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ON THE ZEROS AND ASYMPTOTIC BEHAVIOR OF MINIMIZERS TO THE GINZBURG-LANDAU FUNCTIONAL WITH VARIABLE COEFFICIENT

Ding Shijin
(Department of Mathematics, Suzhou University, Suzhou 215006, China)
Liu Zuhan
(Department of Mathematics, Normal College, Yangzhou University, Yangzhou 225002, China)

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Abstract In this paper a partial answer to the fourth open problem of Bethuel-Brezis-Hélein [1] is given. When the boundary datum has topological degree ±1, the asymptotic behavior of minimizers of the Ginzburg-Landau functional with variable coefficient $\frac{1}{x_1}$ is given. The singular point is located.

Key Words Ginzburg-Landau functional; asymptotics; vortices.
Classification 35J55, 35Q40.

1. Introduction

Recently, Bethuel-Brezis-Hélein [1-3] have studied the asymptotic behavior for the minimizers u_ε of the following Ginzburg-Landau functional in $H_g^1(\Omega; R^2) \equiv \{v \in H^1(\Omega, R^2), u\mid_{\partial\Omega}= g\}$,

$$E_\varepsilon(u) = \frac{1}{2}\int_\Omega \left[|\nabla u|^2 + \frac{1}{2\varepsilon^2}(1-|u|^2)^2\right] \tag{1.1}$$

where Ω is a simply connected, star-shaped and bounded smooth domain in R^2, $g : \partial\Omega \to S^1$ is a smooth map, ε is a small parameter. They proved that there is a subsequence $\varepsilon_n \downarrow 0$ such that

$$u_{\varepsilon_n} \to u_* \text{ in } C^{1+\alpha}_{\rm loc}(\overline{\Omega}\backslash\{a_1, \cdots, a_{|d|}\}) \text{ and in } C^k_{\rm loc}(\Omega), \quad \forall k \in \mathbb{N}$$

where $d = \deg(g, \partial\Omega)$ denotes the winding number, $u_* : \Omega\backslash\{a_1, \cdots, a_{|d|}\} \to S^1$ is a smooth harmonic map, $a_1, \cdots, a_{|d|}$ are the limit positions of the zeros of u_{ε_n} (zeros of u_{ε_n} are called vortices which correspond to the normal points in superconductor) which minimize the so-called renormalized energy $W(b)$ (see [1]). This problem is related to the phase transition in superconductivity (see [4]).

In their proofs, a key estimate

$$\frac{1}{\varepsilon^2}\int_\Omega (1-|u_\varepsilon|^2)^2 \le C \tag{1.2}$$

is derived from the global Pohozaev identity. From (1.2), for ε small enough, one can obtain the uniform upper bound on the number of zeros of u_ε. Then the precise lower and upper bounds on the energy $E_\varepsilon(u_\varepsilon)$ lead to *a priori* estimate for u_ε in $H^1_{\mathrm{loc}}(\Omega\backslash\{a_1,\cdots a_{|d|}\})$. Finally, they obtained the convergence of u_{ε_n}, subsequence of minimizers, in various norms.

In [5], based on a local version of (1.2), M. Struwe got a similar result to [1] without the restriction of star-shapedness on Ω. There are also some other generations (see [6–10]).

In this paper, we discuss open Problem 4 in [1]. That is,

$$E_\varepsilon(u_\varepsilon)=\frac{1}{2}\int_\Omega \frac{1}{x_1}\left[|\nabla u_\varepsilon|^2+\frac{1}{\varepsilon^2}(1-|u_\varepsilon|^2)^2\right] \tag{1.3}$$

where $\Omega=\{(x_1,x_2)\in R^2|(x_1-1)^2+x_2^2<R^2, 0<R<1\}$, $u_\varepsilon\in H^1_g(\Omega,R^2)$, g is as above. We intend to study the behaviour of minimizers u_{ε_n} as $\varepsilon_n\downarrow 0$. This problem is related to the model of superconducting thin films having variable thickness (see [11]). In contrast with [1], we call our problem Ginzburg-Landau model with variable coefficient.

In our case, some arguments in [1] or [5] do not work. As a try, we only consider a special situation, i.e., $\deg(g,\partial\Omega)=\pm 1$. By a different way, we prove that u_ε has unique zero (in Section 3). To get a uniform estimate, we use Lemma 4.4 to prove that $|u_\varepsilon|\ge\frac{1}{2}$ in $\overline{\Omega}\backslash B(x_\varepsilon, 2\varepsilon^{\beta_1})$, $0<\beta_1<1/2$, x_ε is the unique zero of u_ε. This is much different from [1] in which they prove $|u_\varepsilon|\ge\frac{1}{2}$ in $\overline{\Omega}\backslash B(x_\varepsilon,\lambda_0\varepsilon)$. Next, we prove that $x_\varepsilon\to a=(1+R,0)$ and for any sequence u_{ε_n}, there is a subsequence, still denoted by u_{ε_n}, such that $u_{\varepsilon_n}\to u_*$ in $C^k(K)$, $\forall k\in\mathbb{N}$, $\forall K\subset\subset\Omega$, where u_* is a harmonic map from $\Omega\to S^1$. The Euler equation of (1.3) is

$$\begin{cases} -\Delta u_\varepsilon+\dfrac{1}{x_1}u_{\varepsilon x_1}=\dfrac{1}{\varepsilon^2}u_\varepsilon(1-|u_\varepsilon|^2) & \text{in } \Omega \\ u_\varepsilon\,|_{\partial\Omega}=g \end{cases} \tag{1.4}$$

This paper is organized as follows. In Section 2, we shall discuss the case $\deg(g,\partial\Omega)=0$ which is the base of the case $|\deg(g,\partial\Omega)|=1$; In Section 3, we prove the existence and uniqueness of the zero of u_ε; In Section 4, through a series of *a priori* estimates, we establish the asymptotic behavior of u_ε, i.e., Theorem 4.1, our main result.

2. Results for $\deg(g, \partial\Omega) = 0$

In this section, we assume $\Omega \subset R^2$ is a simply connected bounded smooth domain and star-shaped with respect to a point $x_* \in \Omega$, $b \geq x_1 \geq a > 0$ for any $x = (x_1, x_2) \in \Omega$ (a, b are constants), $g : \partial\Omega \to S^1$ is smooth and

$$\deg(g, \partial\Omega) = 0 \tag{2.1}$$

Let u_ε be the minimizers of $E_\varepsilon(u)$ in $H^1_g(\Omega, R^2)$, i.e.,

$$E(u_\varepsilon) = \inf_{u \in H^1_g(\Omega, R^2)} \left(\frac{1}{2} \int_\Omega \frac{1}{x_1} \left[|\nabla u|^2 + \frac{1}{2\varepsilon^2}(1 - |u|^2)^2 \right] \right) \tag{2.2}$$

We have the following lemma.

Lemma 2.1 *Let (2.1) hold. We have*

$$u_\varepsilon \to u_0 \text{ strongly in } H^1(\Omega, R^2) \tag{2.3}$$

where u_0 satisfies $E(u_0) = \inf_{u \in H^1_g(\Omega, S^1)} \left(\frac{1}{2} \int_\Omega \frac{1}{x_1} |\nabla u|^2 dx_1 dx_2 \right)$

Proof There is a smooth function $\varphi_0 : \partial\Omega \to R$ such that

$$g = e^{i\varphi_0} \quad \text{on } \partial\Omega$$

since Ω is simply connected and $\deg(g, \partial\Omega) = 0$. It is clear that one can minimize $\frac{1}{2} \int_\Omega \frac{1}{x_1} |\nabla u|^2$ in $H^1_g(\Omega, S^1)$ by some u_0 in which similarly to [2], $u_0 = e^{i\varphi_1}$ in Ω, where φ_1 uniquely solves

$$\begin{cases} -\mathrm{div}\left(\frac{1}{x_1} \nabla \varphi_1 \right) = 0 & \text{in } \Omega \\ \varphi_1 = \varphi_0 & \text{on } \partial\Omega \end{cases}$$

Therefore

$$\frac{1}{2} \int_\Omega \frac{1}{x_1} |\nabla u_\varepsilon|^2 + \frac{1}{4\varepsilon^2} \int_\Omega \frac{1}{x_1} (1 - |u_\varepsilon|^2)^2 \leq \frac{1}{2} \int_\Omega \frac{1}{x_1} |\nabla u_0|^2 < \infty \tag{2.4}$$

and then there is a subsequence $\varepsilon_n \downarrow 0$ such that

$$u_{\varepsilon_n} \rightharpoonup u \quad \text{weakly in } H^1$$

(2.4) and lower semi-continuity imply

$$\frac{1}{2} \int_\Omega \frac{1}{x_1} |\nabla u|^2 \leq \frac{1}{2} \int_\Omega \frac{1}{x_1} |\nabla u_0|^2$$

On the other hand, we also have

$$\int_\Omega (1 - |u_\varepsilon|^2)^2 \leq C\varepsilon^2$$

which implies $|u| = 1$ a.e. and $u \in H^1_g(\Omega, S^1)$. Moreover, from the minimizing property and (2.4) we deduce that

$$\lim_{n\to\infty} \int_\Omega \frac{1}{x_1} |\nabla u_{\varepsilon_n}|^2 = \int_\Omega \frac{1}{x_1} |\nabla u_0|^2$$

and $u_{\varepsilon_n} \to u_0$ in $H^1(\Omega)$ since $0 < a \le x_1 \le b$. The convergence of the full sequence is a consequence of the uniqueness of u_0.

By modifying the proofs of Lemmas A.1 and A.2 in [2], one can prove

Lemma 2.2 *Under the assumptions of this section we have*

$$|u_\varepsilon| \le 1, \quad |\nabla u_\varepsilon| \le \frac{C}{\varepsilon} \quad \text{in } \Omega \tag{2.5}$$

The following two lemmas can be proved by the same method as in [1].

Lemma 2.3 *Let u_ε be a minimizer of* (2.2). *Then*

$$\int_{\partial\Omega} \left|\frac{\partial u_\varepsilon}{\partial n}\right|^2 \le C = C(g, \Omega) \tag{2.6}$$

Lemma 2.4 *There exist positive constants λ_0, μ_0 depending only on g and Ω such that if u_ε is as above satisfying*

$$\frac{1}{\varepsilon^2} \int_{\Omega\cap B_{2l}} (1 - |u_\varepsilon|^2)^2 \le \mu_0 \quad \text{when } \frac{l}{\varepsilon} \ge \lambda_0, \ l \le 1 \tag{2.7}$$

then

$$|u_\varepsilon(x)| \ge \frac{1}{2}, \quad \forall x \in \Omega \cap B_l \tag{2.8}$$

where B_l is a ball with radius $l > 0$.

Proof See the proof of Theorem III.3 of [1].

Corollary 2.5 *There exists $\varepsilon_0 > 0$ such that for $\varepsilon \le \varepsilon_0$*

$$|u_\varepsilon| \ge \frac{1}{2} \quad \text{in } \overline{\Omega} \tag{2.9}$$

Proof Since $u_\varepsilon \to u_0$ in H^1, we have

$$\frac{1}{\varepsilon^2} \int_\Omega (1 - |u_\varepsilon|^2)^2 \to 0 \tag{2.10}$$

and (2.9) follows from Lemma 2.4.

Now, we can prove the following theorem by the same method as that in [2].

Theorem 2.6 *We have, as $\varepsilon \to 0$,*

$$u_\varepsilon \to u_0 \quad \text{in } C^{1+\alpha}(\overline{\Omega}), \quad \forall \alpha \in (0,1) \tag{2.11}$$

$$\|\Delta u_\varepsilon\|_{L^\infty(\Omega)} \le C \tag{2.12}$$

$$u_{\varepsilon_n} \to u_* \quad in\ C^k(K), \quad \forall k \in \mathbb{N}, \quad \forall K \subset\subset \Omega \tag{2.13}$$

We now turn to the minimization problem (2.2) with g replaced by g_ε where $g_\varepsilon : \partial\Omega \to R^2$ and $g_\varepsilon \to g$ uniformly on $\partial\Omega$ as well as

$$\|g_\varepsilon\|_{L^\infty(\partial\Omega)} \le 1$$

$$\|g_\varepsilon\|_{H^1(\partial\Omega)} \le C$$

$$\int_{\partial\Omega} (1 - |g_\varepsilon|^2)^2 \le C\varepsilon^2 \tag{2.14}$$

It is clear that $|g| = 1$ on $\partial\Omega$ and $\deg(g, \partial\Omega)$ is well defined. In what follows, we denote by u_ε the corresponding minimizers. We still assume $\deg(g, \partial\Omega) = 0$. Then g can be written as

$$g = e^{i\varphi_0} \quad \text{on } \partial\Omega \tag{2.15}$$

where $\varphi_0 : \partial\Omega \to R$ is a continuous function and $\varphi_0 \in H^1(\partial\Omega)$. Let

$$u_0 = e^{i\varphi_1}$$

$$\begin{cases} -\nabla \cdot \left(\dfrac{1}{x_1}\nabla\varphi_1\right) = 0 & \text{in } \Omega \\ \varphi_1 \mid_{\partial\Omega} = \varphi_0 \end{cases} \tag{2.16}$$

We have

Theorem 2.7 *Under the above assumptions, there hold*

$$u_\varepsilon \to u_0, \quad strongly\ in\ H^1(\Omega) \tag{2.17}$$

$$u_\varepsilon \to u_0, \quad uniformly \quad on\ \overline{\Omega} \tag{2.18}$$

$$u_\varepsilon \to u_0, \quad in\ C^k_{\mathrm{loc}}(\Omega), \quad \forall k \in \mathbb{N} \tag{2.19}$$

Proof The method proving Theorem 2 in [2] now can be applied.

3. Zeros of Minimizers

In this section, we discuss the zeros of the solutions of (1.4). Assume $\deg(g, \partial\Omega) = \pm 1$, we prove that the solution u_ε minimizing (2.2) has unique zero. The argument is similar to that of [12].

Let $Y(s)$ $(0 \le s < 2\pi R)$ be a one-to-one parameterization of $\partial\Omega$ with arclength. Consider Dirichlet data, g, in $C^{2+\alpha}(\partial\Omega, R^2)$. In polar coordinate, we have

$$g(Y(s)) = (\cos\theta(s), \sin\theta(s)) \tag{3.1}$$

and we assume that

$$\theta'(s) \neq 0 \text{ for } 0 \leq s < 2\pi R, \quad |\theta(2\pi R) - \theta(0)| = 2\pi \tag{3.2}$$

Thus $g(Y(s))$ crosses each ray $\theta = \theta_0$ exactly once as s increases from zero to $2\pi R$. In the following, we set $y_1 = x_1 - 1$, $y_2 = x_2$ and still denote them by x_1, x_2.

Lemma 3.1 *There exists at least one minimizer for $E_\varepsilon(\cdot)$ in $H^1_g(\Omega, R^2)$ which must be a weak solution of*

$$\begin{cases} -\Delta u + \dfrac{1}{1+x_1} u_{x_1} = \dfrac{1}{\varepsilon^2} u(1 - |u|^2) & \text{in } \Omega \\ u = g & \text{on } \partial\Omega \end{cases} \tag{3.3}$$

Moreover, any weak solution $\tilde{u}$ of (3.3) in $H^1(\Omega, R^2)$ is of class $C^{2+\alpha}(\overline{\Omega}, R^2)$ and

$$\|\tilde{u}\|_{C^{2+\alpha}(\overline{\Omega}, R^2)} \leq C(\|\tilde{u}\|_{H^1(\Omega)}, \|g\|_{C^{2+\alpha}(\partial\Omega)}) \tag{3.4}$$

Proof The general theory of variational problems ([13, Chapter I]) implies the existence of a minimizer u in $H^1_g(\Omega, R^2)$. In addition, $u \in L^p(\Omega, R^2)$ follows from imbedding theorem for any $p < +\infty$ since $\Omega \subset R^2$. And clearly, u solves (3.3).

Equation (3.4) follows from standard elliptic estimates.

For $\alpha \in \mathbb{R}$, $u = (u_1, u_2)$, a minimizer of $E_\varepsilon(\cdot)$ in $H^1_g(\Omega, R^2)$, set

$$w_\alpha(X) = -u_1(X) \sin\alpha + u_2(X) \cos\alpha$$

$$N_\alpha \equiv \{x \in \overline{\Omega} \mid w_\alpha(X) = 0\}$$

Lemma 3.2 *For each α, N_α is a C^1 imbedded curve in $\overline{\Omega}$, which contacts $\partial\Omega$ at two distinct points.*

Proof First, consider $N_\alpha \cap \partial\Omega$.

From (3.2) we have $N_\alpha \cap \partial\Omega = \{p_1, p_2\}$. Let $Y(s_1) = p_1$, $Y(s_2) = p_2$ we can assume, without loss of generality, that $\theta(s_1) = \alpha + \pi$, $\theta(s_2) = \alpha$, then

$$w_\alpha(Y(s)) = [-\cos\theta(s) \sin\alpha + \sin\theta(s) \cos\alpha]$$

Hence

$$\frac{\partial}{\partial s} w_\alpha(Y(s)) \, |_{s_1} = \theta'(s_1) \neq 0$$

$$\frac{\partial}{\partial s} w_\alpha(Y(s)) \, |_{s_2} = \theta'(s_2) \neq 0$$

Therefore, there are neighborhoods O_1 and O_2 of p_1 and p_2, respectively, such that $N_\alpha \cap O_1$ and $N_\alpha \cap O_2$ are C^1 curves intersecting $\partial\Omega$ at p_1 and p_2.

Note that w_α is a $C^{2+\alpha}$ solution of

$$\Delta w_\alpha - \frac{1}{1+x_1} w_{\alpha x_1} + \frac{1}{\varepsilon^2}(1-|u|^2) w_\alpha = 0 \quad \text{in } \Omega$$

and $\frac{1}{\varepsilon^2}(1-|u|)^2$, $\frac{1}{1+x_1}$ are continuous. It follows from Hartman and Wintner's classical results ([14, Th1-2 and Cor. 1]) that the set $K_\alpha = \{x \in \Omega \mid w_\alpha = 0, \nabla w_\alpha = 0\}$ is locally finite. Our previous analysis near $\partial\Omega$ then implies that K_α is either empty or a finite subset of Ω. It also follows from [14] and our analysis near $\partial\Omega$ that N_α consists of a finite number of C^1 arcs along which $\nabla w_\alpha \neq 0$ except at their endpoints in Ω; moreover, the arcs may intersect only at these (interior) endpoints. Exactly two endpoints of these arcs are at $\partial\Omega$, and the rest make up K_α.

Finally, we note that at least four distinct arcs in N_α meet at each point in K_α. This follows from Hartman and Wintner's analysis of w_α near x_0 in K_α: indeed, they show that for some integer n there is a homogeneous harmonic polynomial, H_n, of order n so that

$$w_\alpha(x) - H_n(x-x_0) = o(|x-x_0|^n)$$

and

$$\nabla w_\alpha(x) - \nabla H_n(x-x_0) = o(|x-x_0|^n)$$

(see (5) and (5′) of Section 1 in [14]). This demonstrates that the nodal set of w_α has the same structure near x_0 as that of the harmonic function $H(x-x_0)$.

Now, the proof left over is just the same as that in [12], we omit it.

With the help of Lemmas 3.1 and 3.2, we can prove as in [12] the following

Theorem 3.3 *Under conditions (3.1), (3.2), the minimizer u_ε of E_ε in H^1_g has unique zero $x_\varepsilon \in \Omega$, (for $0<\varepsilon<1$) with sign (deg $(g, \partial\Omega)$) as its degree.*

4. Main Result and Its Proof

In this section, we prove our main result of this paper under the conditions (3.1) and (3.2).

Theorem 4.1 *Let (3.1) and (3.2) be fulfilled. Then $x_\varepsilon \to a = (1+R, 0)$ (as $\varepsilon \to 0$). And, for any $K \subset\subset \Omega$, we have, for some $\varepsilon_n \downarrow 0$,*

$$u_{\varepsilon_n} \to u_* \quad in\ C^k(K), \quad \forall k \in \mathbb{N} \tag{4.1}$$

where u_{ε_n} is the minimizer of (2.2), and u_* satisfies

$$\begin{cases} -\nabla \cdot \left(\dfrac{1}{x_1}\nabla u_*\right) = \dfrac{1}{x_1} u_* |\nabla u_*|^2 & in\ \Omega \\ |u_*| = 1 & in\ \Omega \end{cases} \tag{4.2}$$

To prove this theorem, we need several lemmas. We first give an upper bound for $E_\varepsilon(\cdot)$.

From now on, we always assume that (3.1) and (3.2) hold. For simplicity, we assume $\deg(g, \partial\Omega) = 1$.

Lemma 4.2 *For any $\sigma_0 \in (0,1)$, there is a constant $C_1 = C_1(\Omega, g, \sigma_0)$, such that for $0 < \varepsilon \le 1$*

$$\inf_{u\in H_g^1(\Omega,R^2)} E_\varepsilon(u) \le \frac{1}{1+R-\sigma_0}\pi|\log\varepsilon| + C_1 \tag{4.3}$$

Proof Given $\sigma_0 \in (0,1)$, we may find a ball $B_\rho(x_0) \subset\subset \Omega$ such that $x_1 \in (1+R_0-\sigma_0, 1+R)$, $\forall x = (x_1, x_2) \in B_\rho(x_0)$. Consider new domain $\widetilde{\Omega} = \Omega\backslash B_\rho(x_0)$ and new boundary data $\widetilde{g}(x) : \widetilde{g}(x) = g(x)$ on $\partial\Omega, \widetilde{g}(x) = g_1(x) = \dfrac{x-x_0}{|x-x_0|}$, on $\partial B_\rho(x_0)$. Then $\deg(\widetilde{g}, \partial\widetilde{\Omega}) = 0$ since $\deg(g, \partial\Omega) = 1$. This implies that there is a map $\widetilde{u} \in H_{\widetilde{g}}^1(\widetilde{\Omega}, S^1)$. Therefore

$$E_\varepsilon(\widetilde{u}, \widetilde{\Omega}) \le C(\rho, g)$$

On the other hand, for $\varepsilon > 0$, $\rho > 0$ small enough, let v_ρ be the minimizer of

$$I(\varepsilon,\rho) = \inf_{v\in H_{g_1}^1(B_\rho(x_0),R^2)} \left[\frac{1}{2}\int_{B_\rho(x_0)} |\nabla v|^2 + \frac{1}{4\varepsilon^2}\int_{B_\rho(x_0)} (1-|v|^2)^2\right]$$

It follows from [1] that

$$I(\varepsilon,\rho) \le \pi|\log\varepsilon| + C(\rho)$$

Therefore, taking $v = \begin{cases} \widetilde{u} & \text{in } \widetilde{\Omega} \\ v_\rho & \text{in } B_\rho(x_0) \end{cases}$ as a comparison function, we have

$$\begin{aligned}\inf_{u\in H_g^1(\Omega,R^2)} E_\varepsilon(u) &\le E_\varepsilon(v,\Omega) = E_\varepsilon(\widetilde{u},\widetilde{\Omega}) + E_\varepsilon(v_\rho, B_\rho(x_0)) \\ &\le C(\rho) + \frac{1}{1+R-\sigma_0} I(\varepsilon,\rho) \\ &\le \frac{1}{1+R-\sigma_0}\pi|\log\varepsilon| + C_1(\Omega, g, \sigma_0)\end{aligned}$$

Lemma 4.3 *Any critical point $u_\varepsilon \in H_g^1(\Omega, R^2)$ of $E_\varepsilon(\cdot)$ satisfies*

$$|u_\varepsilon| \le 1, \quad |\nabla u_\varepsilon| \le C/\varepsilon, \quad in\ \Omega \tag{4.4}$$

with a uniform constant C depending only on g and Ω.

Proof See the proof of Lemma 2.2.

For each $\varepsilon > 0$, any minimizer u_ε has exactly one zero $x_\varepsilon \in \Omega$. We denote for $\rho > 0$,

$$f(\rho) = \rho\int_{\partial B_\rho(x)\cap\Omega} \frac{1}{x_1}\left[|\nabla u_\varepsilon|^2 + \frac{1}{2\varepsilon^2}(1-|u_\varepsilon|^2)^2\right] do$$

where *do* denotes the arc-length measure.

Lemma 4.4 *For $0 < \varepsilon < e^{-1}$, there exists $\beta_1 \in [\alpha, 2\alpha]$ for some $\alpha \in (0,1)$ such that*

$$\begin{aligned}\frac{1}{1+R}\varepsilon^{\beta_1}\int_{\partial B_{\varepsilon^{\beta_1}}\cap\Omega}\Big[|\nabla u|^2+\frac{1}{2\varepsilon^2}(1-|u|^2)^2\Big]do\\ \le f(\varepsilon^{\beta_1}) = \varepsilon^{\beta_1}\int_{\partial B_{\varepsilon^{\beta_1}}\cap\Omega}\frac{1}{x_1}\Big[|\nabla u|^2+\frac{1}{2\varepsilon^2}(1-|u|^2)^2\Big]do\\ \le C(\alpha)\end{aligned} \tag{4.5}$$

Proof From Fubini's theorem we have

$$\begin{aligned}E_\varepsilon(u_\varepsilon) &\ge \frac{1}{2}\int_{\varepsilon^{2\alpha}}^{\varepsilon^\alpha} f(\rho)\frac{d\rho}{\rho}\\ &\ge \frac{\alpha}{2}|\log\varepsilon| \inf_{\varepsilon^{2\alpha}\le\rho\le\varepsilon^\alpha} f(\rho)\\ &= \frac{\alpha}{2}|\log\varepsilon| f(\varepsilon^{\beta_1})\end{aligned}$$

and (4.5) follows from Lemma 4.2.

One of the key steps in the following discussion is to prove

Proposition 4.5 *For $0 < \beta_1 < \frac{1}{2}$, let $\Omega_\varepsilon = \Omega \backslash B_{2\varepsilon^{\beta_1}}(x_\varepsilon)$. Then*

$$|u_\varepsilon(x)| \ge \frac{1}{2} \quad in\ \Omega_\varepsilon \tag{4.6}$$

for $0 < \varepsilon \le \varepsilon_0 = \varepsilon_1 \wedge \frac{1}{2(1+R)} \wedge e^{-1}$, where ε_1 is determined in the following, x_ε is the unique zero of u_ε.

The proof of this proposition is based on the following two lemmas.

Lemma 4.6 *Let $\tilde{u}_\varepsilon$ be a minimizer of the functional*

$$F_\varepsilon(\tilde{u}) = \frac{1}{2}\int_B \frac{1}{x_0+\varepsilon^\beta x_1}\Big[|\nabla\tilde{u}|^2+\frac{1}{2\varepsilon^2}(1-|\tilde{u}|^2)^2\Big], \quad 0<\beta<1$$

with $\tilde{u} = g_\varepsilon$ on ∂B, where $B = B_{\rho_0}(0)$. Suppose

$$\int_{\partial B}\Big[|D_T g_\varepsilon|^2 + \frac{1}{2\varepsilon^2}(|g_\varepsilon|^2-1)^2\Big] \le C_1 \tag{4.7}$$

for some constant C_1, and $0 < \varepsilon < \frac{1}{2(1+R)}$, $\frac{1}{1+R} < x_0 < \frac{1}{1-R}$. Then, for all sufficiently small $\varepsilon > 0$ (depending only on C_1), we have

$$F_\varepsilon(\tilde{u}_\varepsilon) \le C_2 = C_2(C_1, R) \tag{4.8}$$

whenever $\deg(g_\varepsilon, \partial B) = 0$.

Proof From (4.7) it follows that $g_\varepsilon \in C^{1/2}(\partial B)$ and $|g_\varepsilon| \geq 1 - C\varepsilon^{1/4}$ for a constant C depending on C_1. We may assume $g_\varepsilon \to g$ uniformly on ∂B. In particular, deg $(g, \partial B)$ is well defined. Taking a special comparison function $V_\varepsilon = \eta_\varepsilon e^{i\psi_\varepsilon}$ where η_ε and ψ_ε are determined by

$$\begin{cases} -\varepsilon^2 \Delta \eta_\varepsilon + \eta_\varepsilon = 1 & \text{on } B \\ \eta_\varepsilon = |g_\varepsilon| & \text{on } \partial B \end{cases} \qquad \begin{cases} -\Delta \psi_\varepsilon = 0 & \text{on } B \\ \psi_\varepsilon = \varphi_\varepsilon & \text{on } \partial B \end{cases}$$

respectively, in which $\varphi_\varepsilon : \partial B \to R$ is defined by $e^{i\varphi_\varepsilon} = g_\varepsilon / |g_\varepsilon|$, we may choose φ_ε such that $\varphi_\varepsilon \to \varphi_0$ uniformly on ∂B, where $e^{i\varphi_0} = g$ on ∂B. We then deduce

$$\begin{aligned} F_\varepsilon(\tilde{u}_\varepsilon) &\leq \frac{1}{2}\int_B \frac{1}{x_0 + \varepsilon^\beta x_1}|\nabla \psi_\varepsilon|^2 + C\varepsilon \\ &\leq \frac{1}{2}\int_B \frac{1}{x_0}|\nabla \psi_0|^2 + C\varepsilon \\ &\equiv C_2 \end{aligned}$$

Lemma 4.7 *With the same hypothesis as that of Lemma* 4.5 *and* deg $(g_\varepsilon, \partial B) = 0$, *there holds*

$$|\tilde{u}_\varepsilon(x)| \geq \frac{3}{4} \quad in\ B$$

whenever $0 < \varepsilon \leq \varepsilon_1$ *for some* ε_1 *depending only on* R.

Proof If not, we may have a sequence $\varepsilon_n \downarrow 0$, $\frac{1}{1+R} < x_{0n} < \frac{1}{1-R}$, $x_{0n} \to x_0$ $(n \to \infty)$, and a sequence of minimizers $\tilde{u}_{\varepsilon_n} = \tilde{u}_n$ with boundary data g_n satisfying (4.7) and deg $(g_n, \partial B) = 0$. Moreover, $\inf_B |\tilde{u}_n| \leq 3/4$.

Since $|g_n| \to 1$, $\|g_n\|_{C^{1/2}(\partial B)} \leq C$, we see that $|\tilde{u}_n| \geq \frac{4}{5} > \frac{3}{4}$ whenever $1 - |x| \leq C_0\varepsilon_n$, for some C_0. Indeed, the function $\widetilde{V}_n(x) = \tilde{u}_n(\varepsilon_n x)$ satisfies

(a) $|\widetilde{V}_n(x) - \widetilde{V}_n(y)| \leq C|x-y|^{1/2}$, for $|x - y| < 1$, $x, y \in \frac{1}{\varepsilon_n}B$,

(b) $|\nabla \widetilde{V}_n| \leq C/R$ for $R \in (0,1)$ and $|x| \leq \frac{1}{\varepsilon_n} - R$.

Both (a) and (b) follows from the standard elliptic estimates.

Hence, if $|\tilde{u}_n(x)| \leq \frac{3}{4}$, then there is a ball $\{x : |x - x_n| \leq \eta\varepsilon_n\} \subset B$, for some $\eta > 0$ with $|\tilde{u}_n(x)| \leq \frac{4}{5}$ for all $x : |x - x_n| \leq \eta\varepsilon_n$. Therefore

$$\begin{aligned} &\int_B \frac{1}{x_{0n} + x_1\varepsilon_n^\beta} \cdot \frac{1}{\varepsilon_n^2}(1 - |\tilde{u}_n|^2)^2 dx \\ &\geq \frac{R+1}{2}\int_B \frac{1}{\varepsilon_n^2}(1 - |\tilde{u}_n|^2)^2 dx \end{aligned}$$

$$\geq C(\eta, R)$$

$$> 0$$

By Lemma 4.6, $E_{\varepsilon_n}(\tilde{u}_n) \leq E_{\varepsilon_n}(V_{\varepsilon_n}) \leq C_2$. Since $g_n \to g = e^{i\varphi_0}$ weakly in $H^1(\partial B)$, $\int_B \frac{1}{x_{0n} + \varepsilon_n x_1}|\nabla \psi_n|^2 dx$ converges to $\int_B \frac{1}{x_0}|\nabla \psi_0|^2$, where ψ_0 is the harmonic extension of φ_0, thus

$$\overline{\lim}\, E_{\varepsilon_n}(V_{\varepsilon_n}) \leq \frac{1}{2}\int_B \frac{1}{x_0}|\nabla \psi_0|^2 dx \tag{4.9}$$

On the other hand, $\tilde{u}_n \rightharpoonup \tilde{u}$ weakly in $H^1(B)$ with $\tilde{u} = g$ on ∂B and $|u| = 1$ a.e. in B, we have

$$\underline{\lim}\, E_{\varepsilon_n}(\tilde{u}_n) \geq C(\eta, R) + \lim_n \frac{1}{2}\int_B \frac{1}{x_{0n} + \varepsilon_n^\beta x_1}|\nabla \tilde{u}_n|^2$$

$$\geq C(\eta, R) + \frac{1}{2}\int_B \frac{1}{x_0}|\nabla \psi_0|^2$$

therefore, we obtain a contradiction since $C(\eta, R) > 0$.

Remark Both Lemma 4.6 and Lemma 4.7 remain true when we replace B by a bounded Lipschitz domain with Lipschitz constant independent of ε.

Now we prove Proposition 4.5.

For any $x_0 \in \Omega_\varepsilon = \Omega \backslash B_{2\varepsilon^{\beta_1}}(x_\varepsilon)$, consider a functional on $B_{\varepsilon^{\beta_1}}(x_0) \backslash B_{\varepsilon^{2\beta_1}}(x_0) \equiv D$.

It follows from (4.5) that there exists $\lambda_\varepsilon \in [\varepsilon^{2\beta_1}, \varepsilon^{\beta_1}]$ such that

$$\lambda_\varepsilon \int_{\partial B_{\lambda_\varepsilon}(x_0) \cap \Omega_\varepsilon} \left[\frac{1}{2}|\nabla u_\varepsilon|^2 + \frac{1}{4\varepsilon^2}(1 - |u_\varepsilon|^2)^2\right] \leq C(\beta_1)$$

and $\lambda_\varepsilon^{-1}(D \cap \Omega_\varepsilon - x_0)$ is a Lipschitz domain with Lipschitz constant independent of ε. On $\lambda_\varepsilon^{-1}(D \cap \Omega_\varepsilon - x_0) = D_\varepsilon$, function $u_\varepsilon(\lambda_\varepsilon x + x_0)$ minimizes the functional of the form

$$\int_{D_\varepsilon} \frac{1}{x_{01} + \lambda_\varepsilon x_1}\left[|\nabla u|^2 + \frac{1}{2(\varepsilon/\lambda_\varepsilon)^2}(1 - |u|^2)^2\right]$$

with boundary data g_ε on ∂D_ε satisfying

$$\int_{\partial D_\varepsilon} \left[|D_T g_\varepsilon|^2 + \frac{1}{2(\varepsilon/\lambda_\varepsilon)^2}(1 - |g_\varepsilon|^2)^2\right] \leq C(\beta_1) \tag{4.10}$$

Since $|u_\varepsilon| > 0$ on $\Omega_\varepsilon \cap D$, one has $\deg(g_\varepsilon, \partial D_\varepsilon) = 0$. Then Lemma 4.7 leads to $|u_\varepsilon(x)| \geq \frac{1}{2}$ in $D \cap \Omega_\varepsilon$ for $\varepsilon \ll 1$ since $\varepsilon/\lambda_\varepsilon \leq \varepsilon^{1-2\beta_1} \to 0$.

For $0 < \varepsilon < \varepsilon_0$ and minimizers u_ε of E_ε, consider the set $\Sigma_\varepsilon = \left\{x \in \Omega : |u_\varepsilon(x)| \leq \frac{1}{2}\right\}$, then

$$\Sigma_\varepsilon \subset B(x_\varepsilon, \varepsilon^{\beta_1})$$

The same proof in [6, Theorem 2] gives

Lemma 4.8 *There exists a number $J_0 \in \mathbb{N}$ such that for any collection of disjoint balls $B(x_j^\varepsilon, \varepsilon/5)$, $x_j^\varepsilon \in \Omega$, $1 \le j \le J$, with $|u_\varepsilon(x_j^\varepsilon)| < \frac{1}{2}$, there holds $J \le J_0$.*

Now consider the cover $\left\{B\left(x, \frac{\varepsilon}{5}\right)\right\}_{x \in \Sigma_\varepsilon}$ of Σ_ε. By Vitali's covering Lemma, we can find a collection of disjoint balls $B\left(x_j^\varepsilon, \frac{\varepsilon}{5}\right)$, $x_j^\varepsilon \in \Sigma_\varepsilon$, $1 \le j \le J$ such that

$$\Sigma_\varepsilon \subset \bigcup_{j=1}^{J} B(x_j^\varepsilon, \varepsilon)$$

By Lemma 4.8, we have $J \le J_0$ with J_0 independent of ε.

As in [1], we may find $\lambda \ge 1$ such that $\bigcup_{j=1}^{J} B(x_j^\varepsilon, \varepsilon) \subset \bigcup_{j=1}^{J_1} B(x_j^\varepsilon, \lambda\varepsilon)$ with $J_1 \le J$ and $B(x_j^\varepsilon, 2\lambda\varepsilon)$ disjoint where λ is independent of ε.

Lemma 4.9 ([6, Theorem 2]) *There is a constant $C = C(\Omega, g)$ such that*

$$\frac{1}{\varepsilon^2}\int_\Omega (1 - |u_\varepsilon|^2)^2 \le C \tag{4.11}$$

uniformly in $0 < \varepsilon \le \varepsilon_1$, for some $\varepsilon_1 > 0$.

Now, we prove the first claim in Theorem 4.1, i.e., for the unique zero x_ε of u_ε.

$$x_\varepsilon \to a = (1 + R, 0) \quad \text{as } \varepsilon \to 0$$

We argue by contradiction. If the claim fails, then for some $\sigma_0 > 0$, there exists a subsequence $\varepsilon_n \to 0$ such that $x_{\varepsilon_n} \to a_1 \ne a$, $a_1 \in \overline{\Omega}$.

In order to make use of Theorem 4 in [15] and Corollary II.1 in [1], we proceed as follows since a_1 may belong to $\partial\Omega$.

Extend g to $\overline{g}$ defined on $\Omega' = B_{R'}((1,0))$ $(R < R' < 1)$ such that $\overline{g} : \Omega' \backslash \Omega \to S^1$, $\overline{g}\mid_{\partial\Omega} = g$ and $\overline{g}$ satisfies (3.1) and (3.2) as well as $\deg(\overline{g}, \partial\Omega') = 1$. u_ε and $\frac{1}{x_1}$ are also extended such that $u_\varepsilon = \overline{g}$ on $\Omega' \backslash \Omega$.

Hence

$$E_{\varepsilon_n}(u_{\varepsilon_n}, \Omega' \backslash \Omega) \le C$$

with C independent of n.

From the assumption on a_1, we may find $\rho > 0$ small such that for some $\sigma_0 > 0$, $\frac{1}{x_1} \ge \frac{1}{1 + R - 2\sigma_0}$ in $B(a_1, \rho)$. Since $x_{\varepsilon_n} \to a_1$, we have $x_j^{\varepsilon_n} \to a_1$ $(n \to \infty)$. Then $B(x_j^{\varepsilon_n}, \lambda\varepsilon_n) \subset B(a_1, \rho)$, $j = 1, \cdots, J_1$, for n large enough. Applying Theorem 4 in [15] and Corollary II.1 in [1], we have

$$E_\varepsilon(u_\varepsilon, \Omega') \ge E_\varepsilon(u_\varepsilon, B(a_1, \rho))$$

$$\geq \frac{1}{1+R-2\sigma_0}\pi \log \frac{\rho}{\varepsilon_n} - C$$

Hence,

$$E_\varepsilon(u_\varepsilon, \Omega) = E_\varepsilon(u_\varepsilon, \Omega') - E_\varepsilon(u_\varepsilon, \Omega' \backslash \Omega)$$
$$\geq \frac{1}{1+R-2\sigma_0}\pi \log \frac{\rho}{\varepsilon_n} - C$$

Combining this with (4.3) it is led to a contradiction:

$$\sigma_0 |\ln \varepsilon_n| \leq C, \quad \text{independent of } n$$

Now, we prove the convergence in Theorem 4.1.

We should keep in mind that we have found disjoint balls $B(x_j^\varepsilon, \lambda\varepsilon)$, $1 \leq j \leq J_1$, $J_1 \leq J_0$ such that

$$\begin{cases} |u_\varepsilon(x)| \geq \dfrac{1}{2}, \quad \forall x \in \Omega \Big\backslash \bigcup\limits_{j \in J^\varepsilon} B(x_j^\varepsilon, \lambda\varepsilon), & J^\varepsilon = \{1, \cdots, J_1\} \\ \overline{B(x_j^\varepsilon, \lambda\varepsilon)} \cap \overline{B(x_i^\varepsilon, \lambda\varepsilon)} = \emptyset, & \forall i, j = 1, \cdots, J_1, \quad i \neq j \end{cases} \tag{4.12}$$

Define $\omega_j = B(x_i^\varepsilon, \lambda\varepsilon)$, and

$$\Omega_\varepsilon = \Omega \Big\backslash \bigcup_{j \in J^\varepsilon} \omega_j$$
$$\tilde{\Omega}_\varepsilon = \Omega \Big\backslash \bigcup_{j \in K} \omega_j$$

where $K = \{i \in J^\varepsilon : \partial\Omega \cap \omega_i \neq \emptyset\}$, $L = J^\varepsilon \backslash K$.

Note that, if we write locally on Ω_ε, $u_\varepsilon = \rho_\varepsilon e^{i\psi_\varepsilon}$, with $\rho_\varepsilon = |u_\varepsilon|$, then we have

$$\begin{cases} \operatorname{div}\left(\dfrac{1}{x_1}\rho_\varepsilon^2 \nabla\psi_\varepsilon\right) = 0 & \text{in } \Omega_\varepsilon \\ -\nabla \cdot \left(\dfrac{1}{x_1}\nabla\rho_\varepsilon\right) + \dfrac{1}{x_1^2}\rho_{\varepsilon x_1} + \rho_\varepsilon |\nabla\psi_\varepsilon|^2 = \dfrac{1}{x_1}\rho_\varepsilon(1-\rho_\varepsilon^2) & \text{in } \Omega_\varepsilon \end{cases} \tag{4.13}$$

However, we must note that we cannot write (4.13) globally since ρ_ε vanishes at some point in Ω, the corresponding ψ_ε then need not be defined as a single-valued function. To overcome this difficulty, we proceed as follows.

Let Φ_ε be the solution of the linear problem

$$\operatorname{div}\left(\frac{x_1}{\rho_\varepsilon^2}\nabla\Phi_\varepsilon\right) = 0 \quad \text{in } \Omega_\varepsilon \tag{4.14}$$

$$\Phi_\varepsilon = \text{constant} = c_i \quad \text{on } \partial\omega_i, \quad i \in L \tag{4.15}$$

$$\Phi_\varepsilon = 0 \quad \text{on } \partial\tilde{\Omega}_\varepsilon \tag{4.16}$$

$$\int_{\partial\omega} \frac{x_1}{\rho_\varepsilon^2}\frac{\partial\Phi_\varepsilon}{\partial\nu} = 2\pi\delta_i, \quad \delta_i = \deg(u_\varepsilon, \partial\omega), \quad i \in L \tag{4.17}$$

We recall that $\rho_\varepsilon \geq \frac{1}{2}$ in Ω_ε by (4.12), hence (4.14) is elliptic and Φ_ε exists and is unique.

It is obvious that

$$\frac{\partial}{\partial x_1}\Big(\frac{x_1}{\rho_\varepsilon^2}u_\varepsilon \times \Big(\frac{1}{x_1}u_\varepsilon\Big)_{x_2}\Big) - \frac{\partial}{\partial x_2}\Big(\frac{x_1}{\rho_\varepsilon^2}u_\varepsilon \times \Big(\frac{1}{x_1}u_\varepsilon\Big)_{x_1}\Big) = 0 \quad \text{in } \Omega_\varepsilon \tag{4.18}$$

If set

$$D = \Big(\frac{x_1}{\rho_\varepsilon^2}\Big[-u_\varepsilon \times \Big(\frac{1}{x_1}u_\varepsilon\Big)_{x_2} + \Phi_{\varepsilon x_1}\Big], \frac{x_1}{\rho_\varepsilon^2}\Big[u_\varepsilon \times \Big(\frac{1}{x_1}u_\varepsilon\Big)_{x_1} + \Phi_{\varepsilon x_2}\Big]\Big)$$

then, by (4.14) and (4.18)

$$\operatorname{div} D = 0 \quad \text{and} \quad \int_{\partial\omega_i} D \cdot \nu = 0$$

By Lemma I.1 in [1], there is a function H_ε defined in Ω_ε such that

$$D = \Big(-\frac{\partial H_\varepsilon}{\partial x_2}, \frac{\partial H_\varepsilon}{\partial x_1}\Big)$$

that is,

$$\begin{cases} \dfrac{1}{x_1}u_\varepsilon \times u_{\varepsilon x_1} + \Phi_{\varepsilon x_2} = \dfrac{1}{x_1}\rho^2 H_{\varepsilon x_1} \\ \dfrac{1}{x_1}u_\varepsilon \times u_{\varepsilon x_2} - \Phi_{\varepsilon x_1} = \dfrac{1}{x_1}\rho_\varepsilon^2 H_{\varepsilon x_2} \end{cases} \text{in } \Omega_\varepsilon \tag{4.19}$$

We have from the fact $\operatorname{div}\Big(\frac{1}{x_1}\nabla u_\varepsilon\Big) \times u_\varepsilon = 0$ that

$$\operatorname{div}\Big(\frac{1}{x_1}\rho_\varepsilon^2 \nabla H_\varepsilon\Big) = 0 \quad \text{in } \Omega_\varepsilon \tag{4.20}$$

From (4.19) it follows that

$$|u_\varepsilon \times \nabla u_\varepsilon| \leq |\nabla \Phi_\varepsilon| + |\nabla H_\varepsilon| \quad \text{in } \Omega_\varepsilon \tag{4.21}$$

Finally, we claim that

$$|\nabla u_\varepsilon| \leq |\nabla \rho_\varepsilon| + \frac{1}{\rho}|u_\varepsilon \times \nabla u_\varepsilon| \tag{4.22}$$

Indeed, if we locally write $u_\varepsilon = \rho_\varepsilon e^{i\psi}$, we easily see that

$$u_\varepsilon \times \nabla u_\varepsilon = \rho_\varepsilon^2 |\nabla \psi| \tag{4.23}$$

and

$$|\nabla u_\varepsilon| \leq |\nabla \rho_\varepsilon| + \rho_\varepsilon |\nabla \psi|$$

These imply (4.22). Furthermore, from (4.21) and (4.22) we deduce

$$|\nabla u_\varepsilon| \leq 4[|\nabla \Phi_\varepsilon| + |\nabla H_\varepsilon| + |\nabla \rho_\varepsilon|] \quad \text{in } \Omega_\varepsilon \tag{4.24}$$

To get estimates on $|\nabla u_\varepsilon|$, it suffices to estimate $|\nabla\Phi_\varepsilon|, |\nabla H_\varepsilon|$ and $|\nabla\rho_\varepsilon|$ respectively. This is what we are to do in the following.

Lemma 4.10 ([1], Lemma X.7]) *Let $1 < p < 2$. There is a constant $C = C(p, R)$ such that*

$$\left(\int_{\Omega_\varepsilon} |\nabla\Phi_\varepsilon|^p\right)^{1/p} \le C(p, R)|\Omega_\varepsilon|^{\frac{1}{p}-\frac{1}{2}} \tag{4.25}$$

Lemma 4.11 ([1], Lemma X.13]) *For $1 < p < 2$, there are constants α and C independent of ε such that*

$$\int_{\Omega_\varepsilon} |\nabla\rho_\varepsilon|^p \le C\varepsilon^\alpha \tag{4.26}$$

Lemma 4.12 *For any $K \subset\subset \Omega$, there exists a constant C_K independent of ε such that*

$$\int_K |\nabla H_\varepsilon|^2 \le C_K \tag{4.27}$$

Proof Recall that H_ε satisfies

$$\operatorname{div}\left(\frac{1}{x_1}\rho_\varepsilon^2 \nabla H_\varepsilon\right) = 0 \quad \text{in } \Omega_\varepsilon$$

we claim that $\int_{\partial\omega_i} \frac{1}{x_1}\rho^2\frac{\partial H_\varepsilon}{\partial\nu} = 0$, $i \in L$. For simplicity we drop ε.

Recall also that

$$\frac{\partial}{\partial x_1}\left(u \times \frac{1}{x_1}u_{x_1}\right) + \frac{\partial}{\partial x_2}\left(u \times \frac{1}{x_1}u_{x_2}\right) = 0 \quad \text{in } \Omega_\varepsilon$$

Integrate it over ω_i to obtain

$$\int_{\partial\omega_i} u \times \frac{1}{x_1}\frac{\partial u}{\partial\nu} = 0$$

On the other hand, by (4.19) and $\frac{\partial\Phi}{\partial\tau} = 0$ on $\partial\omega_i$ because of (4.15), we obtain

$$u \times \frac{1}{x_1}\frac{\partial u}{\partial\nu} = \frac{1}{x_1}\rho^2\frac{\partial H}{\partial\nu} \quad \text{on } \partial\omega_i, \quad i \in L$$

the claim follows. Invoke Lemma X.4 in [1] to assert that

$$\sup_{\Omega_\varepsilon} H - \inf_{\Omega_\varepsilon} H \le C \quad \text{independent of } \varepsilon$$

Set $H_0 = \inf_{\Omega_\varepsilon} H$, $\varphi \in C_0^\infty(\Omega)$, $0 \le \varphi \le 1$, $\varphi \equiv 1$ in K, $\varphi \equiv 0$ in $\Omega\backslash K'$, where $K \subset\subset K' \subset\subset \Omega$ and $K' \subset\subset \Omega_\varepsilon$ for ε small enough, multiply (4.20) by $(H - H_0)\varphi^2$ and integrate over Ω_ε, we get

$$\int_{\Omega_\varepsilon} \varphi^2\frac{1}{x_1}\rho^2|\nabla H|^2 = -2\int_{\Omega_\varepsilon} \varphi\frac{1}{x_1}\rho^2(H - H_0)\nabla H \cdot \nabla\varphi$$

On the other hand, since $\sup_{\Omega_\varepsilon} H - \inf_{\Omega_\varepsilon} \leq C$, we have

$$\left| \int_{\Omega_\varepsilon} \varphi \frac{1}{x_1} \rho^2 \nabla H \cdot \nabla \varphi \cdot (H - H_0) \right| \leq \frac{1}{2} \int_{\Omega_\varepsilon} \varphi^2 \frac{1}{x_1} \rho^2 |\nabla H|^2 + C \int_{\Omega_\varepsilon} \frac{1}{x_1} |\nabla \varphi|^2$$

Therefore,

$$\frac{1}{2} \int_{\Omega_\varepsilon} \varphi^2 \frac{1}{x_1} \rho^2 |\nabla H|^2 \leq C_K$$

i.e.,

$$\int_K |\nabla H|^2 \leq C_K$$

Hence, we get

$$\int_K |\nabla u_\varepsilon|^p \leq C_K, \quad \forall K \subset\subset \Omega, \quad \forall 1 < p < 2$$

Then, we may extract a further subsequence, still denoted by $\varepsilon_n \to 0$, such that

$$u_{\varepsilon_n} \to u_* \quad \text{weakly in } W^{1,p}_{\text{loc}}$$

From Lemma 4.2 we know

$$\int_\Omega (1 - |u_\varepsilon|^2)^2 \leq C\varepsilon^2 (1 + |\log \varepsilon|) \to 0$$

therefore $|u_{\varepsilon_n}| \to 1$ in L^2 and $|u_*| = 1$ a.e., i.e.,

$$u_* \in W^{1,r}_g(\Omega, S^1) \quad \text{for all } 1 < r < 2$$

Note that Φ_ε and H_ε are only defined on Ω_ε, we extend them in Ω by setting

$$\begin{cases} \Phi_\varepsilon = C_i & \text{in } \omega_i, \quad i \in L \\ \Phi_\varepsilon = 0 & \text{in } \Omega \backslash \widetilde{\Omega}_\varepsilon \end{cases} \tag{4.28}$$

and

$$\begin{cases} \nabla \cdot \left(\dfrac{1}{x_1} \nabla \widetilde{H}_\varepsilon \right) = 0 & \text{in } \omega_i, \\ \widetilde{H}_\varepsilon = H_\varepsilon & \text{on } \partial\omega_i, \end{cases} \quad i \in L \tag{4.29}$$

We still denote them by Φ_ε and H_ε.

It is clear that $\Phi_\varepsilon = 0$ on $\partial\Omega$ and

$$\int_\Omega |\nabla \Phi_\varepsilon|^p \leq C_p, \ \forall 1 < p < 2 \tag{4.30}$$

By the trace theorem together with Lemma 4.12, and definition of H_ε we see (as in Lemma 3 in [15]) that

$$\int_{\omega_i} |\nabla H_\varepsilon|^2 \leq C, \quad i \in L$$

where C depends only on g and Ω. Combining this inequality with Lemma 4.12, we still have

$$\int_K |\nabla H_\varepsilon|^2 \le C, \quad \forall K \subset\subset \Omega \quad \text{and } \varepsilon \text{ small enough} \tag{4.31}$$

In view of (4.30)–(4.31), we may extract a further subsequence $\varepsilon_n \to 0$ such that

$$\Phi_{\varepsilon_n} \rightharpoonup \Phi_* \quad \text{weakly in } W^{1,p}(\Omega), \quad 1 < p < 2$$
$$H_{\varepsilon_n} \rightharpoonup H_* \quad \text{weakly in } H^1_{\text{loc}}(\Omega) \tag{4.32}$$

and

$$\begin{cases} u_* \times u_{*x_1} + x_1 \Phi_{*x_2} = H_{*x_1} \\ u_* \times u_{*x_2} + x_1 \Phi_{*x_1} = H_{*x_2} \end{cases} \tag{4.33}$$

where u_*, Φ_*, H_* are smooth in Ω.

Lemma 4.13 *For any $K \subset\subset \Omega$, we have*

$$u_{\varepsilon_n} \to u_* \quad \textit{strongly in } H^1(K) \tag{4.34}$$
$$-\nabla \cdot \left(\frac{1}{x_1} \nabla u_* \right) = \frac{1}{x_1} u_* |\nabla u_*|^2 \quad \textit{in } \Omega \tag{4.35}$$

Proof We only need to prove,

$$\Phi_{\varepsilon_n} \to \Phi_* \quad \text{strongly in } H^1(K) \tag{4.36}$$
$$H_{\varepsilon_n} \to H_* \quad \text{strongly in } H^1(K) \tag{4.37}$$
$$\rho_{\varepsilon_n} \to 1 \quad \text{strongly in } H^1(K) \tag{4.38}$$

Let $\xi \in C_0^\infty(\Omega)$, $\xi \equiv 1$ in K. For n sufficiently large, the support of ξ is in Ω_{ε_n} and therefore we may multiply (4.14) by $\xi(\Phi_{\varepsilon_n} - \Phi_*)$ and integrate over Ω to obtain

$$\int_\Omega \frac{1}{\rho^2_{\varepsilon_n}} x_1 \xi |\nabla \Phi_{\varepsilon_n}|^2 + \frac{1}{\rho^2_{\varepsilon_n}} x_1 (\Phi_{\varepsilon_n} - \Phi_*) \nabla \Phi_{\varepsilon_n} \cdot \nabla \xi$$
$$= \int_{\Omega'} \frac{x_1}{\rho^2_{\varepsilon_n}} \xi \nabla \Phi_{\varepsilon_n} \cdot \nabla \Phi_* \tag{4.39}$$

However, (4.32) and Sobolev imbedding theorem guarantee

$$\|\Phi_{\varepsilon_n} - \Phi_*\|_{L^q} \to 0, \quad \text{as } n \to \infty, \quad \forall q < +\infty \tag{4.40}$$

hence,

$$\int_\Omega \frac{x_1}{\rho^2_{\varepsilon_n}} (\Phi_{\varepsilon_n} - \Phi_*) \nabla \Phi_{\varepsilon_n} \cdot \nabla \xi \to 0, \quad \text{as } n \to +\infty \tag{4.41}$$

On the other hand, we have

$$\int_\Omega \frac{x_1}{\rho^2_{\varepsilon_n}} \xi \nabla \Phi_{\varepsilon_n} \cdot \nabla \Phi_* \to \int_{\Omega'} x_1 \xi |\nabla \Phi_*|^2, \quad \text{as } n \to +\infty \tag{4.42}$$

Hence, we obtain

$$\int_\Omega \frac{1}{\rho_{\varepsilon_n}^2} x_1\xi|\nabla\Phi_{\varepsilon_n}|^2 \to \int_\Omega x_1\xi|\nabla\Phi_*|^2 \tag{4.43}$$

Since $\rho_{\varepsilon_n} \le 1$, it follows that

$$\int_\Omega x_1\xi|\nabla\Phi_{\varepsilon_n}|^2 \le \int_\Omega x_1\xi|\nabla\Phi_*|^2 + o(1)$$

And therefore, by lower semi-continuity and $x_1 \ge a_0 > 0$, we deduce that

$$\nabla\Phi_{\varepsilon_n} \to \nabla\Phi_* \quad \text{strongly in } L^2(K)$$

Similarly, using the equation (4.20), we have

$$\int_\Omega \frac{1}{x_1}\rho_{\varepsilon_n}^2 \cdot \xi|\nabla H_{\varepsilon_n}|^2 \to \int_\Omega \frac{1}{x_1}\xi|\nabla H_*|^2 \quad \text{as } n \to +\infty \tag{4.44}$$

$$\begin{aligned}\int_\Omega \frac{1}{x_1}\rho_{\varepsilon_n}^2\xi|\nabla(H_{\varepsilon_n} - H_*)|^2 &= \int_\Omega \frac{1}{x_1}\rho_{\varepsilon_n}^2\xi|\nabla H_{\varepsilon_n}|^2 \\ &\quad - 2\int_\Omega \frac{1}{x_1}\rho_{\varepsilon_n}^2\xi\nabla H_{\varepsilon_n}\cdot\nabla H_* + \int_\Omega \frac{1}{x_1}\rho_{\varepsilon_n}^2\xi|\nabla H_*|^2\end{aligned} \tag{4.45}$$

Note that

$$\int_\Omega \frac{1}{x_1}\rho_{\varepsilon_n}^2\xi\nabla H_{\varepsilon_n}\cdot\nabla H_* \to \int_\Omega \frac{1}{x_1}\xi|\nabla H_*|^2 \tag{4.46}$$

Combining (4.44)–(4.46), we obtain (4.37).

Finally, testing $(4.13)_2$ by $\xi(1-\rho_{\varepsilon_n})$ and using (4.23), we obtain

$$\begin{aligned}&\int_\Omega \frac{1}{x_1}\xi|\nabla\rho_{\varepsilon_n}|^2 - \int_\Omega \frac{1}{x_1}(1-\rho_{\varepsilon_n})\nabla\rho_{\varepsilon_n}\cdot\nabla\xi \\ &= \int_\Omega \xi\frac{(1-\rho_{\varepsilon_n})}{\rho_{\varepsilon_n}^3}|u_{\varepsilon_n}\times\nabla u_{\varepsilon_n}|^2 - \frac{1}{\varepsilon_n^2}\int_\Omega \frac{\xi\rho_{\varepsilon_n}}{x_1}(1-\rho_{\varepsilon_n}^2)(1+\rho_{\varepsilon_n}) \\ &\quad - \int_\Omega \frac{\xi}{x_1^2}(\rho_{\varepsilon_n})_{x_1}(1-\rho_{\varepsilon_n})\end{aligned} \tag{4.47}$$

Since $\rho_{\varepsilon_n} \to 1$ in $W^{1,p}$, we are led to (apply (4.21))

$$\int_\Omega \frac{\xi}{x_1}|\nabla\rho_{\varepsilon_n}|^2 \le C\int_\Omega \xi(1-\rho_{\varepsilon_n})(|\nabla H_{\varepsilon_n}|^2 + |\nabla\Phi_{\varepsilon_n}|^2) + o(1) \tag{4.48}$$

Using (4.36), (4.37), the fact $\rho_{\varepsilon_n} \to 1$, a.e. and Lebesgue's dominated convergence theorem, we see that the right-hand side of (4.48) tends to zero as $n \to +\infty$. This proves $\int_\Omega \xi|\nabla\rho_{\varepsilon_n}|^2 \to 0$ and hence (4.38).

Now, we prove (4.1) and (4.2).

Step 1 For any $K \subset\subset \Omega$, we have

$$u_{\varepsilon_n} \to u_* \quad \text{in } H^1(K) \tag{4.49}$$

Proof By (4.36)–(4.38), (4.19) and (4.33), we know

$$u_{\varepsilon_n} \times \nabla u_{\varepsilon_n} \to u_* \times \nabla u_* \quad \text{in } L^2(K) \tag{4.50}$$

On K we may write locally

$$u_{\varepsilon_n} = \rho_{\varepsilon_n} e^{i\psi_{\varepsilon_n}} \text{ and } u_* = e^{i\psi_*} \tag{4.51}$$

so that

$$u_{\varepsilon_n} \times \nabla u_{\varepsilon_n} = \rho^2_{\varepsilon_n} \nabla \psi_{\varepsilon_n}, \quad u_* \times \nabla u_* = \nabla \psi_* \tag{4.52}$$

Hence, by (4.50) and (4.38) we have

$$\nabla \psi_{\varepsilon_n} \to \nabla \psi_* \quad \text{in } L^2(K) \tag{4.53}$$

and (4.49) follows from (4.51), (4.53) and (4.38).

Step 2 Finally, (4.1) follows from Step 1, Fubini's Theorem and theorem 2.7 by the method in [1].

Step 3 (4.2) follows from above estimates and convergence as well as the fact $-\nabla \cdot \left(\frac{1}{x_1} \nabla u_\varepsilon \times u_\varepsilon\right) = 0$.

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