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REGULARITY RESULTS FOR MINIMIZERS OF CERTAIN FUNCTIONALS HAVING NONQUADRATIC GROWTH WITH OBSTACLES

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Abstract We prove partial regularity for minimizers of degenerate variational integrals $\int_\Omega F(x,u,Du)dx$ with obstacles of either the form

(i) $$\mu_f = \{u \in H^{1,m}(\Omega,\mathbb{R}^N) | u^N \ge f_1(u^1,\cdots,u^{N-1}) + f_2(x) \text{ a.e.}\}$$

or

(ii) $$\mu_N = \{u \in H^{1,m}(\Omega,\mathbb{R}^N) \mid u^i(x) \ge h^i(x), \text{ a.e.}; i = 1,\cdots,N\}$$

The typical mode of variational integrals is given by

$$\int_\Omega \left[a^{\alpha\beta}(x,u)b_{ij}(x,u)D_\alpha u^i D_\beta u^i\right]^{\frac{m}{2}} dx, \quad m \ge 2$$

Key Words Degenerate variational integral; obstacle; partial regularity.
Classification 49N60, 35J50, 35J70.

1. Introduction

Let Ω be a bounded open set in $\mathbb{R}^n$, $u = (u^1,\cdots,u^N)$ be in general a vector valued function, $N \ge 1$ and $Du = \{D_\alpha u^i\}$, $\alpha = 1,\cdots,n$; $i = 1,\cdots,N$, stands for the gradient of u. We deal with variational integrals

$$\mathcal{F}(u,\Omega) := \int_\Omega F(x,u,Du)dx \tag{1.1}$$

where the integrand $F(x,u,p)$ grows polynomially like $|p|^m$.

More precisely we assume that

$$F(x,u,p) = g(x,u,a^{\alpha\beta}(x,u)b_{ij}(x,u)p^i_\alpha p^j_\beta) \tag{1.2}$$

where $(a^{\alpha\beta})$ and (b_{ij}) are symmetric positive definite matrices and satisfies

H.1 For some positive λ, Λ and for all x, u, p we have

$$\lambda|p|^m \leq F(x,u,p) \leq \Lambda|p|^m \tag{1.3}$$

where $m \geq 2$.

H.2 $F(x,u,p)$ is of class C^2 with respect to p and

$$|F_{pp}(x,u,p)| \leq C_1|p|^{m-2}$$

$$|F_{pp}(x,u,p) - F_{pp}(x,u,q)| \leq C_2(|p|^2+|q|^2)^{\frac{m-2}{2}-\frac{\alpha}{2}}|p-q|^\alpha$$

for some positive α.

H.3 The integrand $F(x,u,p)$ is elliptic in the sense that

$$F_{p^i_\alpha p^j_\beta(x,u,p)}\xi^\alpha_i\xi^\beta_j \geq |p|^{m-2}|\xi|^2, \quad \forall\xi \in \mathbb{R}^{nN} \tag{1.4}$$

H.4 The function $|p|^{-m}F(x,u,p)$ is Hölder-continuous in (x,u) uniformly with respect to p, i.e.

$$|F(x,u,p) - F(y,v,p)| \leq C|p|^m\eta(|u|, |x-y|+|u-v|)$$

where $\eta(t,s) = K(t)\min(s^\delta, L)$ for some $\delta, 0<\delta<1$, and $L>0$ and where $K(t)$ is an increasing function. Without loss of generality, we may assume that $\eta(t,s)$ is concave is s for fixed t.

H.5 We assume that $g(x,u,t)$ is an increasing function in t for each fixed $(x,u) \in \Omega \times \mathbb{R}^N$.

A particular example of the above functional is given by the p-energy functional

$$\mathcal{F}(u;\Omega) = \int_\Omega [a^{\alpha\beta}(x,u)b_{ij}(x,u)D_\alpha u^i D_\beta u^j]^{\frac{m}{2}}dx, \quad m \geq 2 \tag{1.5}$$

where $(a^{\alpha\beta})$ and (b_{ij}) are symmetric positive definite matrices.

We recall that a minimizer for the functional (1.1) is a function $u \in H^{1,m}(\Omega, \mathbb{R}^N)$ such that

$$\mathcal{F}(u;\Omega) \leq \mathcal{F}(u+\phi;\Omega)$$

for all $\phi \in H^{1,m}_0(\Omega, \mathbb{R}^N)$.

The functional (1.5) denotes the p-energy of maps between two Riemannian manifolds which the images lie in a single chart (with $p=m$). The critical point of (1.5) is called a p-harmonic map. When $m=2$, the partial regularity of minimizing harmonic maps between Riemannian manifolds was obtained by Schoen and Uhlenbeck in [1], independently by Giaquinta and Giusti ([2], [3]) for the case where the images lie in a single chart. For $m = p > 2$, the regularity of minimizers of a special functional, in which the integrand does not depend on x and u, was first obtained by Uhlenbeck in [4]. Later the regularity results of p-harmonic maps which minimize the p-energy functional were obtained in [5–8].

Definition *A minimizer for the functional* (1.1) *with an obstacle μ is a function $u \in H^{1,m}(\Omega, \mathbb{R}^N) \cap \mu$ such that*

$$\mathcal{F}(u; \operatorname{supp}\phi) \leq \mathcal{F}(u + \phi; \operatorname{supp}\phi)$$

for all $\phi \in H_0^{1,m}(\Omega, \mathbb{R}^N)$ such that the $\operatorname{supp}\phi \subset\subset \Omega$ *and $u + \phi \in \mu$ where μ is a given obstacle.*

In this paper we shall prove the regularity of minimizing the functional (1.1) with the following two kinds of obstacles: either (i)

$$\mu_f = \{u \in H^{1,2}(\Omega, \mathbb{R}^N) | u^N \geq f_1(u^1, \cdots, u^{N-1}) + f_2(x) \text{ a.e.}, u - u_0 \in H^{1,m}(\Omega, \mathbb{R}^N)\}$$

or (ii)

$$\mu_N = \{u \in H^{1,2}(\Omega, \mathbb{R}^N) | u^i(x) \geq h^i(x) \text{ a.e. } i = 1, \cdots, N; u - u_0 \in H^{1,m}(\Omega, \mathbb{R}^N)\}$$

where $f_1(y^1, \cdots, y^{N-1})$ is a given function on $\mathbb{R}^{N-1}$, f_2 and $h^i(x)$ $(i = 1, \cdots, N)$ are given functions on Ω and u_0 is a given boundary value function.

The main results of this paper is roughly described as follows: Assume that the integrand $F(x, u, p)$ satisfies assumptions H.1–H.5 and has the form (1.2), then each minimizer of the functional (1.1) with the obstacle type (i) or (ii) belongs to $C^{0,\alpha}(\Omega_0, \mathbb{R}^N)$ for some $0 < \alpha < 1$ where $\Omega_0 \subset \Omega$ is open and $\mathbb{H}^{n-q}(\Omega \backslash \Omega_0) = 0$ for some $q > m$ where $\mathbb{H}^{n-q}$ denotes the $n - q$ dimensional Hausdorff measure. Moreover when $p = 2$, the minimizer $u \in C^{1,\alpha}(\Omega, \mathbb{R}^N)$.

Minimizing a functional with an obstacle we only obtain variational inequalities, not the Euler-Lagrange equation. For example, minimizing the functional (1.5) with an obstacle μ, we obtain corresponding weak variational inequalities in the following form:

$$\int_\Omega [a^{\alpha\beta}(x, u) b_{ij}(x, u) D_\alpha u^i D_\beta u^j]^{\frac{m-2}{2}} a^{\alpha\beta}(x, u) b_{ij}(x, u) D_\alpha u^i D_\beta \phi^j dx \geq 0$$

for all $\phi \in W_0^{1,m}(\Omega; \mathbb{R}^N)$ such that $u + \phi \in \mu$.

The regularity of the minimizing problem with the graphic obstacle of the form (i) was studied in [9–14] only for $m = 2$, but comparatively little is known for the regularity theory of minimizers of the non-quadratic growth functional with the obstacle. The obstacle of the form (ii) is naturally extended from scalar case to vector case. Some results in vector case were mentioned by Giaquinta in his book (see PP.241–243 in [15]) by using variational inequalities to treat the problem minimizing a functional with obstacle (ii). In general the problem of regularity for solutions of variational inequalities is greatly open. The only results in [16] have been obtained in vector case only for diagonal variational inequalities in which the method is based on one in scalar case [17]. It will be more difficult to deals with the regularity of solutions of variational inequalities than to do Euler-Lagrange equations. In this paper, we deal directly with the functional (1.1) instead of working with variational inequalities. This technique is called the "Direct method". By proving a existence theorem (see Theorem 2.2), we consider a obstacle problem of minimizing a functional

$$F^0(v;\Omega)=\int_{B_R(x_0)}F(x_0,u_0,Du)dx$$

with the obstacle of the form either (i) or (ii). For the obstacle problem of the degenerate functional of the form like p-energy, another key point is to extend the Uhlenbeck's theorem (see [4]) to the obstacle problem. We prove a regularity theorem (Theorem 3.3) by a induction method. Finally, comparing $\mathcal{F}^0(v,\Omega)$ with $\mathcal{F}(u,\Omega)$ and using L^p-estimates we prove our main results.

Notation *We adopt the standard notations of Giaquinta's book* [15]. *Moreover we use the following notations:*

$$\tilde{u}(x)=(u^1(x),\cdots,u^{N-1}(x)),\quad and\quad [h(x)]_{x_0,R}=\frac{1}{|B_R(x_0)|}\int_{B_R(x_0)}h(x)dx$$

where $h(x)$ is any function on x.

2. L^p-Estimates

In this section we discuss the L^p-estimate for minimizers of the obstacle problem, compared with [18], [14].

Now let us define a Q-minima of the functional $\mathcal{F}$ with an obstacle μ.

H.1′ We assume that for some positive constants λ,Λ and for all x,u,p we have

$$\lambda(-\theta^m+|p|^m)\le F(x,u,p)\le\Lambda(\theta^m+|p|^m)\tag{2.1}$$

where $\theta\ge 0$ and $m\ge 2$.

Definition 2.1 *Consider the functional* (1.1) *and assume that the assumption H.1′ holds. We say that* $u \in H^{1,m}_{\rm loc}(\Omega, \mathbb{R}^N) \cap \mu$ *is a Q-minima for* $\mathcal{F}$ *with an obstacle* μ *in* Ω *(with a constant Q) if*

$$\mathcal{F}(u; \operatorname{supp}\phi) \le Q\mathcal{F}(u+\phi; \operatorname{supp}\phi) \tag{2.2}$$

for all ϕ *with* $\operatorname{supp}\phi \subset\subset \Omega$, *and* $u+\phi \in \mu$.

Then we have

Theorem 2.2 *Let* $u \in H^{1,m}_{\rm loc}(\Omega, \mathbb{R}^N)$ *be a Q-minima for the functional* (1.1) *with a graphic obstacle* μ_f *of the form* (i). *Suppose that the function* $f_1(\widetilde{u})$ *and* $f_2(x)$ *are* C^1*-continuous function with* $|Df_1| \le L$ *and* $|Df_2(x)| \le L$ *for some constant* $L > 0$. *Then there exists an exponent* $q > m$ *such that* $u \in H^{1,q}_{\rm loc}(\Omega, \mathbb{R}^N)$. *Moreover for all* $x_0 \in \Omega$ *and* $R < \operatorname{dist}(x_0, \partial\Omega)$ *the following estimate holds:*

$$\Big(\int_{B_{\frac{R}{2}}(x_0)} (1+|Du|)^q dx\Big)^{\frac{1}{q}} \le C\Big(\int_{B_R(x_0)} (1+|Du|)^m dx\Big)^{\frac{1}{m}} \tag{2.3}$$

where $C \ge 0$ *is a constant in* (1.1).

Proof For simplicities, we denote $f(x,\widetilde{u}) = f_1(\widetilde{u}) + f_2(x)$. Choose as test function in (2.2)

$$\widetilde{\phi}(x) = -\eta(\widetilde{u}(x) - \widetilde{u}_R)$$

$$\phi^N = -\eta(u^N(x) - u^N_R) + f(x, \widetilde{u} + \widetilde{\phi}) - (1-\eta)f(x,\widetilde{u}) - \eta[f(\widetilde{u})]_R$$

where $\eta \in C_0^\infty(B_R)$, $0 \le \eta \le 1$, $\eta \equiv 1$ on B_t $(t < R)$, $|\nabla\eta| \le \dfrac{C}{R-t}$.

It is easy to check that $u + \phi \in \mu_f$. Setting

$$\widetilde{\Phi} = (1-\eta)(u - u_R), \quad \Phi^N = u^N - u^N_R + \phi^N$$

we have

$$u - u_R = -\phi + \Phi$$

where $\Phi := (\widetilde{\Phi}, \Phi^N)$.

Then we obtain from the definition of a Q-minima

$$\begin{aligned}\int_{B_R} |D\phi|^m dx &= \int_{B_R} |Du - D\Phi|^m dx \\ &\le C\int_{B_R} |Du|^m dx + C\int_{B_R} |D\Phi|^m dx \\ &\le CQ\int_{B_R} |D(u+\phi)|^m dx + C\int_{B_R} |D\Phi|^m dx + (\lambda+\Lambda)\theta^m |B_R|\end{aligned}$$

$$\leq C\int_{B_R}|D\widetilde{\Phi}|^m dx + C\int_{B_R}|D\Phi^N|^m dx + C\theta^m|B_R| \tag{2.4}$$

From the definition of the test function Φ, we have

$$\begin{aligned}\int_{B_R}|D\,\Phi^N|^m dx &= \int_{B_R}|D(u^N+\phi^N)|^m dx\\
&= \int_{B_R}|D[u^N-\eta(u^N-u_R^N)+f(\widetilde{u}+\widetilde{\phi})-(1-\eta)f(\widetilde{u})-\eta f(\widetilde{u})_R]|^m dx\\
&\leq \int_{B_R}|D\{(1-\eta)[u^N-u_R^N-f(\widetilde{u})+f(\widetilde{u})_R]|^m dx + C\int_{B_R}|D[f(\widetilde{u}+\widetilde{\phi})]|^m dx\\
&\leq C\int_{B_R\setminus B_t}|D[u^N-f(\widetilde{u})]|^m dx + \frac{C}{(R-t)^m}\int_{B_R}|u^N-f(\widetilde{u})-[u_R^N-f(\widetilde{u})_R]|^m dx\\
&\quad + C\int_{B_R}|D[f(\widetilde{u}+\widetilde{\phi})|^m dx \end{aligned}\tag{2.5}$$

Using the assumption of f_1 and f_2, we have

$$\int_{B_R\setminus B_t}|D[f(\widetilde{u})]|^m dx \leq C\int_{B_R\setminus B_t}|D\widetilde{u}|^m dx + CR^n \tag{2.6}$$

$$\int_{B_R}|D[f(\widetilde{u}+\widetilde{\phi})]|^m dx \leq C\int_{B_R}|D(\widetilde{u}+\widetilde{\phi})|^m dx + CR^n \tag{2.7}$$

Then from (2.5), (2.6) and (2.7) we have

$$\begin{aligned}\int_{B_R}|D\Phi^N|^m dx \leq C\int_{B_R\setminus B_t}|Du|^m dx + C\int_{B_R}|D\widetilde{\Phi}|^m dx\\
+\frac{C}{(s-t)^m}\int_{B_S}|u^N-f(\widetilde{u})-[u_R^N-f(\widetilde{u})_R]^2 dx\end{aligned}\tag{2.8}$$

Noticing $D\widetilde{\Phi}=(1-\eta)D\widetilde{u}-(\widetilde{u}-\widetilde{u}_R)D\eta$, we have

$$|D\widetilde{\Phi}|^m \leq C|D\widetilde{u}|^m(1-\eta)^m + C|\widetilde{u}-\widetilde{u}_R|^m|D\eta|^m$$

By the definition of η, we obtain from (2.4)-(2.8)

$$\begin{aligned}\int_{B_t}|D\,u|^m dx &\leq \int_{B_R}|D\phi|^m dx\\
&\leq C_1\int_{B_R\setminus B_t}|Du|^m dx + \frac{C_1}{(R-t)^m}\int_{B_R}|\widetilde{u}-\widetilde{u}_R|^m dx\\
&\quad+\frac{C_1}{(R-t)^m}\int_{B_R}|u^N-f(\widetilde{u})-u_R^N+f(\widetilde{u})_R|^m dx + C|B_R|\end{aligned}\tag{2.9}$$

Now we fill the hole, i.e. we sum t_0 (2.9) C_1 times the left-hand side and obtain

$$\int_{B_t}|Du|^m dx \leq \theta_1\int_{B_R}|Du|^m dx + \frac{C}{(s-t)^m}\int_{B_R}|\widetilde{u}-\widetilde{u}_R|^m dx$$

$$+\frac{C}{(s-t)^m}\int_{B_R}|u^N-f(\tilde{u})-u_R^N+f(\tilde{u})_R|^m+C|B_R| \tag{2.10}$$

where $\theta_1=\dfrac{C_1}{1+C_1}<1$.

We would like to eliminate the first term on the right-hand side of (2.10) to have

$$\int_{B_{\frac{R}{2}}}|Du|^m dx\le CR^{-m}\int_{B_R}|u^N-f(\tilde{u})-u_R^N+f(\tilde{u})_R|^m dx$$
$$+CR^{-m}\int_{B_R}|\tilde{u}-\tilde{u}_R|^m dx+C(\theta)R^n \tag{2.11}$$

by means of Lemma 3.1 of Chapter V of [15]. Then, through simple application of the Sobolev-Poincare inequality, (2.11) gives

$$\frac{1}{|B_{\frac{R}{2}}|}\int_{B_{\frac{R}{2}}}(1+|Du|)^m dx\le C\Big(\frac{1}{|B_R|}\int_{B_R}(1+|Du|)^q dx\Big)^{\frac{m}{q}},\quad q=\frac{mn}{m+n}<n$$

Then the result follows from proposition of Chapter V of [15].

In a similar way we have

Theorem 2.3 *Let $u\in H^{1,m}_{\rm loc}(\Omega,\mathbb{R}^N)$ be a Q-minima for the functional* (1.1) *with a graphic obstacle μ_f of the form* (ii). *Suppose that the function $h_i(x)$, $i=1,\cdots,N$ in the obstacle* (ii) *belong to the space C^1 and $|Dh_i|\le L$ for some constant $L>0$. Then there exists an exponent $q>m$ such that $u\in H^{1,q}_{\rm loc}(\Omega,\mathbb{R}^N)$. Moreover for all $x_0\in\Omega$ and $R<\operatorname{dist}(x_0,\partial\Omega)$ the following estimate holds:*

$$\Big(\int_{B_{\frac{R}{2}}(x_0)}(1+|Du|)^q dx\Big)^{\frac{1}{q}}\le C\Big(\int_{B_R(x_0)}(1+|Du|)^m dx\Big)^{\frac{1}{m}} \tag{2.12}$$

where $C\ge 0$ is a constant.

Next we prove the existence of minimizers for the functional $\mathcal{F}$ with the obstacle of type (i) or (ii). Let us first state a semicontinuity theorem due to Acerbi-Fusco [19].

Theorem 2.4 *Let $F(x,u,p)$ be a Caratheodary function. Assume that*

$$|F(x,u,p)|\le(1+|u|^m+|p|^m),\quad m\ge 1,\quad \lambda>0$$

and that F be quasi-convex. Then $\mathcal{F}(u;\Omega)$ is weakly sequentially lower semicontinuous in $H^{1,q}(\Omega,\mathbb{R}^N)$ for $q>m$.

Then we prove a existence theorem for the obstacle problem.

Theorem 2.5 *Let $F(x,u,p)$ be a Caratheodary function. Assume that* (2.1) *holds and that F be quasi-convex. Then there exists a minimum point for the functional* (1.1) *with the obstacle of either μ_f or μ_N. Moreover the minimizer $u\in H^{1,q}_{\rm loc}$ for $q>m$.*

Proof The proof of theorem is almost derived by the work of Marcellini and Sbordone in [20]. Let us sketch the proof here. Let $V = \{v \in H^{1,m}(\Omega, \mathbb{R}^N) \mid v \in \mu, v - u_0 \in H_0^{1,m}(\Omega, \mathbb{R}^N)\}$ where $\mu = \mu_f$ or μ_N and Let $d(u,v) = \int_\Omega |Du - Dv| dx$, then obviously $\mathcal{F}$ is lower semicontinuous in $\{u \in H^{1,1}(\Omega) : u = u_0 \ on \ \partial\Omega\}$. Let $\{v_k\}$ be a minimizing sequence, and let $\{u_k\}$ be the corresponding (minimizing) sequence given by Ekeland's Variational principle

$$\mathcal{F}(u_k; \Omega) \leq \mathcal{F}(w; \Omega) + \eta_k \int_\Omega |Du_k - Dw| dx$$

where $w \in V$. Hence the function u_k which we obtain is a Q-minima for a functional of the same type with the obstacle μ where the constant Q is independent of η_k by choosing η_k small enough. More precisely there exists a minimizing sequence $\{u_k\}$ of Q-minima with uniform constant Q. Theorem 2.2 implies

$$\|u_k\|_{H^{1,q}(\widetilde{\Omega}, \mathbb{R}^N)} \leq \text{const}$$

for any $\widetilde{\Omega} \subset\subset \Omega$, with $q > m$ where the constant depends on $\widetilde{\Omega}$, but it is independent of k. Noticing that $u_k \to u$ *a.e. on* Ω, we have $u \in \mu$. Then by means of theorem 2.3 we obtain the results required.

3. An Extension of the Uhlenbeck's Theorem

First, we collect some results and Lemmas from [7].

It is not difficult to show that there exist two positive constants $C_0(\delta), C_1(\delta)$ such that

$$C_0(\delta) \leq \frac{\int_0^1 |ta + (1-t)b|^\delta dt}{(|a|^2 + |b|^2)^{\frac{\delta}{2}}} \leq C_1(\delta) \tag{3.1}$$

$$C_0(\delta) \leq \frac{\int_0^1 |ta + (1-t)b|^\delta dt}{(|a|^2 + |b-a|^2)^{\frac{\delta}{2}}} \leq C_1(\delta) \tag{3.2}$$

for $a, b \in \mathbb{R}^k$.

For $\delta > 0$, $p \in \mathbb{R}^k$, define the vector valued function

$$V_\delta(p) := |p|^\delta p$$

Then by using (3.1) and (3.2) it is not difficult to show that there exist positive constants $C_2(\delta), C_3(\delta)$ such that

$$C_2(\delta) \leq \frac{|V_\delta(p) - V_\delta(q)|}{(|p|^2 + |q|^2)^{\frac{\delta}{2}} |p - q|} \leq C_3(\delta) \tag{3.3}$$

for all $p, q \in \mathbb{R}^k$.

We shall write

$$V(p) = |p|^{\frac{m-2}{2}} p$$

We shall also use the standard notation

$$\phi_{x_0,R} = \frac{1}{|B_R|}\int_{B_R(x_0)} \phi dx$$

for the average over the ball $B_R(x_0)$ in $\mathbb{R}^n$ of the vector valued function $\phi : B_R(x_0) \to \mathbb{R}^N$.

For any $p > 1$ there exists a constant C_4 such that for any $\lambda \in \mathbb{R}^{nN}$

$$\begin{aligned}\frac{1}{|B_R|}\int_{B_R(x_0)} |V(\phi) - V(\phi_{x_0,R})|^p dx &\le C_4 \frac{1}{|B_R|}\int_{B_R(x_0)} |V(\phi) - V(\phi)_{x_0,R}|^p dx \\ &\le C_5 \frac{1}{|B_R|}\int_{B_R(x_0)} |V(\phi) - V(\lambda)|^p dx \qquad (3.4)\end{aligned}$$

where $C_5(\delta)$ is also constant.

We have the following very interesting results due to K. Uhlenbeck [4], or see [7]:

Theorem 3.1 *Let $u \in H^{1,m}_{\text{loc}}(\Omega, \mathbb{R}^N)$ be a minimizer of the functional*

$$\mathcal{F}_0(u;\Omega) = \int_\Omega F_0(Du)dx$$

where the integrand F_0 satisfying assumptions H.1, ..., H.5 *and moreover*

$$F_0(Du) = g(a^{\alpha\beta} b_{ij} D_\alpha u^i D_\beta u^j)$$

where $(a^{\alpha\beta}), (b_{ij})$ are symmetric positive definite constant matrices. Then Du is locally Hölder-continuous function with some exponent, $0 < \delta < 1$. Moreover for every $x_0 \in \Omega$, for all ρ, R. $0 < \rho < R < \text{dist}\,(x_0, \partial\Omega)$ we have

$$\sup_{B_{\frac{R}{2}}(x_0)} |Du|^m \le \left[\frac{1}{B_R(x_0)}\int_{B_R(x_0)} |Du|^m dx\right] \qquad (3.5)$$

$$\Phi(x_0,\rho) \le C\Big(\frac{\rho}{R}\Big)^{2\sigma}\Phi(x_0.R)$$

where $\Phi(x_0, r) := \dfrac{1}{|B_r(x_0)|}\displaystyle\int_{B_r(x_0)} |V(Du) - V(Du)_{x_0,r}|^2 dx.$

Lemma 3.2 *Assume that $A = (a^{\alpha\beta})$, $B = (b_{ij})$ are symmetric positive definite constant matrices, Then we have*

$$\sum_{\alpha,\beta=1}^{n}\sum_{\tilde{i}\,\tilde{j}=1}^{N-1} a^{\alpha\beta} b_{\tilde{i}\tilde{j}} D_\alpha V^{\tilde{i}} D_\beta V^{\tilde{j}} \le \sum_{\alpha,\beta=1}^{n}\sum_{i,j=1}^{N} a^{\alpha\beta} b_{ij} D_\alpha V^i D_\beta V^j$$

Proof Setting $\widetilde{B} = (b_{\widetilde{i}\,\widetilde{j}})_{N-1,N-1}$ and $\widetilde{H} = (b_{N1}, \cdots, b_{NN-1})$, we have

$$B = \begin{pmatrix} \widetilde{B} & \widetilde{H} \\ \widetilde{H}^T & b_{NN} \end{pmatrix}$$

Hence

$$\begin{pmatrix} I & 0 \\ -\widetilde{H}^T\widetilde{B}^{-1} & 1 \end{pmatrix}\begin{pmatrix} \widetilde{B} & \widetilde{H} \\ \widetilde{H}^T & b_{NN} \end{pmatrix}\begin{pmatrix} I & \widetilde{B}^{-1}\widetilde{H} \\ 0 & 1 \end{pmatrix} = \begin{bmatrix} \widetilde{B} & 0 \\ 0 & b^*_{NN} \end{bmatrix}$$

where $b^*_{NN} = b_{NN} - \widetilde{H}^T\widetilde{B}^{-1}\widetilde{H} > 0$ is a constant.

Let $Z_0^T = \begin{pmatrix} I & 0 \\ -\widetilde{H}^T\widetilde{B}^{-1} & 1 \end{pmatrix}$, and let $V^* := (V^1, \cdots, V^{N-1}, V^{*N}) = (\widetilde{V}, V^{*N})$. Where $V^{*N} = V^N - (\widetilde{H}^T\widetilde{B}^{-1})\widetilde{V}$. Thus

$$\begin{aligned} \sum_{\alpha,\beta=1}^{n}\sum_{i,j=1}^{N} a^{\alpha\beta}b_{ij}D_\alpha V^i D_\beta V^j &= \sum_{\alpha,\beta=1}^{n} a^{\alpha\beta}D_\alpha V^T B D_\beta V \\ &= \sum_{\alpha,\beta=1}^{n} a^{\alpha\beta}D_\alpha V^{*N} Z_0^T B Z_0 D_\beta V^* = \sum_{\alpha\beta=1}^{n} a^{\alpha\beta}D_\alpha V^{*T}\begin{pmatrix} \widetilde{B} & 0 \\ 0 & b^*_{NN} \end{pmatrix} D_\beta V^* \\ &= \sum_{\alpha,\beta=1}^{n}\sum_{\widetilde{i},\widetilde{j}=1}^{N-1} a^{\alpha\beta}b_{\widetilde{i}\,\widetilde{j}}D_\alpha V^{\widetilde{i}}V^{\widetilde{j}} + \sum_{\alpha,\beta=1}^{n} a^{\alpha\beta}b^*_{NN}D_\alpha V^{*N}D_\beta V^{*N} \\ &\geq \sum_{\alpha,\beta=1}^{n}\sum_{\widetilde{i},\widetilde{j}=1}^{N-1} a^{\alpha\beta}b_{\widetilde{i}\,\widetilde{j}}D_\alpha V^{\widetilde{i}}V^{\widetilde{j}} \end{aligned}$$

Now, we extend the Uhlenbeck's theorem to the obstacle problem.

Theorem 3.3 *Suppose that U is a minimum for the functional $\mathcal{F}_0$ in Ω with the obstacle μ_N of the form*

$$\mu_1 = \{u \in H^{1,m}(\Omega, \mathbb{R}^N) | u - u_0 \in H_0^{1,m}(\Omega, \mathbb{R}^N), u^N \geq h^N \ a.e.\}$$

where the integral $\mathcal{F}_0$ is defined as in Theorem 3.1 and u_0 is a given boundary function with $u_0 \in \mu_1$. Let F_0 satisfy the same assumptions of Theorem 3.1 and assume that $|Dh^N| \in L^{m,n-m+m\alpha}$ where $L^{m,n-m+m\alpha}$ denotes the Morrey space ([15]). Then the minimizer $U \in C^{0,\alpha}_{\text{loc}}(\Omega, \mathbb{R}^N)$ and we have

$$\int_{B_\rho(x_0)} H(DU)dx \leq C\Big(\Big(\frac{\rho}{R}\Big)^n + \varepsilon\Big)\int_{B_{\frac{R}{2}}(x_0)} H(DU)dx + C\int_{B_{\frac{R}{2}}(x_0)} |Dh^N|^m dx \quad (3.6)$$

where $H(p) := |p|^m$.

Proof Let $V \in H^{1,2}(B_{\frac{R}{2}}(x_0), \mathbb{R}^N)$ be a minimizer of the functional

$$\int_{B_{\frac{R}{2}}(x_0)} F_0(Dv)dx$$

for all $v - U \in H_0^{1,2}(B_{\frac{R}{2}}, \mathbb{R}^N)$.

Then we have from [15] and [4]

$$\sup_{B_{\frac{R}{4}}(x_0)} H(DV) \le C \frac{1}{|B_{\frac{R}{2}}(x_0)|} \int_{B_{\frac{R}{2}}(x_0)} H(DV)dx$$

Hence for all $0 < \rho < \frac{R}{2}$, we obtain

$$\int_{B_\rho(x_0)} H(DV)dx \le C\Big(\frac{\rho}{R}\Big)^n \int_{B_{\frac{R}{2}}(x_0)} H(DV)dx \tag{3.7}$$

Therefore for all $\rho < \frac{R}{2}$, we have

$$\int_{B_\rho(x_0)} H(DU)dx \le C\Big(\frac{\rho}{R}\Big)^n \int_{B_R(x_0)} H(DU)dx + C\int_{B_{\frac{R}{2}}(x_0)} |D(U-V)|^m dx \tag{3.8}$$

In fact

$$\begin{aligned} H(DU) &= [H^{\frac{1}{m}}(DV) + H^{\frac{1}{m}}(DU) - H^{\frac{1}{m}}(DV)]^m \\ &\le C(m)H(DV) + C(m)[H^{\frac{1}{m}}(DU) - H^{\frac{1}{m}}(DV)]^m \end{aligned}$$

Next we estimate the last term in (3.8). Setting $V^N \vee h^N := \max\{V^N, h^N\} = (V^N - h^N)^+ + h^N$, we denote

$$V^\star := (V^1, \cdots, V^{N_1}, V^N \vee h^N)$$

We know that $V^\star \in \mu_1$ and

$$V^\star\,|_{\partial B_{\frac{R}{2}}(x_0)} = U\,|_{\partial B_{\frac{R}{2}}(x_0)} = V\,|_{\partial B_{\frac{R}{2}}(x_0)}$$

Using the formula

$$g(p) - g(q) = g_p(q)(p-q) + \int_0^1 (1-t)g_{pp}((1-t)q + tp)dt(p-q)(p-q) \tag{3.9}$$

taking the ellipticity into account, and applying estimate (3.1)–(3.3) and the Euler-Lagrange equation for V

$$\int_{B_{\frac{R}{2}}(x_0)} F_{0p}(DV - DU)dx = 0$$

we obtain

$$\int_{B_{\frac{R}{2}}(x_0)} |D(U-V)|^m dx \le C\int_{B_{\frac{R}{2}}(x_0)} [F_0(DU) - F_0(DV)]dx$$

$$=C\int_{B_{\frac{R}{2}}(x_0)}[F_0(DU)-F_0(DV^*)]dx+C\int_{B_{\frac{R}{2}}(x_0)}[F_0(DV^*)-F_0(DV)]dx$$
$$:=I_1+I_2$$

It is easy to get

$$I_1\le 0$$

Moreover we have

$$I_2=\int_{B_{\frac{R}{2}}(x_0)}[F_0(DV^*)-F_0(DV)]dx=\int_{B_{\frac{R}{2}}(x_0)\cap\{V^N>h^N\}}[F_0(DV^*)-F_0(DV)]dx$$
$$+\int_{B_{\frac{R}{2}}(x_0)\cap\{V^N\le h^N\}}[F_0(DV^*)-F_0(DV)]dx$$
$$:=I_3+I_4$$

Notice that $D(V^N\vee h^N)=DV^N$ for $x\in B_{\frac{R}{2}}(x_0)\cap\{V^N>h^N\}$ and $D(V^N\vee h^N)=Dh^N$ for $x\in B_{\frac{R}{2}}(x_0)\cap\{V^N\le h^N\}$. Hence we have $I_3=0$ and

$$I_4=\int_{B_{\frac{R}{2}}(x_0)\cap\{V^N\le h^N\}}[F_0(DV^*)-F_0(DV)]dx$$
$$=\int_{B_{\frac{R}{2}}(x_0)\cap\{V^N\le h^N\}}[F_0(DV^*)-g(a^{\alpha\beta}b_{\tilde{i}\tilde{j}}D_\alpha V^{\tilde{i}}D_\beta V^{\tilde{j}})]$$
$$+\int_{B_{\frac{R}{2}}(x_0)\cap\{V^N\le h^N\}}[g(a^{\alpha\beta}b_{\tilde{i}\tilde{j}}D_\alpha V^{\tilde{i}}D_\beta V^{\tilde{j}})-F_0(DV)]dx$$
$$:=I_5+I_6$$

By assumption H.5 and Lemma 3.2 we have

$$I_6=\int_{B_{\frac{R}{2}}(x_0)\cap\{V^N\le h^N\}}g(a^{\alpha\beta}b_{\tilde{i}\tilde{j}}D_\alpha V^{\tilde{i}}D_\beta V^{\tilde{j}}-F_0(DV)]dx\le 0$$

By the definition of V^*, we have

$$I_5=\int_{B_{\frac{R}{2}}(x_0)\cap\{V^N\le h^N\}}[g(a^{\alpha\beta}b_{\tilde{i}\tilde{j}}D_\alpha V^{\tilde{i}}D_\beta V^{\tilde{j}}+2a^{\alpha\beta}b_{iN}D_\alpha V^{\tilde{i}}D_\beta h^N$$
$$+a^{\alpha\beta}b_{NN}D_\alpha h^N D_\beta h^N)-g(a^{\alpha\beta}b_{\tilde{i}\tilde{j}}D_\alpha V^{\tilde{i}}D_\beta V^{\tilde{j}})]dx$$
$$\le C\int_{B_{\frac{R}{2}}(x_0)}(|DV|^{m-2}|Dh^N|dx+|Dh^N|^m]dx$$
$$\le\varepsilon\int_{B_{\frac{R}{2}}(x_0)}|DV|^m dx+C\int_{B_{\frac{R}{2}}(x_0)}|Dh^N|^m dx$$

Concluding the above estimates, we have

$$\int_{B_\rho(x_0)}|\nabla U|^m dx \le C\Big[\Big(\frac{\rho}{R}\Big)^n\int_{B_R(x_0)}|\nabla U|^m dx+\varepsilon\Big]+CR^{n-m+m\alpha}$$

Through Lemma 2.1, Chapter III of [15], we obtain the results required.

Theorem 3.4 *Suppose that U is a minimum for the functional $\mathcal{F}_0$ in Ω with the obstacle μ_N of the form*

$$\mu_N\{u\in H^{1,m}(\Omega,\mathbb{R}^N)|u-U_0\in H_0^{1,m}(\Omega,\mathbb{R}^N), u^i\ge h^i \ a.e.;\ i=1,\cdots,N\}$$

where the integral $\mathcal{F}_0$ is defined as in Theorem 3.1. Let F_0 satisfy the same assumptions of Theorem 3.1 and $u_0\in u_N$. Let $|\nabla h|\in L^{m,n-m+m\alpha}(\Omega)$ with $h=(h^1,\cdots,h^N)$. Then the minimizer $U\in C^{0,\alpha}_{\rm loc}(\Omega,\mathbb{R}^N)$ and we have the same estimate (3.6) for U.

Proof Let $V\in H^{1,2}(B_{\frac{R}{2}}(x_0),\mathbb{R}^N)$ be a minimizer of the functional

$$\int_{B_{\frac{R}{2}}(x_0)}F_0(Dv)dx$$

for all $v-u\in H_0^{1,2}(B_{\frac{R}{2}},\mathbb{R}^N)$. We denote obstacles

$$\mu_i=\{u\in H^{1,m}(\Omega,\mathbb{R}^N)|u^{N-j+1}\ge h^{N-j+1}, j=1,\cdots,i\}$$

for $1\le i\le N$. We know that $\mu_N\subset\cdots\subset\mu_1$.

Let $U_{(i)}$ be a minimizer for the functional in $B_{\frac{R}{2}}(x_0)$ with the obstacle μ_i. We prove that for $1\le i\le N$

$$\int_{B_\rho(x_0)}H(DU_{(i)})dx\le C\Big[\Big(\frac{\rho}{B}\Big)^n+\varepsilon\Big]\int_{B_R(x_0)}H(DU_{(i)})dx+C\int_{B_{\frac{R}{2}}(x_0)}|Dh|^m dx \quad (3.10)$$

where ε is a small constant and

$$\int_{B_{\frac{R}{2}}(x_0)}|D(U_{(i)}-V)|^m dx\le C\int_{B_{\frac{R}{2}}(x_0)}|DU_{(i)}|^{m-2}|Dh|^2dx+C\int_{B_{\frac{R}{2}}(x_0)}|Dh|^m dx \quad (3.11)$$

Now we use the induction method to prove (3.10) and (3.11).

From Theorem 3.2 we know that (3.10) and (3.11) holds for $i=1$. Suppose that (3.11) is true for $1\le j\le i$. We shall prove that (3.11) holds for $j=i+1$. By the definition of the minimum $U_{(i)}$, we have

$$\int_{B_{\frac{R}{2}}(x_0)}F_{0p}(DU_{(i)})(DU_{(i+1)}-DU_{(i)})dx\ge 0$$

Combining this inequality with (3.9) and the assumption H.4, we have

$$\begin{aligned}\int_{B_{\frac{R}{2}}(x_0)}|D(U_{(i+1)}-U_{(i)})|^m dx &\le C\int_{B_{\frac{R}{2}}(x_0)}[F_0(DU_{(i+1)})-F_0(DU_{(i)})]dx\\ &= C\int_{B_{\frac{R}{2}}(x_0)}[F_0(DU_{(i+1)})-F_0(DV^\star_{(i+1)})]dx\\ &\quad+\int_{B_{\frac{R}{2}}(x_0)}[F_0(DDV^\star_{(i+1)})-F_0(DU_{(i)})]dx \qquad (3.12)\end{aligned}$$

Setting

$$V^\star_{(i+1)}=(U^1_{(i)},\dots,U^{N-i-1}_{(i)},U^{(N-i)}_{(i)},U^{N-i+1}_{(i)},\dots,U^N_{(i)})$$

we know that $V^\star_{(i+1)}\in\mu_{i+1}$. Using (3.12) and H.4, we repeat the similar proof as in Theorem 3.2 to obtain

$$\int_{B_{\frac{R}{2}}(x_0)}|D(U_{(i)}-U_{(i+1)})|^m dx\le\int_{B_{\frac{R}{2}}(x_0)}|DU|^{m-2}|Dh|^2dx+C\int_{B_{\frac{R}{2}}(x_0)}|Dh|^m dx$$

This proves that (3.11) holds for $j=i+1$. Then (3.10) is also true for $j=i+1$.

Theorem 3.5 *Let $m=2$. Let u be a minimizer of the functional*

$$\mathcal{F}=\int_\Omega a^{\alpha\beta}b_{ij}D_\alpha u^i D_\beta u^j$$

with the obstacle μ_N. Assume that $(a^{\alpha\beta}$ and (b_{ij}) are symmetric positive definite constant matrices. Suppose that $Dh\in C^{1,\alpha}(\Omega)$ for some $\alpha>0$ where $h=(h^1,\cdots,h^N)$. Then the minimizer $u\in C^{1,\alpha}_{\rm loc}(\Omega,\mathbb{R}^N)$.

Proof We use the induction method to prove this theorem. Assume μ_i be obstacles defined in Theorem 3.4.

We first prove the case of $i=1$. For each $v=(v^1,\cdots,v^N)$, we transform v into $v^\star$ by

$$\tilde{v}^\star=\tilde{v},\quad v^{\star N}=v^N-h^N$$

then the obstacle problem

$$\mathcal{F}(u,\Omega)\to\min_{v\in\mu_1}\mathcal{F}(v,\Omega)$$

is equivalent to the following new obstacle problem

$$\mathcal{F}^\star(u^\star,\Omega)\to\min_{v\in\mu_1^\star}\mathcal{F}^\star(v^\star,\Omega)$$

where $\mathcal{F}^\star=\int_\Omega[a^{\alpha\beta}b_{ij}D_\alpha v^{\star i}D_\beta v^{\star j}+2a^{\alpha\beta}b_{Nj}D_\alpha v^{\star j}D_\beta h^N+a^{\alpha\beta}b_{NN}D_\alpha h^N D_\beta h^N]dx$ and $\mu_1^\star=\{v\in H^{1,2}(\Omega,\mathbb{R}^N)|u^\star-v\in H^{1,2}_0(\Omega,\mathbb{R}^N),v^N\ge 0\}$.

For the sake of simplicity, we still denote $\mathcal{F}, u^*$ and μ_1^* by $\mathcal{F}$, u and μ_1. Define

$$\mathcal{F}^0(v, B_R(x_0)) = \int_{B_R(x_0)} F^0(Dv)dx$$

$$= \int_{B_R(x_0)} [a^{\alpha\beta} b_{ij} D_\alpha v^i D_\beta v^j + 2a^{\alpha\beta} b_{Ni} D_\alpha v^i (D_\beta h^N)_{x_0,R} + a^{\alpha\beta} b_{NN} D_\alpha h^N D_\beta h^N] dx$$

Let U_1 be a solution of the following obstacle problem

$$\mathcal{F}^0(U_1, B_R(x_0)) = \min_{v \in \mu_1} \mathcal{F}^0(v, B_R(x_0))$$

Let U_2 be a minimum of the functional $\mathcal{F}^0(v, B_R(x_0))$. By the standard theory of regularity, we have

$$\int_{B_\rho(x_0)} |DU_2 - (DU_2)_{x_0,\rho}|^2 dx \le C\Big(\frac{\rho}{R}\Big)^{n+2\sigma} \int_{B_R(x_0)} |DU_2 - (DU_2)_{x_0,R}|^2 dx$$

Then

$$\int_{B_\rho(x_0)} |DU_1 - (DU_1)_{x_0,\rho}|^2 dx \le C\Big(\frac{\rho}{R}\Big)^{n+2\sigma} \int_{B_R(x_0)} |DU_1 - (DU_1)_{x_0,R}|^2 dx + \int_{B_R(x_0)} |Dw|^2 dx$$

where $w = U_1 - U_2$.

Using the Euler-Lagrange equation and the ellipticities and $U_1 |_{\partial B_R(x_0)} = U_2 |_{\partial B_R(x_0)}$, we have

$$\int_{B_R(x_0)} |Dw|^2 dx \le \mathcal{F}^0(U_1, B_R(x_0)) - \mathcal{F}^0(U_2, B_R(x_0))$$

$$= \int_{B_R(x_0)} [a^{\alpha\beta} b_{ij} D_\alpha U_1^i D_\beta U_1^j - a^{\alpha\beta} b_{ij} D_\alpha U_2^i D_\beta U_2^j] dx$$

Using the same proof in Theorem 3.3 we obtain

$$\int_{B_R(x_0)} |Dw|^2 dx = 0$$

This implies that

$$\int_{B_\rho(x_0)} |DU_1 - (DU_1)_{x_0,\rho}|^2 dx \le C\Big(\frac{\rho}{R}\Big)^{n+2\sigma} \int_{B_R(x_0)} |DU_1 - (DU_1)_{x_0,R}|^2 dx$$

Then

$$\int_{B_\rho(x_0)} |Du - (Du)_{x_0,\rho}|^2 dx \le C\Big(\frac{\rho}{R}\Big)^{n+2\sigma} \int_{B_R(x_0)} |Du - (Du)_{x_0,R}|^2 dx + \int_{B_R(x_0)} |D(u - U_1)|^2 dx \tag{3.13}$$

On the other hand,

$$\begin{aligned}\int_{B_R(x_0)}|Du-DU_1|^2dx &\le \mathcal{F}^0(u,B_R(x_0))-\mathcal{F}^0(U_1,B_R(x_0))\\ &\le \int_{B_R(x_0)}[F^0(Du)-F(x,Du)]dx+\int_{B_R(x_0)}[F(x,Du)-F(x,DU_1)]dx\\ &\quad+\int_{B_R(x_0)}[F(x,DU_1)-F^0(DU_1)]dx\\ &:= I_1+I_2+I_3 \end{aligned} \tag{3.14}$$

where for all $v^\star$, we define

$$F(x,Du^\star):=a^{\alpha\beta}b_{ij}D_\alpha v^{\star j}+2a^{\alpha\beta}b_{Nj}D_\beta v^{\star j}D_\beta h^N+a^{\alpha\beta}b_{NN}D_\alpha h^N D_\beta h^N$$

Since $u^\star$ is a minimizer of $\mathcal{F}^\star$, we have that $I_2\le 0$. Noticing

$$\int_{B_R(x_0)}[Dh^N-(Dh)^N)_{x_0,R}]dx=0$$

we have

$$\begin{aligned}I_1&=\int_{B_R(x_0)}2a^{\alpha\beta}B_{Nj}D_\alpha u^\star[(D_\beta h^N)_{x_0,R}-D_\beta h^N]dx\\ &=\int_{B_R(x_0)}2a^{\alpha\beta}B_{Nj}[D_\alpha u^\star-(D_\alpha u^\star)_{x_0,R}][(D_\beta h^N)_{x_0,R}-D_\beta h^N]dx\\ &\le C\varepsilon\int_{B_R(x_0)}|Du^\star-(Du^\star)_{x_0,R}|^2dx+\int_{B_R(x_0)}|Dh^N-(Dh^N)_{x_0,R}|^2dx \end{aligned}\tag{3.15}$$

Similarly, we have

$$\begin{aligned}I_3&\le\int_{B_R(x_0)}2a^{\alpha\beta}b_{Nj}[D_\alpha U_1-(D_\alpha U_1)_{x_0,R}][(Dh^N)_{x_0,R}-Dh^N]dx\\ &\le C\varepsilon\int_{B_R(x_0)}|Du^\star-(Du^\star)_{x_0,R}|^2dx+C\varepsilon\int_{B_R(x_0)}|Du^\star-DU_1|^2dx\\ &\quad+C\int_{B_R(x_0)}|Dh^n-(Dh^N)_{x_0,R}|^2dx \end{aligned}\tag{3.16}$$

Using (3.13)–(3.16) and Lemma 2.1 of [15, p. 86] together with Campanato's characterization of Hölder-continuous function [15, p.70] we have that Du is locally Hölder continuous. Using the same method in Theorem 3.4, we obtain the result required.

4. Partial Regularities for the Obstacle

In this section, we consider the functional of the form (1.1). Assume that the integrand $F(x,u,p)$ satisfies assumptions H.1, $\cdots$, H.5 and is of the special form (1.2).

We have the following results

Theorem 4.1 *Let $u \in H^{1,m}(\Omega, \mathbb{R}^N) \cap \mu_f$ be a minimizer of the functional* (1.1) *with the graphic obstacle* (i). *Suppose that the integrand $F(x,u,p)$ satisfies the main assumptions* H.1, $\cdots$, H.5 *and* (1.2). *Assume that the function $f_1(\widetilde{u})$ and $f_2(x)$ in the obstacle* (i) *belongs to space $C^1(\mathbb{R}^{N_1})$, $|Df_1(y)| \le L$ and $|Df_2| \le L$ for some $L > 0$. Then there exists an open set $\Omega_0 \subset \Omega$ such that $u \in C^{0,\tau}(\Omega_0)$ for any $\tau \in (0,1)$. Moreover $\Omega \backslash \Omega_0 \subset \Sigma_1 \cap \Sigma_2$ where*

$$\Sigma_1 = \left\{ x \in \Omega : \sup_R |u_{x,R}| = +\infty \right\}$$

$$\Sigma_2 = \left\{ x \in \Omega : R^{m-n} \int_{B_R(x)} |Du|^m dy > 0 \right\}$$

In particular $\mathcal{H}^{n-m-\varepsilon}(\Omega \backslash \Omega_0) = 0$ for some positive $\varepsilon > 0$.

Proof For each $v = (v^1, \cdots, v^N)$, we transform v into $v^\star$ by

$$\widetilde{v}^\star = \widetilde{v}, \quad v^{\star N} = v^N - f(\widetilde{v}) \tag{4.1}$$

Then the obstacle problem

$$\mathcal{F}(u;\Omega) \to \min_{v \in \mu_f} \mathcal{F}(v;\Omega)$$

is equivalent to the following new obstacle problem

$$\mathcal{F}^\star(u^\star;\Omega) \to \min_{v^\star \in \mu_f^\star} \mathcal{F}^\star(v^\star;\Omega) \tag{4.2}$$

Where the integrand $F^\star$ satisfies the same main assumptions H.1, $\cdots$, H.5,

$$F^\star(x,u,p) = g(x,u,a^{\alpha\beta}(x,u) b^\star_{ij}(x,u) p^i_\alpha p^j_\beta)$$

and

$$\mu_f^\star = \{ v \in \mathbb{R}^N | V^N \ge f_2 \text{ a.e., } v - u^\star \in W_0^{1,m}(\Omega, \mathbb{R}^N) \}$$

Therefore, we shall discuss the new obstacle problem (4.2) instead of the obstacle problem (4.1). For the sake of simplicity we still denote $u^\star, F^\star, \mathcal{F}$ and $\mu_f^\star$ by $u, F, \mathcal{F}$ and μ_f.

Let $x_0 \in \Omega$, $R < \operatorname{dist}(x_0, \partial\Omega)$, $u_0 = u_{x_0, \frac{R}{2}}$ and

$$F^0(p) = F(x_0, u_0, p) = g(x_0, u_0, a^{\alpha\beta}(x_0, u_0) b_{ij}(x_0, u_0) p^i_\alpha p^j_\beta)$$

using Theorem 2.5, there exists a $v \in H^{1,m}(B_{\frac{R}{2}}(x_0); \mathbb{R}^N)$ such that it is a minimizer of the following obstacle problem

$$\int_{B_{\frac{R}{2}}(x_0)} F^0(DV)dx \to \min_{v \in \mu_f} \int_{B_{\frac{R}{2}}(x_0)} F^0(DV)dx$$

where $\mu_f = \{v \in H^{1,m}(B_{\frac{R}{2}}(x_0), \mathbb{R}^N) | v^N \geq f_2, v - u \in H_0^{1,m}(B_{\frac{R}{2}}(x_0), \mathbb{R}^N)\}$ and $F^0(p)$ satisfies the hypothesis of Theorem 3.1.

From Theorem 3.3 we have

$$\int_{B_\rho(x_0)} H(DU)dx \leq C\Big(\frac{\rho}{R}\Big)^n \int_{B_{\frac{R}{2}}(x_0)} H(DU)dx$$

for $0 < \rho < \dfrac{R}{2}$. Then we get

$$\int_{B_\rho(x_0)} H(DU)dx \leq C\Big(\frac{\rho}{R}\Big)^n \int_{B_{\frac{R}{2}}(x_0)} H(Du)dx + C\int_{B_{\frac{R}{2}}(x_0)} |D(u-U)|^m dx$$

Using the formula (3.9), the ellipticity assumption and variational inequalities, we obtain

$$\begin{aligned}\int_{B_\rho(x_0)} |D(u-U)|^m dx &\leq C\int_{B_\rho(x_0)} [F^0(Du) - F^0(DU)]dx \\ &\leq \int_{B_\rho(x_0)} [F^0(Du) - F(x,u,Du)]dx + \int_{B_\rho(x_0)} [F(x,u,Du) - F(x,U,DU)]dx \\ &\quad + \int_{B_\rho(x_0)} [F(x,U,DU) - F^0(DU)]dx\end{aligned}$$

In a similar way as done in [7], we conclude that

$$\int_{B_\rho(x_0)} H(Du)dx \leq C\Big[\Big(\frac{\rho}{R}\Big)^n + \chi(x_0,R)\Big] \int_{B_R(x_0)} H(Du)dx$$

where $\chi(x_0,R) = \eta\Big(C|u_{x_0,R}|, R + C\Big(R^{m-n}\int_{B_R(x_0)} |Du|^m dx\Big)^{\frac{1}{m}}\Big)^{\frac{1}{m}}$. Then result required follows from (4.5)–(4.6) in a standard way, see e.g. [15, pp. 170–171, 105–106].

Theorem 4.2 *Let $u \in H^{1,m}(\Omega, \mathbb{R}^N) \cap \mu^N$ be a minimizer of the functional* (1.1) *with the obstacle μ^N of the form* (ii). *Suppose that the integrand $F(x,u,p)$ satisfies the main assumptions* H.1, $\cdots$, H.5 *and* (1.2). *Assume that the function $h(x)$ in the obstacle* (ii) *is C^1-continuous and $|Dh| \leq L$ for some constant $L > 0$. Then there exists an open set $\Omega_0 \subset \Omega$ such that $u \in C^{0,\tau}(\Omega_0)$ for any $\tau \in (0,1)$. Moreover $\Omega \backslash \Omega_0 \subset \Sigma_1 \cap \Sigma_2$, where*

$$\Sigma_1 = \Big\{x \in \Omega : \sup_R |u_{x,R}| = +\infty\Big\}$$

$$\Sigma_2 = \Big\{x \in \Omega : R^{m-n}\int_{B_R(x)} |Du|^m dy > 0\Big\}$$

In particular $\mathcal{H}^{n-m-\varepsilon}(\Omega \backslash \Omega_0) = 0$ for some positive $\varepsilon > 0$.

Theorem 4.3 *When $m - 2$, let*

$$F(x,u,p) = a^{\alpha\beta}(x,u)b_{ij}(x,u)p_\alpha^i p_\beta^j$$

Suppose that all assumptions of Theorem 4.2 hold and $Dh \in C^{0,\alpha}$. *Then* $Du \in C^{0,\alpha}(\Omega_0, \mathbb{R}^N)$.

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