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HYPERBOLIC PHENOMENA IN A DEGENERATE PARABOLIC EQUATION

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Abstract M. Bertsch and R. Dal Passo [1] considered the equation $u_t = (\varphi(u)\psi(u_x))_x$, where $\varphi > 0$ and ψ is a strictly increasing function with $\lim_{s\to\infty}\psi(s) = \psi_\infty < \infty$. They have solved the associated Cauchy problem for an increasing initial function. Furthermore, they discussed to what extent the solution behaves like the solution of the first order conservation law $u_t = \psi_\infty(\varphi(u))_x$. The condition $\varphi > 0$ is essential in their paper. In the present paper, we study the above equation under the degenerate condition $\varphi(0) = 0$. The solution also possesses some hyperbolic phenomena like those pointed out in [1].

Key Words Degenerate parabolic equation; entropy condition.

Classification 35K65.

1. Introduction

We consider the problem

$$\text{(I)} \qquad \begin{cases} u_t = (\varphi(u)\psi(u_x))_x, & x \in R, t \in (0,\infty) \\ u(x,0) = u_0(x), & x \in R \end{cases}$$

where $\varphi : R^+ \to R^+$ is smooth, $\varphi \in C[0,+\infty)$, $\varphi(0) = 0$, $\varphi'(s) > 0$ $(s > 0)$ and $\lim_{s\to 0}+\frac{s}{\varphi(s)} = 0$. $\psi : R \to R$ is a smooth, odd function such that $\psi' > 0$ in R and $\lim_{s\to+\infty}\psi(s) = \psi_\infty$.

The initial function $u_0 : R \to R$ is bounded, strictly increasing and

$$\lim_{x\to-\infty} u_0(x) = 0, \quad \lim_{x\to+\infty} u_0(x) = A \tag{1}$$

$$u_0'(x) = O(u_0(x)) \quad \text{as } x \to -\infty \tag{2}$$

$$u_0'(x) = O(A - u_0(x)) \quad \text{as } x \to +\infty \tag{3}$$

For the construction of a solution we use a standard parabolic regularization: Let $\varepsilon > 0$ and u_ε be the unique smooth solution of the problem

$$(\mathrm{I}_\varepsilon)\qquad \begin{cases} u_t = (\varphi_\varepsilon(u)\psi_\varepsilon(u_x))_x, & x \in R, t \in (0,\infty) \\ u(x,0) = u_{0\varepsilon}(x), & x \in R \end{cases}$$

where u_ε is a smooth approximation of u_0 and

$$\varphi_\varepsilon(s) = \varphi(s+\varepsilon), \quad s \in [0,\infty) \tag{4}$$

$$\psi_\varepsilon(s) = \psi(s) + \varepsilon s, \quad s \in R \tag{5}$$

We shall show that

$$u_{\varepsilon_i} \to u \quad \text{in } L^1_{\text{loc}}(R \times [0,\infty)) \quad as\ i \to \infty \tag{6}$$

for some sequence $\varepsilon_i \to 0$ as $i \to \infty$, and

$$u \in L^\infty(R \times R^+) \cap BV_{\text{loc}}(R \times [0,\infty))$$

The main results of this paper can be stated as:

If u_0 is strictly increasing and satisfies (1)–(3), then

(i) u (defined by (6)) is a solution of Problem I.

(ii) u is not necessary to be continuous, even if u_0 is continuous.

(iii) $\psi(u_x)$ is a continuous function (under the convention that $\psi(\infty) = \lim\limits_{s\to\infty} \psi(s) = \psi_\infty$).

(iv) u satisfies an entropy-type condition: if at some point $(x,t) \in R \times R^+$

$$u^+ = u(x^+,t) > u^- = u(x^-;t)$$

then

$$\varphi(s) \le \frac{\varphi(u^+) - \varphi(u^-)}{u^+ - u^-}(s - u^-) + \varphi(u^-) \quad \text{for } s \in [u^-, u^+]$$

(v) The entropy condition is necessary for uniqueness of solutions, i.e., there may exist solutions which do not satisfy the entropy condition.

(vi) Let $C_1 \le u(x,t) \le C_2$ for $(x,t) \in D = (x_1,x_2) \times (t_1,t_2)$ for some $C_1 \le C_2$, $x_1 < x_2$, $0 < t_1 < t_2$. Moreover, if φ is strictly concave in $[C_1, C_2]$, then $u_x \in L^\infty_{\text{loc}}(D)$.

The most striking results are the points (ii), (iv) and (v) which show the hyperbolic character of Problem I. These results are, however, expectable. Because the parabolicity of equation in Problem I is so weak for $u_x \to \infty$ that the solution may become discontinuous and behave like the solution of first-order equation $u_t = \psi_\infty(\varphi(u))_x$ (i.e., the discontinuity satisfies Rankine-Hugoniot condition and the entropy condition).

In order to prove the result which concerns the behaviour of the level curves ("characteristic") of u near a shock front, we need a further assumption

$$\psi'(s) \ge cs^{-2} \quad \text{for } s \le s_0 > 0 \tag{7}$$

The proof of the above results is based on the technique used in [1], by a clever coordinate transformation which makes full use of the special features on u_0 and $\varphi(s)$ (as $s \to 0^+$). We concentrate our attention on the transformed elliptic-parabolic equation. After a detail discussion about the properties of this equation, we get the results of this paper.

2. Main Conditions and Results

First we list the precise hypotheses on our data and the definition of "solution".

H$_1$. $\psi \in C^2(R) \cap L^\infty(R)$, $0 < \psi' \le c_1$ in R for some $c_1 > 0$, $\psi(0) = 0$, $\psi(s) \to \psi_\infty$ as $s \to \infty$.

H$_2$. $\varphi \in C^3(0,\infty) \cap C[0,\infty)$, $\varphi(0) = 0$, $\varphi'(s) > 0$ (for $s > 0$) and $\lim_{s\to 0+} \frac{s}{\varphi(s)} = 0$.

H$_3$. $u_0 \in L^\infty(R)$, u_0 is strictly increasing in R, $\lim_{x\to -\infty} u_0 = 0$, $\lim_{x\to+\infty} u_0 = A$ and

$$u_0'(x) = O(u_0(x)) \quad \text{as } x \to -\infty$$

$$u_0'(x) = O(A - u_0(x)) \quad \text{as } x \to +\infty$$

$\psi(u_0') \in C(R)$ (where $\psi(u_0'(x_0)) = \psi_\infty$ if u_0 is discontinuous at x_0), $\psi(u_0') > 0$ in R and $\psi(u_0'(x)) \to 0$ as $x \to \pm\infty$.

Since we have to solve the approximate problem I_ε $(0 < \varepsilon \le 1)$, we should modify $u_{0\varepsilon}$. According to H_3, we can choose $u_{0\varepsilon}$ satisfying the following conditions:

H$_4$. (i) $u_{0\varepsilon} \in C^\infty(R) \cap W^{3,\infty}(R)$, $u_{0\varepsilon} > 0$ in R. $u_{0\varepsilon} \to u_0$ in $L^1_{\rm loc}(R)$ as $\varepsilon \to 0$.

(ii) $\lim_{x\to\pm\infty} u_{0\varepsilon}(x) = \lim_{x\to\pm\infty} u_0(x)$, $\lim_{x\to\pm\infty} u_{0\varepsilon}'(x) = 0$.

(iii) $u_0(x-1) \le u_{0\varepsilon}(x) \le u_0(x+1)$ in R.

(iv) $\psi_\varepsilon(u_{0\varepsilon}) \le c$ for some $c > 0$.

(v) As $x \to \infty$, $u_{0\varepsilon}'(x) = O(u_{0\varepsilon}(x))$ uniformly with respect to ε; as $x \to +\infty$, $u_{0\varepsilon}'(x) = O(A - u_{0\varepsilon}(x))$ uniformly with respect to ε, and $\psi_\varepsilon(u_{0\varepsilon}') \to \psi(u_0')$ in $C_{\rm loc}(R)$ as $\varepsilon \searrow 0$.

(vi) $\psi_\varepsilon(u_{0\varepsilon}')$ has uniformly positive lower bound in any finite interval.

Definition 2.1 *A function $u \in BV_{\rm loc}(R\times[0,\infty)) \cap L^\infty(R\times R^+)$ is called a solution for Problem* I *if*

(i) For any $t \ge 0$, $u(.,t) \in BV_{\rm loc}(R)$ and there exists a continuous function $\overline{\psi}$: $R\times[0,\infty) \to R$ such that

$$\overline{\psi}(x,t) = \lim_{h\to 0} \psi\Big(\frac{u(x-h,t) - u(x^-,t)}{h}\Big)$$

$$= \lim_{h\to 0} \psi\Big(\frac{u(x+h,t) - u(x^+,t)}{h}\Big)$$

for any $x \in R$ *and* $t \geq 0$.

(ii) *For any* $\chi \in C^1(R \times [0,\infty))$ *with compact support*

$$\iint_{R\times R^+} (u\chi_t - \varphi(u)\overline{\psi}\chi_x)dxdt = -\int_R \chi(x,0)u_0(x)dx$$

Definition 2.2 (Definition of the entropy condition) *We say that a solution* u *of Problem* I *satisfies the entropy condition* (E) *if at any point* $(x,t) \in R \times [0,\infty)$ *in which* u *is discontinuous with respect to* x

$$\varphi(s) \leq \frac{\varphi(u^+) - \varphi(u^-)}{u^+ - u^-}(s - u^-) + \varphi(u^-)$$

for any s *between* u^- *and* u^+, *where* $u^{\pm} = u(x\pm, t)$.

Now we are ready to state our main results.

Theorem 2.1 (Existence) *Under Hypotheses* H_1–H_3, *let* u_ε *be the solution for Problem* I_ε *for any* $\varepsilon > 0$. *Then there exists a sequence* $\varepsilon_i \to 0$ *as* $i \to \infty$ *and there exists a function* $u \in L^\infty(R\times R^+) \cap BV_{\mathrm{loc}}(R\times[0,\infty))$ *such that* $u_{\varepsilon_i} \to u$ *in* $L^1_{\mathrm{loc}}(R\times[0,\infty))$ *as* $i \to \infty$, *and* u *is a solution of Problem* I *which satisfies the entropy condition* (E).

Theorem 2.2 (Nonuniqueness) *Let* ψ *satisfy* H_1. *Then there exist functions* φ *and* u_0 *which satisfy hypotheses* H_2, H_3 *such that Problem* I *has at least one solution which does not satisfy the entropy condition* (E).

Theorem 2.3 (Regularity of solution) *Let hypotheses* H_1–H_3 *be satisfied and* u *be defined by Theorem* 2.1.

(i) *if* $\varphi' \not\leq 0$ *in* R, *then* u *is not necessarily continuous, even if* $u_0 \in C^2(R)$.

(ii) *if* $C_1 \leq u \leq C_2$ *in* $D \equiv (x_1,x_2)\times(t_1,t_2)$ *where* $x_1 < x_2$, $0 < t_1 < t_2$, *and if*

$$\varphi \text{ is strictly concave in } [C_1, C_2]$$

then $u \in C_{2,1}(D)$.

Theorem 2.4 (Behaviour near shock fronts) *Under conditions* H_1–H_3 *and* (1.7), *and* φ *be strictly convex. Let* $\xi : (t_0,t_1) \to (x_0,x_1)$ *be a continuous function* ($x_0 < x_1, 0 < t_0 < t_1$). *If the solution* u *of Problem* I *defined by Theorem* 2.1 *satisfies*

(i) u *is discontinuous at the points* $(\xi(t),t)$, $t \in (t_0,t_1) : u^+(t) - u^-(t) \geq \delta > 0$ *for* $t \in (t_0,t_1)$, *where* $u^{\pm}(t) \equiv u(\xi^{\pm}(t),t)$.

(ii) u *is continuous at the points* $(x,t) \in (x_0,x_1)\times(t_0,t_1)$ *for* $x \neq \xi(t)$.

Then $\xi \in W^{1,\infty}(t_0,t_1)$ *with*

$$\xi'(t) = -\psi_\infty \frac{\varphi(u^+) - \varphi(u^-)}{u^+ - u^-} \quad \textit{for a.e. } t \in (t_0,t_1)$$

and for a.e. $t \in (t_0,t_1)$

$$\frac{u_t(x,t)}{u_x(x,t)} \to -\xi'(t) \quad \textit{as } x \to \xi(t)$$

3. The Transformation of Equation

We shall use the technique in [1] to transform (I) to an elliptic-parabolic problem. In order to determine the boundary value of the transformed problem, we should first prove the following lemma.

Lemma 3.1 *Let H_1–H_4 be satisfied and $K \subset R \times [0, \infty)$ be compact. u_ε is the solution of Problem I_ε ($\varepsilon \in (0,1]$). Then there exist constants a and b which do not depend on ε such that*

$$0 < a \le u_\varepsilon(x\ t) \le b \tag{8}$$

for any $(x,t) \in K$ and $\varepsilon \in (0,1]$.

Proof of Lemma 3.1 Since K is compact in $R \times [0,\infty)$, we can take $x_1 \in R$ such that for any $(x,t) \in K : x \ge x_1 + 1$.

We define

$$\underline{u}(x,t) = \beta[1 - e^{-\alpha\frac{(x-x_1)^2}{t+1}}]$$

where β and α are constants to be determined later.

After a straightforward calculation, we get

$$\underline{u}_t - (\varphi_\varepsilon(\underline{u})\psi_\varepsilon(\underline{u}_x))_x = \Big[- \beta\alpha\frac{(x-x_1)^2}{(t+1)^2} - \varphi'_\varepsilon\psi_\varepsilon\beta\alpha\frac{x-x_1}{t+1}$$
$$+ \varphi_\varepsilon\psi'_\varepsilon\beta\alpha^2\frac{4(x-x_1)^2}{(t+1)^2} - \varphi_\varepsilon\psi'_\varepsilon\frac{2\beta\alpha}{t+1}\Big] e^{-\alpha\frac{(x-x_1)^2}{t+1}}$$

By H_1–H_2 and (4)–(5), we can choose the positive constants β and α so small that for all ε

$$\begin{cases} \underline{u}_t - (\varphi_\varepsilon(\underline{u}_x))_x \le 0 & \text{for } x > x_1, t > 0 \\ \underline{u}(x,t) \le \underline{u}_{0\varepsilon}(x) & \text{for } x > x_1 \\ \underline{u}(x_1,t) = 0 & \text{for } t > 0 \end{cases}$$

In fact we can choose $\beta = u_0(x_1 - 1)$ with $\beta \le u_{0\varepsilon}(x_1)$ and choose $\alpha \le \dfrac{1}{AM(c_1+1)}$ where

$$M = \max\{\varphi(s), 0 \le s \le A+1\} \tag{9}$$

Since $u_\varepsilon \ge 0$, from the Maximum Principle we find that $u_\varepsilon(x,t) \ge u(x,t)$. So we choose

$$a = \min\{\underline{u}(x,t), (x,t) \in K\}$$

The proof of the upper bound is similar.

Let $\varepsilon > 0$ and u_ε be the unique bounded, classical solution of Problem I_ε. Since $u'_{0\varepsilon} > 0$ in R, it follows from Maximum Principle that

$$u_{\varepsilon x} > 0 \quad \text{in } R \times R^+ \tag{10}$$

Thus for any $t \in [0,\infty)$, $\lim\limits_{x\to\pm\infty} u_\varepsilon(x,t)$ exists and by the proof of Lemma 3.1 we can get the following corollary:

Corollary 3.2 *Let u_ε be the solution of* I_ε ($\varepsilon \in (0,1]$). *Then for any* $t > 0$, $\lim\limits_{x\to-\infty} u_\varepsilon(x,t) = 0$, $\lim\limits_{x\to+\infty} u_\varepsilon(x,t) = A$.

Proof In fact, by the proof of Lemma 3.1, for any $(x_0,t) \in R\times R^+$ there exists a constant α such that

$$u_\varepsilon(x,t) \geq u_0(x_0-1)(1-e^{-\alpha\frac{(x-x_0)^2}{t+1}}) \quad \text{for } x > x_0, \quad \varepsilon \in (0,1]$$

Let $x \to +\infty$ then

$$\lim_{x\to+\infty} u_\varepsilon(x,t) \geq u_0(x_0-1)$$

By (8) and H_3, we get

$$\lim_{x\to+\infty} u_\varepsilon(x,t) = \lim_{x\to+\infty} u_0(x) = A$$

Similarly,

$$\lim_{x\to-\infty} u_\varepsilon(x,t) = \lim_{x\to-\infty} u_0(x) = 0$$

Inequality (10) guarantees the following coordinate transformation $(x,t) \to (y,t)$ by

$$y = u_\varepsilon(x,t)$$

The conclusion of Corollary 3.2 implies $0 < y < A$.

We define

$$v_\varepsilon(y,t) = \psi_\varepsilon(u_{\varepsilon x}(x,t)) \quad \text{for } 0 < y < A, \quad t > 0 \tag{11}$$

Since, by H_4 $u'_{0\varepsilon} \to 0$ as $|x| \to \infty$, it follows from a standard barrier argument that

$$u_{\varepsilon x}(x,t) \to 0 \quad as\ |x| \to \infty, \quad t > 0$$

and thus that

$$v_\varepsilon(y,t) \to 0 \quad as\ y \searrow 0, \quad and\ y \nearrow A$$

We derive the equation for v_ε from I_ε and (11):

$$v_t = \psi'_\varepsilon(u_{\varepsilon x})(u_{\varepsilon x})^2(\varphi_\varepsilon(u_\varepsilon)v)_{yy}$$

Define

$$c_\varepsilon(v) = -\int_v^\infty \frac{1}{\psi'_\varepsilon(\psi_\varepsilon^{-1}(s))(\psi_\varepsilon^{-1}(s))^2}ds = -\frac{1}{\psi_\varepsilon^{-1}(v)} \quad \text{for } v > 0 \tag{12}$$

We find that

$$c'_\varepsilon(v_\varepsilon) = \frac{1}{u_{\varepsilon x}^2\psi'_\varepsilon(u_{\varepsilon x})} \tag{13}$$

and hence $v_\varepsilon(y,t)$ satisfies the equation $c_\varepsilon(v_\varepsilon)_t = (\varphi v_\varepsilon)_{yy}$. Let

$$w_{0\varepsilon}(y) = c_\varepsilon(\psi_\varepsilon(u'_{0\varepsilon}(x)) \tag{14}$$

then

$$c_\varepsilon(v)_t = (\varphi_\varepsilon(y)v)_{yy} \tag{15}$$

we see that $v_\varepsilon(y,t)$ is a solution for the problem

$$(\mathrm{II}_\varepsilon)\qquad \begin{cases} c_\varepsilon(v)_t = (\varphi_\varepsilon(y)v)_{yy} & \text{in } (0,A)\times R^+ \\ v(0,t) = v(A,t) = 0 & \text{for } t>0 \\ c_\varepsilon(v(y,0)) = w_{0\varepsilon}(y) & \text{for } 0<y<A \end{cases}$$

Equation (15) is a parabolic equation which degenerates at $v=0$ and $v=\infty$.

$$c'_\varepsilon(s)\to\infty \quad \text{as } s\to 0$$
$$c'_\varepsilon(s)\to 0 \quad \text{as } s\to\infty$$

According to (5) and H_1, it follows that if $s>\psi_\infty$, then $\psi_\varepsilon^{-1}(s)\to\infty$ as $\varepsilon\searrow 0$. Hence, by (12)

$$c_\varepsilon\to 0 \quad \text{in } C[\psi_\infty,\infty) \quad \text{as } \varepsilon\to 0$$

We define

$$c(s) = \lim_{\varepsilon\to 0} c_\varepsilon(s) \quad \text{for } s>0$$

i.e.

$$c(s) = \begin{cases} -\dfrac{1}{\psi^{-1}(s)} & \text{for } s\in[0,\psi_\infty) \\ 0 & \text{for } s\ge\psi_\infty \end{cases} \tag{16}$$

By (12), (16) it can be seen that

$$c\in C(R^+)\cap C^2(0,\psi_\infty), \quad c_\varepsilon\in C^2(R^+) \quad \text{for } \varepsilon\in(0,1]$$

and

$$c_\varepsilon\to c \quad \text{in } C_{\mathrm{loc}}(R^+)\cap C^2_{\mathrm{loc}}(0,\psi_\infty)$$

Naturally, we discuss the limit problem as $\varepsilon\to 0$:

$$(\mathrm{II})\qquad \begin{cases} c(v)_t = (\varphi(y)v)_{yy} & \text{in } (0,A)\times R^+ \\ v(0,t) = v(A,t) = 0 & \text{for } t>0 \\ c(v(y,0)) = w_0(y) & \text{for } 0<y<A \end{cases}$$

where $w_0(y) = \lim_{\varepsilon\searrow 0} w_{0\varepsilon}(y)$ $(0<y<A)$.

Observe that the equation of Problem II is of parabolic type if $c'>0$, i.e., if $v<\psi_\infty$, and of elliptic type if $c'=0$, i.e., if $v>\psi_\infty$, and $\varphi(y)=0$ on the boundary $y=0$.

It is needed to define what we mean by a solution of Problem II.

Definition *A solution $v : (0,A)\times[0,\infty)\to R$ is called to be a solution of Problem II if*

(i) $v\in L^2(0,T;H^1_{\rm loc}(0,A))\cap L^\infty((0,A)\times(0,T))$ *for all* $T>0$.

(ii) $v>0$ *a.e. in* $(0,A)\times R^+$ *and* $c(v)\in L^\infty_{\rm loc}((0,A)\times(0,\infty))$.

(iii) *for all* $T>0$, *the function* $q_T(y)=\int_0^T v(y,t)dt$ $(0<y<A)$ *satisfies* $q_T(y)\to 0$ *as* $y\searrow 0$ *and* $y\nearrow A$.

(iv) *For any* $\chi\in H^1((0,A)\times(0,\infty))\cap C([0,A]\times[0,\infty))$ *with compact support in* $(0,A)\times[0,\infty)$

$$\iint_{(0,A)\times R^+}(c(v)\chi_t-(\varphi v)_y\chi_y)dydt=-\int_0^A\chi(y,0)w_0(y)dy$$

Remark By virtue of Hölder's inequality, $q_T\in H^1_{\rm loc}(0,A)$. Hence $q_T\in C(0,A)$.

The main results of Problem $(\mathrm{II}_\varepsilon)$ *and* II.

Theorem 3.3 *Under hypotheses H_1–H_4, let v_ε be the solution of Problem II_ε. Then there exists a sequence $\varepsilon_i\to 0$ as $i\to\infty$ and a function $v\in L^\infty_{\rm loc}(0,\infty;H^1_{\rm loc}(0,A))$ such that $v_{\varepsilon_i}\to v$ weakly in $L^2(0,T;H^1_{\rm loc}(0,A))$ as $i\to\infty$, where v is a solution of Problem* II. *In addition, the following assertions are valid.*

(i) *The functions $c_\varepsilon(v_\varepsilon)$ are locally equicontinuous in $(0,A)\times[0,\infty)$.*

(ii) *For any $T>0$ there exist functions $g_1(y;T)$ and $g_2(y;T)$ which are continuous in $[0,A]$, vanish at $y=0$ and $y=A$, and are strictly positive in $(0,A)$ such that*

$$g_1(y;T)\le v_\varepsilon(y,t)\le g_2(y;T)\quad for\ y\in[0,A],\quad t\in[0,T]$$

(iii) *For some $\delta_1>0$ and $\varepsilon\in(0,1]$, and for some compact set $K\subset(0,A)\times(0,\infty)$ let*

$$K_\delta=\{(y,t)\in K:v_\varepsilon(y,t)<\psi_\infty-\delta\}$$

Then $\|v_\varepsilon\|_{C^{2,1}}(K_{\delta_1})<C$ for some C which depends only on δ_1 and K.

(iv) *$\{v_\varepsilon\}$ is uniformly bounded in $(0,A)\times R^+$.*

Theorem 3.4 *The solution v of Problem* II *defined by Theorem* 3.3 *has the following properties:*

(i) *$v\in L^\infty_{\rm loc}((0,\infty);H^1_{\rm loc}(0,A))$ and thus v is locally Hölder continuous with respect to y in $(0,A)\times[0,\infty)$;*

(ii) *$c(v)$ is continuous in $(0,A)\times[0,\infty)$;*

(iii) *v is a classical solution in the set $D=\{(y,t)\in(0,A)\times R^+:c(v(y,t))<0\}$.*

(iv) *v satisfies "the entropy condition": If for some $t_0>0$, $c(v(y,t_0))\equiv 0$ $(y_0\le y\le y_1)$, and if for any $\delta>0$, $c(v(.,t_0))\not\equiv 0$ in $(y_0-\delta,y_0)$ and in $(y_1,y_1+\delta)$, then*

$$\varphi(y)\le\frac{\varphi(y_1)-\varphi(y_0)}{y_1-y_0}(y-y_0)+\varphi(y_0)\quad for\ y_0\le y\le y_1$$

(v) $v \in C(([0,\sigma) \cup (A-\sigma, A)) \times R^+)$ *for some* $\sigma > 0$.

In order to prove Theorem 3.3–3.4, we need the following lemma.

Lemma 3.5 *Let H_1–H_4 be satisfied. Then Problem II_ε has a unique classical solution*

$$v_\varepsilon \in C([0,A] \times [0,\infty)) \cap C^{2,1}((0,A) \times (0,\infty)) \quad \text{for } \varepsilon \in (0,1]$$

Moreover, v_ε has the following properties:

(i) *v_ε is uniformly bounded in $[0,A] \times R^+$;*

(ii) *for any $T > 0$ there exists a function $g_1(\cdot;T) \in C[0,A]$ which is strictly positive in $(0,A)$ such that for any $\varepsilon \in (0,1]$.*

$$v_\varepsilon(y,t) \ge g_1(y;t) \quad \text{for } y \in [0,A], \quad t \in [0,T]$$

(iii) *There exists a function $g_2 \in C([0,A])$ such that*

$$g_2(0) = g_2(A) = 0 \quad \text{and } v_\varepsilon \le g_2 \quad \text{in } [0,A] \times [0,\infty)$$

Proof Since $\varphi_\varepsilon(s)$ is positive in $[0,+\infty)$, by using a similar argument as that in [1], the first part of the lemma can be proved easily. What we should do is to prove the properties (i) (ii) and (iii) as *a priori* estimates. The proof of (i)–(iii) is based on the construction of comparison functions.

Proof of (i) By (14) and H_4 follows that for any $y \in (0,A)$

$$v_\varepsilon(y,0) = c_\varepsilon^{-1}(w_{0\varepsilon}) = \psi_\varepsilon(u'_{0\varepsilon}(x)) \le m_0 \tag{17}$$

where m_0 is a constant which does not depend on ε.

By H_1 and H_4, as $y \to 0^+$ there exists a constant c_2 which does not depend on ε such that

$$v_\varepsilon(y,0) = \psi_\varepsilon(u'_{0\varepsilon}) \le (c_1+1)u'_{0\varepsilon} \le c_2 u_{0\varepsilon} = c_2 y \tag{18}$$

Define

$$\overline{v}(y) = \frac{my}{\varphi_\varepsilon(y)} = \frac{my}{\varphi(y+\varepsilon)}$$

where m is a constant to be determined later.

Then $\overline{v}(y)$ satisfies $(\varphi_\varepsilon(y)\overline{v}(y))'' = 0$. By (9), $\overline{v}(y) \ge \dfrac{m}{M}y$, from (17)–(18), we can choose m so large that $\overline{v}(y) \ge v_\varepsilon(y,0)$ in $[0,A]$.

Hence

$$\overline{v}(y) \ge v_\varepsilon(y,t) \quad \text{for all } y \in [0,A] \quad \text{and } t \ge 0$$

Further, from $\overline{v}(y) = \dfrac{my}{\varphi(y+\varepsilon)} \le \dfrac{m(y+\varepsilon)}{\varphi(y+\varepsilon)}$, and H_2 we know that $\overline{v}(y)$ is uniformly bounded in $[0,A]$.

Proof of (ii) Let $0 < y_0 < y_1 < A$, by H_4 there exists a constant $\delta_0 > 0$ (also $\delta_0 < \frac{\psi_\infty}{2}$) such that $v_\varepsilon(y, 0) \geq \delta_0$ for $y \in [y_0, y_1]$.
Put $z_\varepsilon(y,t) = \varphi_\varepsilon v_\varepsilon$, it satisfies

$$L(z) = z_t - \frac{1}{c'_\varepsilon\left(\frac{z}{\varphi_\varepsilon}\right)}\varphi z_{yy} = 0$$

We look for a subsolution of the form

$$\underline{z}(y,t) = \delta e^{-\lambda t}\sin(w(y-y_0)), \quad w = \frac{\pi}{y_1 - y_0} \quad (\delta, \lambda > 0)$$

By (12), $c'_\varepsilon(v) = \frac{1}{[\psi_\varepsilon^{-1}(v)]^2 \psi'_\varepsilon(\psi_\varepsilon^{-1}(v))}$. Hence there exists a constant B such that $c'_\varepsilon(v)) \geq B$ for $v \in \left(0, \frac{\psi_\infty}{2}\right)$. So we can choose δ small enough (e.g. $\frac{\delta}{\delta_0} \leq \min_{y_0 \leq y \leq A+1} \varphi(y))$ and λ large enough (e.g. $\lambda \geq \frac{w^2 M}{B}$) so that in $(y_0, y_1) \times R^+$:

$$L(\underline{z}) = -\lambda \underline{z} + \frac{w^2}{c'_\varepsilon\left(\frac{\underline{z}}{\varphi_\varepsilon}\right)}\varphi_\varepsilon \underline{z} \leq -\lambda \underline{z} + w^2 M B^{-1} \underline{z} \leq 0$$

Moreover, for $y \in (y_0, y_1)$

$$\underline{z}(y, 0) \leq \delta \leq v_\varepsilon(y,0)\varphi_\varepsilon(y) = z_\varepsilon(y, 0)$$

We obtain from the Maximum Principle that $v_\varepsilon \varphi_\varepsilon \geq \underline{z}$ in $(y_0, y_1) \times R^+$. Hence for $(y,t) \in (y_0, y_1) \times (0, T)$

$$v_\varepsilon(y,t) \geq \frac{\underline{z}(y,t)}{\varphi_\varepsilon(y)} \geq \frac{\underline{z}(y,t)}{M} \geq \frac{\delta}{M} e^{-\lambda T}\sin(w(y-y_0)) \triangleq \underline{z}^1(y;T)$$

Since $\underline{z}'(y;T) = 0$ as $y = y_0$ and $y = y_1$, we extend $\underline{z}'(.;T)$ from (y_0, y_1) to $[0, A]$ by defining $z'(y;T) = 0$ as $y \in [0, y_0] \cup [y_1, A]$. Then $\underline{z}'(.;T)$ is a nonnegative continuous function in $[0, A]$ and strictly positive in (y_0, y_1). Choose

$$y_0 = \frac{A}{2^2} \qquad y_1 = A - \frac{A}{2^2}$$
$$y_2 = \frac{A}{2^4} \qquad y_3 = A - \frac{A}{2^4}$$
$$\vdots \qquad \vdots$$
$$y_{2n} = \frac{A}{2^{2(n+1)}} \qquad y_{2n+1} = A - \frac{A}{2^{2(n+1)}}$$
$$\vdots \qquad \vdots$$

we obtain a sequence of functions $\underline{z}^n(y;T)$, where $\underline{z}^n(y;T)$ is associated with (y_{2n}, y_{2n+1}) just as $\underline{z}^1(y;T)$ with (y_0, y_1). Denote $g_1(y;T) = \sup_n \underline{z}^n(y;T)$, we obtain (ii).

Proof of (iii) From (i) of this lemma $v_\varepsilon \le c_3$ in $[0,A]\times R^+$ for some constant c_3. For any small $\gamma > 0$ let $\overline{v}_\gamma$ be the solution of

$$(\varphi_\varepsilon \overline{v_\gamma})'' = 0 \quad \text{in } (0,\gamma), \overline{v_\gamma}(0) = 0, \quad \overline{v_\gamma}(\gamma) = c_3$$

Then

$$\overline{v_\gamma} = \frac{c_3\varphi(\gamma+\varepsilon)}{\gamma\varphi(y+\gamma)}y \quad \text{for } 0 \le y \le \gamma$$

By H_2, $\varphi(\gamma+\varepsilon) > \varphi(y+\varepsilon)$ for $y\in(0,\gamma)$, hence

$$\overline{v_\gamma} \ge \frac{c_3 y}{\gamma} \tag{19}$$

By H$_4$, as $y \to 0^+$

$$v_\varepsilon(y,0) = \psi_\varepsilon(u'_{0\varepsilon}) = O(u'_{0\varepsilon}) = O(u_{0\varepsilon}) = O(y) \tag{20}$$

Using (19)–(20), we may choose γ so small that

$$\overline{v}_\gamma(y) \le v_\varepsilon(y,0) \quad \text{for } y\in(0,\gamma)$$

Hence, by the Maximum Principle, $v_\varepsilon(y,t) \le \overline{v}_\gamma$ in $[0,\gamma]\times R^+$. By H_2 and (9), $\overline{v_\gamma} \le \dfrac{c_3 M}{\gamma}\dfrac{\varphi(y)}{y}$.

Let

$$\overline{v_\gamma}^{(1)} = \begin{cases} \dfrac{c_3M}{\gamma}\dfrac{\varphi(y)}{y}, & 0 < y < \gamma \\ 0, & y = 0 \end{cases}$$

Then $v_\varepsilon(y,t) \le \overline{v_\gamma}^{(1)}$ in $[0,\gamma]\times R^+$ for $\varepsilon\in(0,1]$.

In the same way, we construct a continuous function $\overline{v_\gamma}^{(2)}$ in $[A-\gamma, A]$ such that $\overline{v_\gamma}^{(2)}(A) = 0$, $\overline{v_\gamma}^{(2)}(A-\gamma) = c_3$ and $v_\varepsilon(y,t) \le \overline{v_\gamma}(y)$ in $[A-\gamma, A]\times R^+$.

We conclude the assertion (iii) by defining

$$g_2(y) = \begin{cases} \overline{v_\gamma}^{(1)}(y), & 0 \le y \le \gamma \\ c_3, & \gamma < y < A-\gamma \\ \overline{v}_\gamma^{(2)}(y), & A-\gamma \le y \le A \end{cases}$$

The proofs of Ths. 3.3 and 3.4 are direct by use of Lemma 3.5 and the results in [1].

4. Existence and Main Properties of the Solution of Problem I

To prove theorem 2.1, we need the following two lemmas.

Lemma 4.1 *Let hypotheses H_1–H_4 be satisfied and u_ε $(0 < \varepsilon \le 1)$ be the solution of Problem* II_ε, then

(i) $u_{\varepsilon x}$ *is uniformly bounded in* $L^\infty([0,T]; L^1_{\mathrm{loc}}(R))$ *for all* $T > 0$.

(ii) $u_{\varepsilon t}$ *is uniformly bounded in* $L^1_{\mathrm{loc}}(R \times [0,\infty))$.

From the compactness of the imbedding $BV_{\mathrm{loc}} \to L^1_{\mathrm{loc}}$, we get the following results as a consequence of Lemma 4.1.

Corollary 4.2 *Let hypotheses H_1–H_4 be satisfied. Then there exists a sequence $\varepsilon_i \to 0$ as $i \to \infty$ and a function $u \in L^\infty(R \times R^+) \cap BV_{\mathrm{loc}}(R \times [0,\infty))$ such that for any $1 \le P \le \infty$*

$$u_{\varepsilon_i} \to u \quad in\ L^P_{\mathrm{loc}}(R \times [0,\infty))$$

Proof of Lemma 4.1 By (8) we get that u_ε has a uniformly positive lower and upper bound. In any compact set $K \subset R \times [0,\infty)$, we can prove Lemma 4.1 just like in [1].

Lemma 4.3 *Let hypotheses H_1–H_4 be satisfied. Define*

$$\overline{\psi_\varepsilon}(s) = \min\{\psi_\varepsilon(s), \psi_\infty\} \quad \text{for } s \in R$$

Then the functions $\overline{\psi_\varepsilon}(u_{\varepsilon x})$ are locally equicontinuous in $R \times [0,\infty)$.

This lemma can be obtained directly from [1].

The proof of Theorem 2.1 follows from Lemma 4.1, 4.3 and results in [1].

Finally, by Lemma 3.1 u_ε has a uniformly positive lower bound, and the assertions in Theorem 2.2–2.4 are all able to be proved locally in [1], thus they are valid under the conditions of this paper.

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