+10\alpha uu_xHu_{xx}+5\alpha u_x^2Hu_x+10u^2u_x^2)dx\\&\quad+2\alpha^2\epsilon\|u_{xxx}(\cdot,t)\|^2_{L_2(R)}+2\alpha^2\beta\|u_{xxx}(\cdot,t)Hu_{xx}(\cdot,t)\|_{L_1(R)}\\&\quad\le-2\alpha^2c_0\|u_{xx}(\cdot,t)\|^2_{L_2(R)}+C_{19}\{\|u(\cdot,t)\|^2_{H^1(R)}+\|f(\cdot,t)\|^2_{H^2(R)}\}\end{aligned}$$

where C_{19} is a constant dependent on the norm $\|c\|_{W_\infty^{(2,0)}(Q_\infty)}$ and $\sup_{t\in R_+}\|u(\cdot,t)\|_{H^1(R)}$, but independent of $\epsilon > 0$.

Hence the lemma is proved.

Lemma 14 *Under the conditions of Lemma 13, there are estimates for the solutions $u_\epsilon(x,t)$ of the Cauchy problem (7), $(6)_0$ as follows:*

$$\|u_{\epsilon t}(\cdot,t)\|_{L_2(R)} + \|u_\epsilon(\cdot,t)\|_{L_\infty(R)} \le \overline{K}_{23}\Big\{e^{-c_0 t}\|\phi\|_{H^2(R)} + \sup_{t\in R_+}\|u(\cdot,t)\|^2_{L_2(R)} + \sup_{t\in R^+}\|f(\cdot,t)\|^2_{H^2(R)}\Big\} \tag{51}$$

where $\overline{K}_{23}$ depends on the norms $\|c\|_{W_\infty^{(2,0)}(Q_\infty)}$ and $\sup_{t\in R_+}\|u_\epsilon(\cdot,t)\|_{L_2(R)}$, but is independent of $\epsilon > 0$.

3. Now we turn to estimate the norms $\|u_{x^M}(\cdot,t)\|_{L_2(R)}$ of derivatives of high order $M \ge 3$ for large time. We have the inequality:

$$\begin{aligned}
&\frac{d}{dt}\int_{-\infty}^{\infty}(2\alpha^2u^2_{x^M} + (4M+2)\alpha u u_x u_{x^{M-1}} Hu_{x^M} - (2M+1)\alpha u_x u_{x^{M-1}} Hu_{x^{M-1}}\\
&\quad + (4M+2)u^2u^2_{x^{M-1}})dx + 4\alpha^2\epsilon\|u_{x^{M+1}}(\cdot,t)\|^2_{L_2(R)} + 4\alpha^2\beta\|u_{x^{M+1}}(\cdot,t)Hu_{x^M}(\cdot,t)\|_{L_1(R)}\\
&\le -4\alpha^2\int_{-\infty}^{\infty} cu^2_{x^M}dx + \alpha^2\overline{C}_{11}\epsilon\eta\|u_{x^{M+1}}(\cdot,t)\|^2_{L_2(R)}\\
&\quad + \alpha^2\overline{C}_{12}\beta\eta\|u_{x^{M+1}}(\cdot,t)Hu_{x^M}(\cdot,t)\|_{L_1(R)} + \overline{C}_{13}\eta\|u_{x^M}(\cdot,t)\|^2_{L_2(R)}\\
&\quad + \overline{C}_{14}(\eta)(\|u(\cdot,t)\|^2_{H^2(R)} + \|f(\cdot,t)\|^2_{H^M(R)})
\end{aligned}$$

Taking $\eta > 0$ sufficiently small, then we obtain the inequality

$$\begin{aligned}
&\frac{d}{dt}\int_{-\infty}^{\infty}(2\alpha^2u^2_{x^M} + (4M+2)\alpha u u_{x^{M-1}} Hu_{x^M} - (2M+1)\alpha u_x u_{x^{M-1}} Hu_{x^{M-1}}\\
&\quad + (4M+2)u^2u^2_{x^{M-1}})dx + 2\alpha^2\epsilon\|u_{x^{M+1}}(\cdot,t)\|^2_{L_2(R)}\\
&\quad + 2\alpha^2\beta\|u_{x^{M+1}}(\cdot,t)Hu_{x^M}(\cdot,t)\|_{L_1(R)}\\
&\le -2\alpha^2c_0\|u_{x^M}(\cdot,t)\|^2_{L_2(R)} + C_{21}(\|u(\cdot,t)\|^2_{H^2(R)} + \|f(\cdot,t)\|^2_{H^M(R)})
\end{aligned} \tag{52}$$

where C_{21} depends on the constants $\alpha > 0$, $\beta \ge 0$ and the norms $\|c\|_{W_\infty^{(M,0)}(Q_\infty)}$ and $\sup_{t\in R_+}\|u(\cdot,t)\|_{H^2(R)}$.

By the same way of proof as before, we can obtain the following lemmas of the estimates from the inequality (52).

Lemma 15 *Suppose that $\alpha > 0$, $\beta \ge 0$, $c(x,t) \in W_\infty^{(M,0)}(Q_\infty)$, $f(x,t) \in L_\infty(R_+; H^M(R))$ and $\phi(x) \in H^M(R)$ for $M \ge 3$. And further assume $c(x,t) \ge c_0 > 0$. Then there are estimates*

$$\|u_{\epsilon x^M}(\cdot,t)\|_{L_2(R)} \le K_{24}\Big\{e^{-c_0 t}\|\phi\|_{H^M(R)} + \sup_{t\in R_+}\|u_\epsilon(\cdot,t)\|_{L_2(R)}$$

$$+ \sup_{t\in R_+} \|f(\cdot,t)\|_{H^M(R)}\Big\} \tag{53}$$

for $t \in R_+$, where K_{24} depends on the norms of c and f and on $\sup_{t\in R_+} \|u_\epsilon(\cdot,t)\|_{L_2(R)}$ and is independent of $\epsilon > 0$.

Lemma 16 *Under the conditions of Lemma 15, there are the estimates for the solutions $u_\epsilon(x,t)$ of the Cauchy problem (7), $(6)_0$ as follows:*

$$\begin{aligned}&\|u_{\epsilon x^{M-1}t}(\cdot,t)\|_{L_2(R)} + \|u_{\epsilon x^{M-1}}(\cdot,t)\|_{L_\infty(R)} + \|u_{\epsilon x^{M-3}t}(\cdot,t)\|_{L_\infty(R)}\\ &\le K_{25}\Big\{e^{-c_0t}\|\phi\|_{H^M(R)} + \sup_{t\in R_+} \|u_\epsilon(\cdot,t)\|_{L_2(R)} + \sup_{t\in R_+} \|f(\cdot,t)\|_{H^M(R)}\Big\}\end{aligned} \tag{54}$$

for any $t \in R_+$ and $M \ge 3$, where K_{25} depends on the norms of c and f and also on $\sup_{t\in R_+} \|u_\epsilon(\cdot,t)\|_{L_2(R)}$, but is independent of $\epsilon > 0$.

4. For the solution $u(x,t)$ of the Cauchy problem (7) for the nonlinear equation $(5)_0$ of Benjamin-Ono type, we have the following lemmas.

Lemma 17 *Under the conditions of Lemma 12, Lemma 13 or Lemma 15, the global solution $u(x,t)$ of the Cauchy problem (7) for the nonlinear equation $(5)_0$ of Benjamin-Ono type has the estimates*

$$\begin{aligned}\|u_{x^M}(\cdot,t)\|_{L_2(R)} \le K_{26}\Big\{&e^{-c_0t}\|\phi\|_{H^M(R)} + \sup_{t\in R_+} \|u(\cdot,t)\|_{L_2(R)}\\ &+ \sup_{t\in R_+} \|f(\cdot,t)\|_{H^M(R)}\Big\}\end{aligned} \tag{55}$$

for any $t \in R_+$ and $M = 1, 2, \cdots$ respectively, where K_{26} is a constant dependent on $\alpha > 0$, $\beta \ge 0$ and the norms of $c(x,t)$ and $f(x,t)$ and also on the norm $\sup_{t\in R_+} \|u(\cdot,t)\|_{L_2(R)}$.

Lemma 18 *Under the conditions of Lemma 17, there are the estimates:*

$$\begin{aligned}&\|u_{x^{M-2}t}(\cdot,t)\|_{L_2(R)} + \|u_{x^{M-1}}(\cdot,t)\|_{L_\infty(R)} + \|u_{x^{M-3}t}(\cdot,t)\|_{L_\infty(R)}\\ &\le K_{27}\Big\{e^{-c_0t}\|\phi\|_{H^M(R)} + \sup_{t\in R_+} \|u(\cdot,t)\|_{L_2(R)} + \sup_{t\in R_+} \|f(\cdot,t)\|_{H^M(R)}\Big\}\end{aligned} \tag{56}$$

for any $t \in R_+$ and $M \ge 2$ or $M \ge 3$ and $M = 2, 3, \cdots$ respectively, where K_{27} is a constant dependent on $\alpha > 0$, $\beta \ge 0$, the norms of c and f and also on the norm $\sup_{t\in R_+} \|u(\cdot,t)\|_{L_2(R)}$.

9. Existence of Global Attractors

1. It is well known that the existence of the global attractor and the estimate of its dimension hinge crucially upon the compactness of ω-limit sets of trajectories. In the previous study of nonlinear parabolic dynamics, this compactness requirement is trivially satisfied and the trajectories themselves are precompact. For an evolution equation of functions defined on an unbounded Euclidean domain, this precompactness property does not hold true. It is our aim in this section to present a method of the weighted function space to overcome this difficulty. It needs to make *a priori* estimates uniformly for t in weighted space to construct the semigroup and its attractor.

Let us consider the following Cauchy problem of equation of Benjamin-Ono type

$$u_t - \alpha u_{xx} + 2uu_x + \beta H u_{xx} + \delta H u_x = g(u) + h(x) \tag{57}$$

$$u\,|_{t=0} = u_0(x), \quad x \in R \tag{58}$$

where $\alpha > 0$, β, δ are real constants.

Let $H_{0,\gamma}$ be a Hilbert space with the norm

$$\|u\|_{0,\gamma}^2 = \int |u(x)|^2 (1+|x|^2)^\gamma dx \tag{59}$$

and let $H_{l,\gamma}$ $(l = 1, 2, \cdots)$ be weighted Sobolev spaces with the norms

$$\|u\|_{l,\gamma}^2 = \sum_{|\alpha|\le l} \|\partial^\alpha u\|_{0,\gamma}^2,\ \partial^\alpha = \partial_1^{\alpha_1}\cdots\partial_n^{\alpha_n},\ |\alpha| = \alpha_1+\cdots+\alpha_n,\ \partial_i = \frac{\partial}{\partial x_i} \tag{60}$$

It is evident that, as $\gamma > 0$, functions $u(x)$ which belong to $H_{0,\gamma}$ are decreasing at infinity faster than the functions from $L_2(R^n)$.

We use in the estimation for the solutions of (57) the function $\phi(x)$,

$$\phi(x) = \phi_\epsilon(x) = (1+|\epsilon x|^2)^\gamma \tag{61}$$

where $0 < \epsilon \le 1$ and ϵ is sufficiently small. This function $\phi(x)$ is in the definition of norm in $H_{0,\gamma}$ with $\epsilon = 1$. For $\epsilon \ne 0$, the following norms are equivalent:

$$C_\epsilon^{-1}\|\phi_\epsilon^{\frac{1}{2}} u\| \le \|u\|_{0,\gamma} \le C_\epsilon \|\phi_\epsilon^{\frac{1}{2}} u\| \tag{62}$$

(here and below $\|\cdot\|$ denotes $\|\cdot\|_{0,0}$). The estimate

$$|\bigtriangledown \phi| < C\epsilon\phi \tag{63}$$

holds. It is easy to verify the inequalities

$$C^{-1}\|u\|_{l,\gamma} \le \|\phi_\epsilon^{\frac{1}{2}} u\|_l \le \|u\|_{l,\gamma}, \quad l = 1, 2 \tag{64}$$

Here and below $\|\cdot\|_l$ denotes $\|\cdot\|_{l,0}$.

Now we give *a priori* estimates for the solutions of the Cauchy problem (57) (58).

Lemma 19 **(A.P. Calderon [16])** *Let $A : R \to R$ be a C^∞ function. Then the operator $[H; A]\partial_x$ maps $L_2(R)$ into $L_2(R)$ with*

$$\|[H, A]\partial_x f\| \le C\|A' f\| \tag{65}$$

where H is the Hilbert transform, i.e.

$$Hf(x) = \frac{1}{\pi} P \int_{-\infty}^{\infty} \frac{f(y)}{x-y} dy = F^{-1}\left(i\,\mathrm{sgn}\xi \widehat{f}(\xi)\right)$$

with "$\wedge$" and F^{-1} denoting the Fourier transform and its invers respectively. $[H\partial_x, A]$ *denotes*

$$[H\partial_x, A]f = [H, A]f_x + H(A'f)$$

and

$$[A, B]f = ABf - BAf$$

is the commutator of singular integral operator A and B.

2. Lemma 20 *Suppose that the following conditions are satisfied:*

(1) $g(u) \in C^1$, $g(0) = 0$, $g'(u) \le -b$, $b > 0$,

(2) $\alpha > |\beta| + \frac{|\delta|}{2}$, $b > \frac{3}{2}|\delta|$,

(3) $u_0(x) \in H_{0,\gamma}$, $h(x) \in H_{0,\gamma}$.

Then for the smooth solution of Cauchy problem (57) (58) we have

$$\begin{aligned} &\|u(\cdot,t)\|_{0,\gamma} \le C_\epsilon\Big(e^{-\frac{|\delta|}{2}t}\|u_0\|_{0,\gamma} + \frac{2}{b|\delta|}(1 - e^{-\frac{1}{2}|\delta|t})\|h\|_{0,\gamma}\Big) \\ &\overline{\lim_{t\to\infty}}\, \|u(\cdot,t)\|_{0,\gamma} \le C_\epsilon \frac{2}{b|\delta|}\|h\|^2_{0,\gamma} = E_0 \\ &\overline{\lim_{t\to\infty}}\, \frac{1}{t}\int_0^t \|u(\cdot,\tau)\|^2_{1,\gamma} d\tau \le E_0 \end{aligned} \tag{66}$$

Proof Multiplying (57) by $\phi_\epsilon(x)u$ and integrating with respect to x, we obtain

$$(\phi_\epsilon u, u_t - \alpha H u_{xx} + 2uu_x + \beta H u_{xx} + \delta H u_x - g(u) - h(x)) = 0 \tag{67}$$

Here we have

$$\begin{aligned} &(\phi_\epsilon u, u_t) = \frac{1}{2}\frac{d}{dt}\|\phi_\epsilon^{\frac{1}{2}} u(\cdot,t)\|^2 \\ &(\phi_\epsilon u, -\alpha u_{xx}) = \alpha(\phi'_\epsilon u, u_x) + \alpha(\phi_\epsilon u_x, u_x) \\ &|(\phi'_\epsilon u, u_x)| \le \gamma\epsilon(\|\phi_\epsilon^{\frac{1}{2}} u\|^2 + \|\phi_\epsilon^{\frac{1}{2}} u_x\|^2) \\ &|(\phi_\epsilon u, 2uu_x)| = \frac{2}{3}|(\phi_\epsilon, (u^3)_x)| = \frac{2}{3}|(\phi'_\epsilon, u^3)| \le \frac{4}{3}\gamma\epsilon\|u\|_\infty\|\phi_\epsilon^{\frac{1}{2}} u\|^2 \\ &(\phi_\epsilon u, \beta H u_{xx}) = -\beta(\phi'_\epsilon u, H u_x) - \beta(\phi_\epsilon u_x, H u_x) \\ &|\beta(\phi'_\epsilon u, H u_x)| \le |\beta|\gamma\epsilon(\|\phi_\epsilon^{\frac{1}{2}} u\|^2 + \|\phi_\epsilon^{\frac{1}{2}} H u_x\|^2) \\ &|\beta(\phi_\epsilon u_x, H u_x)| \le \frac{|\beta|}{2}(\|\phi_\epsilon^{\frac{1}{2}} u_x\|^2 + \|\phi_\epsilon^{\frac{1}{2}} H u_x\|^2) \end{aligned}$$

In view of Lemma 19, it follows

$$\|\phi_\epsilon^{\frac{1}{2}} H u_x\|^2 \le \|\phi_\epsilon^{\frac{1}{2}} u_x\|^2 + c\epsilon\|\phi_\epsilon^{\frac{1}{2}} u\|^2 \tag{68}$$

Hence

$$|(\phi_\epsilon u, \beta H u_{xx})| \leq |\beta|(1+\gamma\epsilon)\|\phi_\epsilon^{\frac{1}{2}} u_x\|^2 + \frac{|\beta|}{2}\epsilon[c+2\gamma(1+c\epsilon)]\|\phi_\epsilon^{\frac{1}{2}} u\|^2$$

$$|(\phi_\epsilon u, \delta H u_x)| \leq |\delta|\|\phi_\epsilon^{\frac{1}{2}} u\|\|\phi_\epsilon^{\frac{1}{2}} H u_x\| \leq \frac{|\delta|}{2}[(1+c\epsilon)\|\phi_\epsilon^{\frac{1}{2}} u\|^2 + \|\phi_\epsilon^{\frac{1}{2}} u_x\|^2]$$

$$(g(u), \phi_\epsilon u) \leq -b\|\phi_\epsilon^{\frac{1}{2}} u\|^2$$

$$(\phi_\epsilon u, h) \leq \|\phi_\epsilon^{\frac{1}{2}} u\|^2\|\phi_\epsilon^{\frac{1}{2}} h\|^2 \leq \frac{b}{2}\|\phi_\epsilon^{\frac{1}{2}} u\|^2 + \frac{1}{2b}\|\phi_\epsilon^{\frac{1}{2}} h\|^2$$

Thus from (67) we have

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|\phi_\epsilon^{\frac{1}{2}} u\|^2 + \Big[\alpha - \alpha\gamma\epsilon - |\beta|(1+\gamma\epsilon) - \frac{|\delta|}{2}\Big]\|\phi_\epsilon^{\frac{1}{2}} u_x\|^2 \\
&\qquad + \Big[b - \alpha\gamma\epsilon - \frac{4}{3}\gamma\epsilon\|u\|_\infty - \frac{|\beta|}{2}\epsilon(c+2\gamma(1+c\epsilon)) - \frac{|\delta|}{2}(c\epsilon+1) - \frac{b}{2}\Big]\|\phi_\epsilon^{\frac{1}{2}} u\|^2 \\
&\leq \frac{1}{2b}\|\phi_\epsilon^{\frac{1}{2}} h\|^2
\end{aligned}$$

For small ϵ and from Lemma 12, $\sup\limits_{t\in R_+}\|u(\cdot,t)\|_\infty \leq \infty$ and the conditions of the lemma, we get

$$\frac{d}{dt}\|\phi_\epsilon^{\frac{1}{2}} u\|^2 + \frac{|\delta|}{2}\|\phi_\epsilon^{\frac{1}{2}} u\|^2 \leq \frac{1}{2b}\|\phi_\epsilon^{\frac{1}{2}} h\|^2 \tag{69}$$

From the inequalities (62) and (69), it follows (66).

Lemma 21 *If $\|\phi_\epsilon^{\frac{1}{2}} u_{x^{j-1}}\| \leq M$ $(j=1,2,\cdots)$, then we have*

$$\|\phi_\epsilon^{\frac{1}{2}} u_{x^j}\| \leq \epsilon_1\|\phi_\epsilon^{\frac{1}{2}} u_{x^{j+1}}\| + C(M,\epsilon_1) \tag{70}$$

where $\epsilon_1 > 0$ is sufficiently small and C is a constant depending on M and $\epsilon_1 > 0$.

Proof By using Sobolev's inequality,

$$\begin{aligned}
\|\phi_\epsilon^{\frac{1}{2}} u_{x^j}\| &= \|(\phi_\epsilon^{\frac{1}{2}} u_{x^{j-1}})_x - (\phi_\epsilon^{\frac{1}{2}})' u_{x^{j-1}}\| \\
&\leq \|(\phi_\epsilon^{\frac{1}{2}} u_{x^{j-1}})_x\| + \gamma\epsilon\|\phi_\epsilon^{\frac{1}{2}} u_{x^{j-1}}\| \leq \|\phi_\epsilon^{\frac{1}{2}} u_{x^{j-1}}\|^{\frac{1}{2}}\|(\phi_\epsilon^{\frac{1}{2}} u_{x^{j-1}})_{xx}\|^{\frac{1}{2}} + \gamma\epsilon\|\phi_\epsilon^{\frac{1}{2}} u_{x^{j-1}}\| \\
&\leq \frac{\epsilon_1}{2}\|(\phi_\epsilon^{\frac{1}{2}} u_{x^{j-1}})_{xx}\| + \Big(\frac{1}{2\epsilon_1} + \gamma\epsilon\Big)M \\
&\leq \frac{\epsilon_1}{2}\{\|\phi_\epsilon^{\frac{1}{2}} u_{x^{j+1}}\| + 2\|(\phi_\epsilon^{\frac{1}{2}})' u_{x^j}\| + \|(\phi_\epsilon^{\frac{1}{2}})'' u_{x^{j-1}}\|\} + \Big(\frac{1}{2\epsilon_1} + \gamma\epsilon\Big)M \\
&\leq \frac{\epsilon_1}{2}\|\phi_\epsilon^{\frac{1}{2}} u_{x^{j+1}}\| + \epsilon_1\gamma\epsilon\|\phi_\epsilon^{\frac{1}{2}} u_{x^j}\| + \Big[\frac{1}{8}\gamma\epsilon^2\epsilon_1(2\gamma+1) + \frac{1}{2\epsilon_1} + \gamma\epsilon\Big]M
\end{aligned} \tag{71}$$

Taking $\epsilon_1\gamma\epsilon = \frac{1}{2}$, we get the inequality (70) immediately.

Lemma 22 *Suppose that the conditions of Lemma 20 are satisfied and assume that $u_0(x) \in H_{1,\gamma}$, $h(x) \in H_{1,\gamma}$. Then for the smooth solution $u(x,t)$ of the Cauchy problem (57) (58) we have*

$$\|u(\cdot,t)\|_{0,\gamma} \leq C_\epsilon\Big[e^{-\frac{|\delta|}{2}t}\|u_0\|_{0,\gamma} + \frac{2}{b|\delta|}(1-e^{-\frac{1}{2}|\delta|t})(\|h\|_{1,\gamma} + \|u\|_{0,\gamma}) \tag{72}$$

$$\overline{\lim_{t\to\infty}}\|u(\cdot,t)\|_{1,\gamma}\le C_\epsilon\frac{2}{b|\delta|}(\|h\|_{1,\gamma}+\|u\|_{0,\gamma})=E_1 \tag{73}$$

$$\overline{\lim_{t\to\infty}}\frac{1}{t}\int_0^t\|u(\cdot,t)\|^2_{2,\gamma}dt\le E_1' \tag{74}$$

Proof Differentiating (57) with respect to x, then taking the inner product with $\phi_\epsilon u_x$, we get

$$(\phi_\epsilon u_x, u_{tx}-\alpha u_{xxx}+2(uu_x)_x+\beta H u_{xxx}+\delta H u_{xx}-g'(u)u_x-h'(x))=0 \tag{75}$$

Here

$$(\phi_\epsilon u_x,u_{xt})=\frac{1}{2}\frac{d}{dt}\|\phi_\epsilon^{\frac12}u_x\|^2$$

$$(\phi_\epsilon u_x,-\alpha u_{xxx})=\alpha\|\phi_\epsilon^{\frac12}u_{xx}\|^2+\alpha(\phi_\epsilon' u_x,u_{xx}),$$

$$|(\phi_\epsilon' u_x,u_{xx})|\le\|(\phi_\epsilon')^{\frac12}u_x\|\|(\phi_\epsilon')^{\frac12}u_{xx}\|\le C\epsilon^2\|\phi_\epsilon^{\frac12}u_x\|\|\phi_\epsilon^{\frac12}u_{xx}\|$$
$$\le\frac{C}{2}\epsilon^2(\|\phi_\epsilon^{\frac12}u_x\|^2+\|\phi_\epsilon^{\frac12}u_{xx}\|^2)$$

$$|(\phi_\epsilon u_x,(2uu_x)_x)|\le 2C\epsilon\|u\|_\infty\|\phi_\epsilon^{\frac12}u_x\|^2+\eta_1\|\phi_\epsilon^{\frac12}u_{xx}\|^2+\frac{1}{2\eta_1}\|\phi_\epsilon^{\frac12}uu_x\|^2$$
$$\le\eta_1\|\phi_\epsilon^{\frac12}u_{xx}\|^2+2C\epsilon\|u\|_\infty\|\phi_\epsilon^{\frac12}u_x\|^2+\frac{\|u\|^2_\infty}{\eta_1}[\epsilon_1^2\|\phi_\epsilon^{\frac12}u_{xx}\|^2+C_1^2(\|\phi_\epsilon^{\frac12}u\|,\epsilon_1)]$$

$$|(\phi_\epsilon u_x,\beta Hu_{xxx})|\le|\beta||(\phi_\epsilon' u_x,Hu_{xx})|+|\beta||(\phi_\epsilon u_{xx},Hu_{xx})|$$
$$\le C|\beta|\epsilon(\|\phi_\epsilon^{\frac12}u_x\|^2+\|\phi_\epsilon^{\frac12}Hu_{xx}\|^2)+\frac{|\beta|}{2}(\|\phi_\epsilon^{\frac12}u_{xx}\|^2+\|\phi_\epsilon^{\frac12}Hu_{xx}\|^2)$$
$$\le C|\beta|\epsilon\|\phi_\epsilon^{\frac12}u_x\|^2+\frac{|\beta|}{2}\ big\|\phi_\epsilon^{\frac12}u_{xx}\|^2+\Big(\frac{|\beta|}{2}+C\epsilon|\beta|\Big)(\|\phi_\epsilon^{\frac12}u_{xx}\|^2+C\epsilon\|\phi_\epsilon^{\frac12}u_x\|^2)$$
$$=(|\beta|+C\epsilon|\beta|)\|\phi_\epsilon^{\frac12}u_{xx}\|^2+C\epsilon\Big(\frac{3|\beta|}{2}+C\epsilon|\beta|\Big)\|\phi_\epsilon^{\frac12}u_x\|^2$$

$$|(\phi_\epsilon u_x,\delta Hu_{xx})|\le\delta\|\phi_\epsilon^{\frac12}u_x\|\|\phi_\epsilon^{\frac12}Hu_{xx}\|\le\eta_2\|\phi_\epsilon^{\frac12}Hu_{xx}\|^2+\frac{\delta^2}{4\eta_2}\|\phi_\epsilon^{\frac12}u_x\|^2$$
$$\le\eta_2\|\phi_\epsilon^{\frac12}u_{xx}\|^2+\Big(\eta_2C\epsilon+\frac{\delta^2}{4\eta_2}\Big)\|\phi_\epsilon^{\frac12}u_x\|^2$$
$$\le\eta_2\|\phi_\epsilon^{\frac12}u_{xx}\|^2+\Big(\eta_2C\epsilon+\frac{\delta^2}{4\eta_2}\Big)(\epsilon_1^2\|\phi_\epsilon^{\frac12}u_{xx}\|^2+C_1^2)$$
$$\le\Big[\eta_2+\epsilon_1^2\Big(\eta_2C\epsilon+\frac{\delta^2}{4\eta_2}\Big)\Big]\|\phi_\epsilon^{\frac12}u_{xx}\|^2+\Big(\eta_2C\epsilon+\frac{\delta^2}{4\eta_2}\Big)C_1^2\le\eta_3\|\phi_\epsilon^{\frac12}u_{xx}\|^2+C_2$$

$$(\phi_\epsilon u_x,g'(u)u_x)\le-b\|\phi_\epsilon^{\frac12}u_x\|^2$$

$$(\phi_\epsilon u_x,h')\le\frac{b}{2}\|\phi_\epsilon^{\frac12}u_x\|^2+\frac{1}{2b}\|\phi_\epsilon^{\frac12}h'\|^2$$

Hence from (95) we have

$$\frac12\frac{d}{dt}\|\phi_\epsilon^{\frac12}u_x\|^2+\Big(\alpha-\frac{C}{2}\epsilon^2-\eta_1-\frac{2\epsilon_1^2\|u\|^2_\infty}{\eta_1}-|\beta|-C|\beta|\epsilon-\eta_3\Big)\|\phi^{\frac12}u_{xx}\|^2$$

$$+\Big(b-\frac{b}{2}-\frac{C}{2}\epsilon^2-2C\epsilon\|u\|_\infty-C\epsilon|\beta|\Big(\frac{3}{2}+C\epsilon\Big)\|\phi_\epsilon^{\frac12}u_x\|^2$$
$$\le\frac{1}{2b}\|\phi_\epsilon^{\frac12}h'\|^2+C_3(\|\phi_\epsilon^{\frac12}u\|^2) \tag{76}$$

For given small η_1 and η_2, then choose sufficiently small ϵ and ϵ_1, from (76) it follows (72)-(74).

Similarly we have

Lemma 23 *Suppose that the conditions of Lemma 22 are satisfied and assume that $u_0(x)\in H^1(R)\cap H_{1,\gamma+\delta_1}$, $h(x)\in H_{1,\gamma+\delta_1}$. Then for the smooth solution $u(x,t)$ of the Cauchy problem (57) (58) we have*

$$\sup_{t\in R_+}\|u(\cdot,t)\|_{1,\gamma+\delta_1}\le E_1 \tag{77}$$

where the constant $\delta_1>0$.

Lemma 24 *Suppose that the conditions of Lemma 22 are satisfied and assume that $u_0(x)\in H_{2,\gamma}$, $h(x)\in H_{2,\gamma}$. Then for the smooth solution $u(x,t)$ of the Cauchy problem (57) (58), we have*

$$\|u(\cdot,t)\|_{2,\gamma}\le C_\epsilon\Big(e^{-\frac{|\delta|}{2}t}\|u_0\|_{2,\gamma}+\frac{2}{b|\delta|}(1-e^{-\frac12|\delta|t})(\|h\|_{2,\gamma}+\|u\|_{1,\gamma}\Big) \tag{78}$$

$$\overline{\lim_{t\to\infty}}\|u(\cdot,t)\|_{2,\gamma}\le C_\epsilon\frac{2}{b|\delta|}(\|h\|_{2,\gamma}+\|u\|_{1,\gamma})=E_2 \tag{79}$$

$$\overline{\lim_{t\to\infty}}\frac{1}{t}\int_0^t\|u(\cdot,t)\|_{3,\gamma}^2dt\le E_2' \tag{80}$$

Proof Differentiating (57) with respect to x two times, then taking the inner product with $\phi_\epsilon u_{xx}$, we get

$$(\phi_\epsilon u_{xx},u_{xxt}-\alpha u_{xxxx}+2(uu_x)_{xx}+\beta Hu_{xxxx}+\delta Hu_{xxx}-(g'(u)u_x)_x-h'')=0 \tag{81}$$

Here we have

$$(\phi_\epsilon u_{xx},u_{xxt})=\frac12\frac{d}{dt}\|\phi_\epsilon^{\frac12}u_{xx}\|^2$$
$$(\phi_\epsilon u_{xx},-\alpha u_{xxxx})=\alpha((\phi_\epsilon u_{xx})_x,u_{xxx})=\alpha(\phi_\epsilon' u_{xx},u_{xxx})+\alpha\|\phi_\epsilon^{\frac12}u_{xxx}\|^2$$
$$|(\phi_\epsilon' u_{xx},u_{xxx})|\le C\epsilon(\|\phi_\epsilon^{\frac12}u_{xx}\|^2+\|\phi_\epsilon^{\frac12}u_{xxx}\|^2)$$
$$(\phi_\epsilon u_{xx},(2uu_x)_{xx})=-(\phi_\epsilon' u_{xx}+\phi_\epsilon u_{xxx},2u_x^2+2uu_{xx})$$
$$|(\phi_\epsilon' u_{xx},2u_x^2+2uu_{xx})|\le C\epsilon[\|\phi_\epsilon^{\frac12}u_{xx}\|^2+\|\phi_\epsilon^{\frac12}u_x^2\|^2+\|\phi_\epsilon^{\frac12}uu_{xx}\|^2]$$
$$\le C\epsilon[\|\phi_\epsilon^{\frac12}u_{xx}\|^2+\|u_x\|_\infty^2\|\phi_\epsilon^{\frac12}u_x\|^2+\|u\|_\infty\|\phi_\epsilon^{\frac12}u_{xx}\|^2]$$
$$\le C\epsilon[\|\phi_\epsilon^{\frac12}u_{xx}\|^2+1]$$
$$|(\phi_\epsilon u_{xxx},2u_x^2+2uu_{xx})|\le|(\phi_\epsilon u_{xxx},2u_x^2)|+|(\phi_\epsilon u_{xxx},2uu_{xx})|$$
$$\le 2\|u_x\|_\infty\|\phi_\epsilon^{\frac12}u_{xxx}\|\|\phi_\epsilon^{\frac12}u_x\|+2\|u\|_\infty\|\phi_\epsilon^{\frac12}u_{xx}\|\|\phi_\epsilon^{\frac12}u_{xxx}\|$$

$$\leq \epsilon_1 \|\phi_\epsilon^{\frac{1}{2}} u_{xxx}\|^2 + C_1$$

Here the boundness of $\sup_{t\in R_+} \|u_x(\cdot,t)\|_{L_\infty(R)}$, $\sup_{t\in R_+} \|u(\cdot,t)\|_{L_\infty(R)}$ and $\sup_{t\in R_+} \|\phi_\epsilon^{\frac{1}{2}} u_x(\cdot,t)\|_{L_2(R)}$ have been used by means of Lemma 13, Lemma 12 and Lemma 23 respectively. Furthermore we have

$$\begin{aligned}
|(\phi_\epsilon u_{xx}, \beta H u_{xxxx})| &\leq |\beta||(\phi_\epsilon' u_{xx}, H u_{xxx})| + |\beta||(\phi_\epsilon u_{xxx}, H u_{xxx})| \\
&\leq C\epsilon(\|\phi_\epsilon^{\frac{1}{2}} u_{xx}\|^2 + \|\phi_\epsilon^{\frac{1}{2}} H u_{xxx}\|^2) + \frac{|\beta|}{2}(\|\phi_\epsilon^{\frac{1}{2}} u_{xxx}\|^2 + \|\phi_\epsilon^{\frac{1}{2}} H u_{xxx}\|^2) \\
&\leq (|\beta| + C\epsilon)\|\phi_\epsilon^{\frac{1}{2}} u_{xxx}\|^2 + C_1\epsilon\|\phi_\epsilon^{\frac{1}{2}} u_{xx}\|^2
\end{aligned}$$

It is similar to the proof of Lemma 23,

$$|(\phi_\epsilon u_{xx}, \delta H u_{xxx})| \leq \eta_1 \|\phi_\epsilon^{\frac{1}{2}} u_{xxx}\|^2 + C_2$$

And also

$$\begin{aligned}
(\phi_\epsilon u_{xx}, (g'(u)u_x)_x) &= (\phi_\epsilon u_{xx}, g''(u)u_x^2 + g'(u)u_{xx}) \\
&\leq -b\|\phi_\epsilon^{\frac{1}{2}} u_{xx}\|^2 + \|g''(u)\|_\infty \|u_x\|_\infty \|\phi_\epsilon^{\frac{1}{2}} u_{xx}\| \|\phi_\epsilon^{\frac{1}{2}} u_x\| \\
&\leq (-b + C\epsilon)\|\phi_\epsilon^{\frac{1}{2}} u_{xx}\|^2 + C_3 \\
|(\phi_\epsilon u_{xx}, h'')| &\leq \frac{b}{2}\|\phi_\epsilon^{\frac{1}{2}} u_{xx}\|^2 + \frac{1}{2b}\|\phi_\epsilon^{\frac{1}{2}} h''\|^2
\end{aligned}$$

Hence from (81) we obtain

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|\phi_\epsilon^{\frac{1}{2}} u_{xx}\|^2 + (\alpha - C\epsilon - \epsilon_1 - \beta - \epsilon_2 - \eta_1)\|\phi_\epsilon^{\frac{1}{2}} u_{xxx}\|^2 \\
&\quad + \Big(\frac{b}{2} - C_5\epsilon\Big)\|\phi_\epsilon^{\frac{1}{2}} u_{xx}\|^2 \leq \frac{1}{2b}\|\phi_\epsilon^{\frac{1}{2}} h''\|^2 + C_4
\end{aligned} \tag{82}$$

As ϵ, ϵ_1, ϵ_2 and η_1 are suitably small, from inequality (82) we can get (78) (79), (80).

3. Lemma 25 *Suppose that $u_1(x,0), u_2(x,0) \in H_{1,\gamma}$, $\alpha > |\beta|$, $g'(u) \leq b$, $b =$ const. and $u_1(x,t)$, $u_2(x,t)$ be any two smooth solutions of the Cauchy problem (57) (58). Then the inequality*

$$\|u_1(\cdot,t) - u_2(\cdot,t)\|_{1,\gamma}^2 \leq C_0 \|u_1(\cdot,0) - u_2(\cdot,0)\|_{1,\gamma}^2, \quad t \in [0,T] \tag{83}$$

holds, where the constant C_0 depends only on T.

Proof First, we estimate $\|u_1(\cdot,t) - u_2(\cdot,t)\|_{0,\gamma}$. Let $w(x,t) = u_1(x,t) - u_2(x,t)$, subtracting the equation (57) for u_1 and u_2, we get

$$\begin{aligned}
&(u_1 - u_2)_t - \alpha(u_1 - u_2)_{xx} + 2u_1u_{1x} - 2u_2u_{2x} + \beta H(u_1 - u_2)_{xx} + \delta H(u_1 - u_2)_x \\
&= g(u_1) - g(u_2)
\end{aligned} \tag{84}$$

Multiplying this equation by $\phi_\epsilon(u_1-u_2)$, and integrating with respect to t and x, we deduce that

$$\frac{1}{2}\frac{d}{dt}\|\phi_\epsilon^{\frac{1}{2}}w(\cdot,t)\|^2|_0^t+\alpha((\phi_\epsilon w)_x,w_x)+(\phi_\epsilon w,(2u_1u_{1x}-2u_2u_{2x}))$$
$$+(\beta Hw_{xx},\phi_\epsilon w)+(\delta Hw_x,\phi_\epsilon w)=(g(u_1)-g(u_2),\phi_\epsilon w)\le b\|\phi_\epsilon^{\frac{1}{2}}w(\cdot,t)\|^2 \quad (85)$$

Here we have

$$\alpha((\phi_\epsilon w)_x,w_x)=\alpha(\phi_\epsilon' w,w_x)+\alpha\|\phi_\epsilon^{\frac{1}{2}}w_x\|^2$$
$$|\alpha(\phi_\epsilon' w,w_x)|\le C\epsilon\|\phi_\epsilon^{\frac{1}{2}}w\|\|\phi_\epsilon^{\frac{1}{2}}w_x\|\le\epsilon\|\phi_\epsilon^{\frac{1}{2}}w_x\|^2+\frac{C^2\epsilon}{4}\|\phi_\epsilon^{\frac{1}{2}}w\|^2$$
$$2u_1u_{1x}-2u_2u_{2x}=2u_1w_x+2u_{2x}w$$
$$|(\phi_\epsilon w,2u_1w_x+2u2xw)|\le|((\phi_\epsilon u_1)_x,w^2)|+((2\phi_\epsilon u_{2x},w^2)|$$
$$\le C\epsilon\|u_1\|_\infty\|\phi_\epsilon^{\frac{1}{2}}w\|^2+\|u_{1x}\|_\infty\|\phi_\epsilon^{\frac{1}{2}}w\|^2+2\|u_{2x}\|\|\phi_\epsilon^{\frac{1}{2}}w\|^2\le C_1\|\phi_\epsilon^{\frac{1}{2}}w\|^2$$
$$|(\beta Hw_{xx},\phi_\epsilon w)|=|\beta(Hw_x,(\phi_\epsilon w)_x)|$$
$$\le|\beta|C\epsilon\|\phi_\epsilon^{\frac{1}{2}}Hw_x\|\|\phi_\epsilon^{\frac{1}{2}}w\|+|\beta|\|\phi_\epsilon^{\frac{1}{2}}w_x\|\|\phi_\epsilon^{\frac{1}{2}}Hw_x\|$$
$$\le(|\beta|+\epsilon)\|\phi_\epsilon^{\frac{1}{2}}w_x\|^2+C_2\|\phi_\epsilon^{\frac{1}{2}}w\|^2$$
$$|(\delta Hw_x,\phi_\epsilon w)|\le|\delta|\|\phi_\epsilon^{\frac{1}{2}}w\|\|\phi_\epsilon^{\frac{1}{2}}Hw_x\|\le\epsilon_1\|\phi_\epsilon^{\frac{1}{2}}w_x\|^2+C_3\|\phi_\epsilon^{\frac{1}{2}}w\|^2$$

Thus from (85) we have

$$\frac{1}{2}\frac{d}{dt}\|\phi_\epsilon^{\frac{1}{2}}w(\cdot,t)\|^2+(\alpha-2\epsilon-|\beta|-\epsilon_1)\|\phi_\epsilon^{\frac{1}{2}}w_x\|^2$$
$$\le\Big(b+\frac{C^2\epsilon}{4}+C_1+C_2+C_3\Big)\|\phi_\epsilon^{\frac{1}{2}}w\|^2$$

For small ϵ and ϵ_1, $\alpha-2\epsilon-|\beta|-\epsilon_1>0$. By use of Gronwall's inequality, we get

$$\|\phi_\epsilon^{\frac{1}{2}}w(t)\|^2\le e^{C_0't}\|\phi_\epsilon^{\frac{1}{2}}w(0)\|^2$$

where $C_0'=2\Big(b+\frac{C^2\epsilon}{4}+C_1+C_2+C_3\Big)$. Let $C_0=Ce^{C_0'T}$, we get

$$\|u_1(t)-u_2(t)\|_{0,\gamma}^2\le C_0\|u_1(0)-u_2(0)\|_{0,\gamma}^2,\quad t\in[0,T]$$

Similarly, we can get (83).

4. Lemma 26 (A.V. Babin, M.I. Vishik) [17] *Let a set B be bounded in $H_{1,\gamma+\delta}$, $\delta>0$ and in $H_{2,\gamma}$. Then B is compact in $H_{1,\gamma}$.*

Theorem 12 (R. Temam [18]) *We assume that H is a metric space and that the nonlinear operators $S(t)$ of H into itself for $t\ge0$ satisfy*

$$S(t+s)=S(t)\cdot S(s),\quad s,t\ge0,\ S(0)=I\ (\textit{Identity in } H) \qquad (86)$$

And also $S(t)$ are continuous and uniformly compact for t large, that means for every bounded set B there exists t_0, which may depend on B, such that $\bigcup_{t\geq t_0} S(t)B$ is relatively compact in H. We also assume that there exists an open set $U \subset H$ and a bounded set B of U such that B is absorbing in U. Then the ω-limit set of $B : A = \omega(B) = \bigcup_{t\geq t_0} S(t)B$ is a compact attractor, which attracts the bounded sets of U. It is the maximal bounded attractor in U. Furthermore, if H is a Banach space and U is convex and connected, then A is connected too.

By use of *a priori* integral estimates and fixed point argument, we have

Theorem 13 *Suppose that the following conditions are satisfied:*

(1) $\alpha > 0, \beta > 0, \delta \leq 0$;

(2) $g(u) \in C^m$, $g'(u) \leq b$, $b =$constant, $g(0) = 0$,

(3) $u_0(x) \in H_{m,\gamma}$, $h(x) \in H_{m,\gamma}$.

Then there exists a unique smooth solution $u(x,t)$ of the Cauchy problem (57) (58):

$$u(x,t) \in L^\infty(0,T; H_{m,\gamma}(R))$$

Theorem 14 *Suppose that the following conditions are satisfied:*

(1) $\alpha > 0, \beta > 0, \delta \leq 0, \alpha > \beta - \frac{\delta}{2}$;

(2) $g(u) \in C^2$, $g'(u) \leq -b$, $b > 0$ is a constant, $g(0) = 0$,

and

$$b + \frac{3}{2}\delta > 0;$$

(3) $u_0(x) \in H_{2,\gamma}$, $h(x) \in H_{2,\gamma}$.

Then there exists a global attractor A of Cauchy problem (57) (58), *i.e., there is a set A, such that*

(1) *A is compact in $H_{1,\gamma}$;*

(2) *$S_t A = A$, $\forall t \geq 0$;*

(3) $\lim_{t\to\infty} \operatorname{dist}(S_t B, A) = 0$, *for any bounded set $B \in H_{1,\gamma}$,*

where

$$\operatorname{dist}(x,y) = \sup_{x\in X}\inf_{y\in Y} \|x-y\|_{1,\gamma}.$$

and $S_t u_0$ is a semi-group operator generated by the Cauchy problem (57) (58).

Proof On account of the result of Theorem 12, we shall prove this theorem by checking the conditions in Theorem 12. Under the assumptions of the theorem, we know that exists a semi-group operator generated by the Cauchy problem (57) (58). Thus we set the Banach space $H = H_{1,\gamma}$ and $S_t : H_{1,\gamma} \to H_{1,\gamma}$. From Lemma 25, it is a continuous (nonlinear) operator from $H_{1,\gamma}$ into itself. Let

$$B = \{u(\cdot,t) \in H_{1,\gamma}(R), \|u\|_{1,\gamma} \leq 2E_1\}$$

be the bounded absorbing set of the semi-group operator S_t from Lemma 22. $B_1 = S_t B$ is bounded in $H_{2,\gamma}$ by Lemma 24. From Lemma 23 and Lemma 37, the absorbing set B_1 is compact in $H_{1,\gamma}$. Hence the ω-limit set of $B, A = \omega(B)$ is a compact attractor for the Cauchy problem (57), (58).

10. Dimensions of Global Attractors

In order to establish the upper bounds of Hausdorff and fractal dimensions for the global attractor of the Cauchy problem (57) (58), we need the following linear variation corresponding to the problem (57) (58)

$$v_t + L(u(t))v = 0 \tag{87}$$

$$v(0) = v_0 \tag{88}$$

where

$$L(u(t))v = -\alpha v_{xx} + 2uv_x + 2u_x v + \beta H v_{xx} + \delta H v_x - g'(u)v \tag{89}$$

Since the solution of problem (57) (58) is sufficiently smooth, we can easily prove that the linear problem (87) and (88) has a global smooth solution as long as the initial data mildly smooth, i.e., there is a solution operator G_t such that $v(t) = G_t v_0$.

It can be verified that the semi-group operator S_t for any $t \geq 0$ is uniformly differentiable on A in the metric of $H_{0,0} = L_2$. Namely, the Frechet derivative $S_t'(u_0)$ exists, and $G_t v_0 = S_t'(u_0)v_0$. In fact, we set

$$w(t) = S_t(u_0 + v_0) - S_t(u_0) - G_t(u_0)v_0 = u_1(t) - u(t) - v(t)$$

Thus we have

$$\begin{aligned} \partial_t w(t) &= L_1(u_1(t)) - L_1(u(t)) + L(u(t))v(t) \\ &= L_1(u(t) + v(t) + w(t)) - L_1(u(t)) + L(u(t))v(t) \end{aligned} \tag{90}$$

$$w(0) = 0 \tag{91}$$

where $u_t = L_1(u)$ is the operator form of the equation (57).

Therefore, (90) can be rewritten in the form

$$\partial_t w + L(u(t))w = \Lambda_0(u, v, w) \tag{92}$$

where

$$\Lambda_0(u, v, w) = L_1(u(t) + v(t) + w(t)) - L_1(u(t)) + L(u(t))(v(t) + w(t)) \tag{93}$$

By applying the theory of linear differential equations, we have the following L_2 estimate

$$\|w(t)\| \leq C\|v_0\|^2 \tag{94}$$

This implies the semi-group operator S_t can be differentiated in $L_2(R)$.

Denote by $v_1(t), v_2(t), \cdots, v_J(t)$ the solution of linear equation (87) corresponding respectively to the initial data $v_1(0) = \xi_1, \cdots, v_J(0) = \xi_J$, here $\xi = (\xi_1, \xi_2, \cdots, \xi_J) \in L_2$, and by the simple computation [19], we can deduce that

$$\frac{d}{dt}\|v_1(t) \wedge \cdots \wedge v_J(t)\|^2 + 2Tr(L(u(t))Q_J)\|v_1(t) \wedge \cdots \wedge v_J(t)\|^2 = 0 \tag{95}$$

where $L(u(t)) = L(S_t u_0)$ is a linear map: $v \to L(u(t))v$, "$\wedge$" denotes the exterior product, Tr the trace of operator, and $Q_J(t)$ the orthographic projection of space $L_2(R)$ to the spanning subspace generated by $v_1(t), \cdots, v_J(t)$. Therefore from (95) we can obtain the change of the volume $\bigwedge_{j=0}^{J} \xi$ of the J dimensional cube by

$$\begin{aligned}\omega_J(t) &= \sup_{u_0 \in A} \sup_{\xi_j \in L_2, \xi_j \le 1} \|v_1(t) \wedge \cdots \wedge v_J(t)\|^2_{\wedge^j L_2} \\ &\le \sup_{u_0 \in A} \exp\Big(-\int_0^t \inf(\operatorname{Tr} L(S_\tau u_0), Q_j(\tau))d\tau \qquad (96)\end{aligned}$$

Noticing the result in [19], we know that ω_j is sub-exponented with respect to t, i.e.

$$\omega_j(t+t') \le \omega_j(t)\omega_j(t'), \quad t, t' \ge 0 \qquad (97)$$

Hence we have

$$\lim_{t\to\infty} \omega_j(t)^{\frac{1}{t}} = \pi_j, \quad j = 1, 2, \cdots, J \qquad (98)$$

and

$$\pi_j \le \exp(-q_J) \qquad (99)$$

where

$$q_J = \overline{\lim_{t\to\infty}} \Big(\inf \frac{1}{t}\int_0^t \inf(\operatorname{Tr} L(S_\tau u_0)\cdot Q_J(\tau)\Big)d\tau \qquad (100)$$

Definition 1 The Hausdorff measure of a set X is defined by

$$n_H(X,d) = \lim_{\epsilon\to 0} n_H(X,d,\epsilon) = \sup_{\epsilon>0} n_H(X,d,\epsilon)$$

where

$$n_H(X,d,\epsilon) = \inf \sum_i r_i^d$$

the infimum takes over the balls that cover the set X with the radius $r_i \le \epsilon$. The Hausdorff dimension of the set X is defined by a number $d_H(X) \in [0,\infty)$, such that

$$n_H(X,d) = 0, \text{ for } d > d_H(X)$$

and

$$n_H(X,d) = \infty, \text{ for } d < d_H(X)$$

Definition 2 The fractal dimension is defined by the number

$$d_F(X) = \lim_{\epsilon\to 0}\sup \frac{\log n_X(\epsilon)}{\log(1/\epsilon)}$$

where $n_X(\epsilon)$ denotes the smallest number of the balls that cover the set X with their radiuses less than or equal to ϵ.

From the results of [18], we see that

$$d_F(X) = \inf\{d > 0, n_F(X, d) = 0\}$$

where

$$n_F(X, d) = \lim_{\epsilon \to 0} \sup (\epsilon^d n_X(\epsilon))$$

Since $n_F(X, d) \geq n_H(X, d)$, we have

$$d_H(X) \leq d_F(X)$$

Theorem 15 [19] *Let A be the attractor of nonlinear evolution equation (such as Navier Stokes equation, Equation (57) etc.) that is bounded in $H^1(R)$. Then if $q_J > 0$ for some J, the Hausdorff dimension of X is no more than J, and its fractal dimension is less than or equal to*

$$J\Big(H \max_{1 \leq l \leq J} \frac{-q_l}{q_J}\Big) \tag{101}$$

Lemma 27 (Sobolev-Lieb-Thiring Inequality, [18]) *Let ϕ_j, $1 \leq j \leq N$ be a finite family of $H^m(R^n)$ which is orthonormal in $L_2(R^n)$ and set, for almost every $x \in R^n$*

$$\rho(x) = \sum_{j=1}^{N} |\phi_j(x)|^2$$

Then there exists a constant k_0 such that

$$\Big(\int_{R^n} \rho(x)^{p/(p-1)} dx\Big)^{2m(p-1)/n} \leq k_0 \sum_{j=1}^{N} \int_{R^n} |D^m \phi_j|^2 dx \tag{102}$$

where the constant k_0 depends on m, n, p and is independent of the ϕ_j and N.

Theorem 16 *Under the conditions of Theorem 14, the Hausdorff and fractal dimensions of the global attractor A of the Cauchy problem (57) (58) are finite, and*

$$d_H(A) \leq J_0, \quad d_F(A) \leq 2J_0 \tag{103}$$

where J_0 is the smallest integer, such that

$$J_0 \geq \Big(b - \frac{\delta^2}{2\alpha}\Big)^{-1} \alpha^{-1/2} \Big(\frac{2^{3/2}}{3^{3/2}} k_0^{2/3} \|u_x\|_{L_{3/2}}^{3/2}\Big) \tag{104}$$

Proof Suppose that $\{\phi_1, \cdots, \phi_J\}$ is an orthogonal basis of the subspace $Q_J L_2(R)$, we have

$$\begin{aligned}
&\mathrm{Tr}\,(L(u(t))Q_J) \\
&= \sum_{j=1}^{J} \{(-\alpha\phi_{jxx} + 2u\phi_{jx} + 2u_x\phi_j + \beta H\phi_{jxx} + \delta H\phi_{jx} - g'(u)\phi_j, \phi_j)\}
\end{aligned}$$

$$= \sum_{j=1}^{J} \{\alpha\|\phi_{jx}\|^2 - (u_x, \phi_j^2) + 2(u_x, \phi_j^2) + \delta(H\phi_{jx}, \phi_j) - (g'(u)\phi_j, \phi_j)\}$$

By Sobolev-Lieb-Thiring inequality (102) it follows

$$\int_R \rho^3(x)dx \le k_0 \sum_{j=1}^{J} \int_R |\phi_{jx}|^2 dx$$

Hence we have

$$\sum_{j=1}^{J} (u_x, \phi_j^2) = (u_x, \rho(x)) \le \|u_x\|_{L_{3/2}} \|\rho\|_{L_3}$$

$$\le \|u_x\|_{L_{3/2}} \Big(k_3 \sum_{j=1}^{J} \int |\phi_{jx}|^2 dx \Big)^{1/3} \le k_0^{1/3} \|u_x\|_{L_{3/2}} \Big(\sum_{j=1}^{J} \|\phi_{jx}\|^2 \Big)^{1/3}$$

And also

$$\Big| \sum_{j=1}^{J} (\delta H\phi_{jx}, \phi_j) \Big| = \Big| \sum_{j=1}^{J} (\delta H\phi_j, \phi_{jx}) \Big| \le \delta \sum_{j=1}^{J} \|\phi_{jx}\| \|H\phi_j\|$$

$$= \delta \sum_{j=1}^{J} \|\phi_{jx}\| \|\phi_j\| \le \frac{\alpha}{2} \sum_{j=1}^{J} \|\phi_{jx}\|^2 + \frac{\delta^2}{2\alpha} J$$

where $\|\phi_j\|^2 = 1$ for $j = 1, \cdots, J$. Hence we get

$$\mathrm{Tr}\,(L(u(t))Q_J) \ge \frac{\alpha}{2} \sum_{j=1}^{J} \|\phi_{jx}\|^2 - k_0^{1/3} \|u_x\|_{L_{3/2}} \Big(\sum_{j=1}^{J} \|\phi_{jx}\|^2 \Big)^{1/3} + \Big(b - \frac{\delta^2}{2\alpha}\Big) J$$

Let $s = \Big(\sum_{j=1}^{J} \|\phi_{jx}\|^2 \Big)^{1/3}$, $f(s) = \frac{\alpha}{2} s^3 - k_0^{1/3} \|u_x\|_{L_{3/2}} s + \Big(b - \frac{\delta^2}{2\alpha}\Big) J$.

For $f'(s) = \frac{3\alpha}{2} s^2 - k_0^{1/3} \|u_x\|_{L_{3/2}} = 0$, there is $s_m = \frac{2^{1/2} k_0^{1/6}}{(3\alpha)^{1/2}} \|u_x\|_{L_{3/2}}^{1/2}$.

Let us choose J, such that

$$f(s_m) = \frac{\alpha}{2} s_m^3 - k_0^{1/3} \|u_x\|_{L_{3/2}} s_m + \Big(b - \frac{\delta^2}{2\alpha}\Big) J > 0$$

This gives

$$\Big(b - \frac{\delta^2}{2\alpha}\Big) J > \frac{2^{3/2} k_0^{2/3}}{3^{3/2} \alpha^{1/2}} \|u_x\|_{L_{3/2}}^{3/2}$$

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