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$\mathcal{L}^{2,\mu}(Q)$-ESTIMATES FOR PARABOLIC EQUATIONS AND APPLICATIONS[1]

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Abstract In this paper we derive *a priori* estimates in the Campanato space $\mathcal{L}^{2,\mu}(Q_T)$ for solutions of the following parabolic equation

$$u_t - \frac{\partial}{\partial x_i}(a_{ij}(x,t)u_{x_j} + a_i u) + b_i u_{x_i} + cu = \frac{\partial}{\partial x_i} f_i + f_0$$

where $\{a_{ij}(x,t)\}$ are assumed to be measurable and satisfy the ellipticity condition. The proof is based on accurate DeGiorgi-Nash-Moser's estimate and a modified Poincare's inequality. These estimates are very useful in the study of the regularity of solutions for some nonlinear problems. As a concrete example, we obtain the classical solvability for a strongly coupled parabolic system arising from the thermistor problem.

Key Words Parabolic equation; *a priori* estimates in Campanato space; DeGiorgi-Nash-Moser's estimate; a modified Poincare's inequality.

Classification 35K20, 35K55.

1. Introduction

Let Ω be a bounded domain in R^n with boundary $S = \partial\Omega$ in C^1 and $Q_T = \Omega \times (0, T]$ with $T > 0$. Consider the following parabolic equation:

$$u_t - \frac{\partial}{\partial x_i}(a_{ij}(x,t)u_{x_j} + a_i u) + b_i u_{x_i} + cu = \frac{\partial}{\partial x_i} f_i + f_0 \qquad (1.1)$$

where a_{ij} satisfies the ellipticity condition:

$$a_0|\xi|^2 \le a_{ij}\xi_i\xi_j \le A_0|\xi|^2 \quad \text{for } \xi \in R^n, \quad 0 < a_0 \le A_0$$

It is well known that the DeGiorgi-Nash-Moser estimate plays an essential role in the study of solvability for nonlinear parabolic equations. However, this estimate is often not enough in dealing with regularity of solutions. On the other hand, the theory of the

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Campanato space $\mathcal{L}^{2,\mu}$ is powerful for investigating regularity of solutions for elliptic equations and systems (cf. [1], [2], etc.). In the present work we would like to derive the $\mathcal{L}^{2,\mu}(Q)$-estimates for weak solutions of the equation (1.1). It will be seen the results are also very useful in applications. The core of the proof is based on accurate DeGiorgi-Nash-Moser's estimates. For elliptic equations, the theory can be found in [2]. The fundamental difference from the elliptic theory is that Poincare's inequality

$$\int_{Q_r}(u-u_{z_0,r})^2dxdt \le Cr^2\int_{Q_{2r}}|\nabla u|^2dxdt$$

does not hold for a general function $u(x,t)\in L^2(0,T;H^1(\Omega))$ (see the notation below). However, by using the equation and combining (elliptic version) Poincare's inequality, we are able to resolve the difficulty. The proof is based on various modifications of elliptic situation.

For convenience we introduce some standard notations: a point (x,t) in Q_T will be denoted by z. The distance between two points $z_1=(x_1,t_1)$ and $z_2=(x_2,t_2)$ is equal to

$$\max\{|x_1-x_2|,|t_1-t_2|^{\frac{1}{2}}\}$$

For $r>0$,

$$B_r(x_0)=\{x\in R^n:|x-x_0|<r\} \text{ and } Q_r(z_0)=B_r(x_0)\times(t_0-r^2,t_0]$$

For a measurable set $A\subset R^n\times[0,T]$ with a finite measure $|A|<\infty$,

$$⨍_A udz=\frac{1}{|A|}\int_A udz$$

In particular, when $A=Q_r(z_0)$,

$$u_{z_0,r}=⨍_{Q_r(z_0)}udz$$

For $\mu>0$, let

$$[u]_{2,\mu,Q_r}=\left(\sup_{z_0\in Q,0<\rho<r}\rho^{-\mu}\int_{Q_\rho(z_0)}|u-u_{z_0,\rho}|^2dz\right)^{\frac{1}{2}}$$

The space $\mathcal{L}^{2,\mu}(Q)$ consists of all functions in $L^2(Q)$ such that

$$[u]_{2,\mu,Q_r}<\infty$$

We understand that $Q\cap Q_r$ should be used in the integration whenever Q_r is not a subset of Q. $\mathcal{L}^{2,\mu}(Q)$ is a Banach space with the norm

$$\|u\|_{2,\mu,Q_r}=\{\|u\|^2_{L^2(Q_r)}+[u]^2_{2,\mu,Q_r}\}^{\frac{1}{2}}$$

We need the following proposition:

Proposition A *The space* $\mathcal{L}^{2,n+2+2\mu}(Q_T)$ *and* $C^{\mu,\frac{\mu}{2}}(\overline{Q_T})$, *where* $\mu \in (0,1)$, *are isomorphic topologically and algebraically.*

If one replaces Q_T by a subset $Q_r(z_0)$, then $u(z)$ is Hölder continuous in a neighborhood of z_0. It is often useful to note the following proposition.

Proposition B *For* $0 < \mu < n+2$, *the norm of* $\mathcal{L}^{2,\mu}(Q_T)$ *is equivalent to the following*

$$\left(\sup_{z_0\in Q, r>0} r^{-\mu}\int_{Q_r(z_0)} u^2 dz\right)^{\frac{1}{2}}$$

The proof of the above propositions can be found in [3], Theorem 2.1 (also see [4]).

2. $\mathcal{L}^{2,\mu}(Q_T)$-theory for Parabolic Equations

We begin with the interior estimates. In this case we always assume that

$$d = \min\{\text{dist}\,(Q_{2r}(z_0), S_T), \text{dist}\,(Q_{2r}, \Omega\times\{t=0\})\} > 0$$

Moreover, in addition to the ellipticity condition the following condition is assumed throughout the paper:

H(A) The coefficients a_i, b_i, $i = 1, 2, \cdots, n$ and c are in $L^\infty(Q_T)$.

For the equation (1.1) along with homogeneous boundary condition and the initial condition $u(x,0) = u_0(x)$, we define a weak solution $u(x,t)$ as in [5]: $u(x,t) \in L^2(0,T;H^1(\Omega))$ is said to be a weak solution if it satisfies

$$\int_{Q_T}[-u\phi_t + (a_{ij}u_{x_j} + a_i u)\phi_{x_i}]dxdt$$
$$= \int_{Q_T}[(b_i u_{x_i} + cu + f_0)\phi - f_i\phi_{x_i}]dxdt + \int_\Omega \phi(x,0)u_0(x)dx \qquad (2.1)$$

for any test function $\phi(x,t) \in H^1(0,T;H^1(\Omega))$ with $\phi(x,t) = 0$ on S_T and $t = T$.

Let $w(x,t)$ be a weak solution of the following parabolic equation:

$$u_t - \frac{\partial}{\partial x_i}(a_{ij}(x,t)u_{x_j}) = 0 \qquad (2.2)$$

that is

$$\iint_{Q_T}[-wv_t + a_{ij}w_{x_j}v_{x_i}]dxdt = 0$$

for any $v(x,t) \in H^1(0,T;H_0^1(\Omega))$ with $v(x,0) = v(x,T) = 0$.

In what follows, a constant which depends only on $\|a_{ij}\|_{L^\infty}, \|a_i\|_{L^\infty} + \|b_i\|_{L^\infty} + \|c\|_{L^\infty}$, a_0, d and the domain Q_T will be denoted by C, it may be different from one line to the next. The following two lemmas are fundamental in order to prove Hölder continuity. Their proofs can be found in [6] (Lemma 1 and Lemma 3).

Lemma 2.1 *Let $w(x,t)$ be a non-negative subsolution of (2.2) and $z_0 \in Q_T$, $Q_{2r}(z_0) \subset Q_T$ $(r > 0)$, then*

$$\operatorname{ess}\sup_{Q_r(z_0)} w(x,t) \le C\rho^{-(\frac{n}{2}+1)}\|w\|_{2,Q_{r+\rho}(z_0)}$$

where $0 < \rho \le r$ and $\|\cdot\|_{2,A}$ represents the norm of $L^2(A)$.

Lemma 2.2 *Let $w(x,t)$ be a non-negative weak solution of (2.2) and $Q_r(z_0) \subset Q_T$. If $E = \{z \in Q_r(z_0) : w(x,t) \ge 1\}$ has measure $\ge K|Q_r(z_0)|$, where $K \in (0,1)$ is a constant, then*

$$\operatorname{ess}\inf_{Q_{r/2}(z_0)} w(x,t) \ge C(K) > 0$$

where the constant $C(K)$ depends on K and $\|a_{ij}\|_{L^\infty(Q_T)}$ and a_0, but not on r.

With the above results on hand, one can derive

Lemma 2.3 *If $w(x,t)$ is a weak solution, then there exist constants $\delta_0 \in (0,1)$ and C such that*

$$\max_{Q_\rho(z_0)} w - \min_{Q_\rho(z_0)} w \le Cr^{-(\frac{n}{2}+1)}\left(\frac{\rho}{r}\right)^{\delta_0}\|w\|_{2,Q_{2r}(z_0)}$$

where $0 < \rho \le r$ and $Q_{2r}(z_0) \subset Q_T$ while δ_0 and C have the same dependency as C above.

The proof is almost identical to the case for elliptic equations, see [2] Theorem 2.14 on page 115 (also [7], Theorem 4 with $k = 0$).

Lemma 2.4 *Let $w(x,t)$ be a weak solution of (2.2). Then for any $\rho \in [0,r]$,*

$$\|\nabla w\|^2_{2,Q_\rho(z_0)} \le C\left(\frac{\rho}{r}\right)^{\mu_0}\|\nabla w\|^2_{2,Q_{2r}(z_0)}$$

where $\mu_0 = n + 2\delta_0$.

Proof To prove the lemma, we need the following fundamental estimate: for a weak solution $w(x,t)$ of (2.2),

$$\iint_{Q_r(z_0)} (w - w_{x_0,r})^2 dz \le Cr^2 \iint_{Q_{2r}(z_0)} |\nabla w|^2 dz \tag{2.3}$$

where the constant C is independent of r and w.

The inequality (2.3) is well known as Poincare's inequality when w is only a function of x (cf. [1]). In the present situation, one can use the equation to control the $L^2(Q_r)$-norm of w_t (See the proof of Lemma 2.6 for this special case, of course the proof of Lemma 2.6 is independent of Lemma 2.4). From the inequality (2.3), we only need to estimate the gradient of u in order to obtain the estimate in the Campanato space $\mathcal{L}^{2,\mu}(Q_T)$.

As $w(x,t) - w_{z_0,r}$ is a weak solution of (2.2), without loss of generality we may assume that $w_{z_0,r} = 0$. We may also assume that z_0 is the origin and use Q_r instead of $Q_r(0)$. By Lemma 2.3, for $z = (x,t) \in Q_{\frac{r}{2}}(0)$,

$$|w(z) - w(0)|^2 \le Cr^{-n-2-2\delta_0}|z|^{2\delta_0}\|w\|^2_{2,Q_r} \le Cr^{-n-2\delta_0}|z|^{2\delta_0}\|\nabla w\|^2_{2,Q_{2r}} \tag{2.4}$$

Let $0<\rho<\frac{r}{4}$. Introduce a cutoff function $g(x,t)$ as follows:

$$g(x,t)\in C^{1,1}(\overline{Q}_r) \text{ and satisfies :}$$
$$0\le g(x,t)\le 1,\quad \operatorname{supp} g\subset Q_{2\rho};\quad g(x,t)=1\quad \text{on } \overline{Q}_\rho$$

Moreover,

$$|\nabla g|\le \frac{2}{\rho},\quad |g_t|\le \frac{4}{\rho^2}$$

Let $v(x,t)=g^2(x,t)[w(z)-w(0)]$. We can use $v(x,t)$ as a test function in the integral identity for $w(x,t)$, although w_t is not necessary in $L^2(Q_r)$. Indeed, otherwise, we can always use the Steklov averaging

$$v_h(x,t)=\frac{1}{h}\int_t^{t+h} v(x,\tau)d\tau$$

to approximate $v(x,t)$ and then take the limit. Now

$$\begin{aligned}\iint_{Q_r} wv_t dxdt &= \iint_{Q_r}[w(z)-w(0)]v_t dxdt+\iint_{Q_r} w(0)v_t dxdt\\ &=\iint_{Q_r}[w(z)-w(0)]^2 gg_t dxdt+\frac{1}{2}\int_{B_r} g^2[w(z)-w(0)]^2dx\\ &\quad+\int_{B_r} w(0)g^2[w(z)-w(0)]dx\\ &\equiv I_1+I_2+I_3\end{aligned}$$

By (2.4) and the construction of g, one has

$$|I_1|\le Cr^{-n-2\delta_0}\rho^{2\delta_0}\rho^{-2}\|\nabla w\|^2_{L^2(Q_{2r})}|Q_{2\rho}|\le C\Big(\frac{\rho}{r}\Big)^{\mu_0}\|\nabla w\|^2_{L^2(Q_{2r})}$$

where $\mu_0=n+2\delta_0$.

Similarly,

$$|I_2|\le C\Big(\frac{\rho}{r}\Big)^{\mu_0}\|\nabla w\|^2_{L^2(Q_{2r})}$$

To estimate I_3, we use Lemma 2.1 and (2.4) to obtain

$$|I_3|\le \|w(0)\|_{L^\infty(B_\rho(0))}\cdot\|w-w(0)\|_{L^\infty(B_{2\rho}(0))}\cdot C\rho^n\le C\Big(\frac{\rho}{r}\Big)^{\mu_0}\|\nabla w\|^2_{L^2(Q_{2r})}$$

It follows that

$$\Big|\iint_{Q_r} wv_t dxdt\Big|\le C\Big(\frac{\rho}{r}\Big)^{\mu_0}\|\nabla w\|^2_{L^2(Q_{2r})}\tag{2.5}$$

On the other hand,

$$\begin{aligned}&\iint_{Q_r} a_{ij}w_{x_i}v_{x_j}dxdt\\ &=\iint_{Q_r} a_{ij}w_{x_i}\{g^2 w_{x_j}+2gg_{x_j}[w-w(0)]\}dxdt\end{aligned}$$

$$\geq \frac{a_0}{2}\iint_{Q_r} g^2|\nabla w|^2dxdt - C\max_{Q_{2\rho}}|w-w(0)|^2\iint_{Q_r}|\nabla g|^2dxdt$$

Combining the above inequality with (2.4)–(2.5), we have

$$\iint_{Q_\rho}|\nabla w|^2dxdt \leq C\Big(\frac{\rho}{r}\Big)^{\mu_0}\|\nabla w\|^2_{L^2(Q_{2r})}$$

When $\frac{r}{4} < \rho < 2r$, clearly we always have

$$\iint_{Q_\rho}|\nabla w|^2dxdt \leq 4^{\mu_0}\Big(\frac{\rho}{r}\Big)^{\mu_0}\|\nabla w\|^2_{L^2(Q_{2r})}$$

This completes the proof.

Lemma 2.5 *Let $\Omega = B_r(0)$ and the functions $f_0(x,t)\in L^2(Q_T)$ and $f_i(x,t)\in L^{2,\mu}(Q_T)$ $(i=1,2,\cdots,n)$. Let $u(x,t)$ be a weak solution of the following equation:*

$$u_t - \frac{\partial}{\partial x_i}(a_{ij}(x,t)u_{x_j}) = \frac{\partial}{\partial x_i}f_i + f_0$$

then

$$\|\nabla u\|^2_{2,Q_\rho} \leq C\bigg[\Big(\frac{\rho}{r}\Big)^{\mu_0}\|\nabla u\|^2_{2,Q_{2r}} + r^2\|f_0\|^2_{2,Q_r} + \sum_{i=1}^n\|f_i\|^2_{2,Q_r}\bigg]$$

where $\mu_0 = n + 2\delta_0$.

Proof Let $h(x,t)\in L^2(0,T;H^1_0(B_r(0))$ solve the equation

$$h_t - (a_{ij}h_{x_i})_{x_j} = (f_i)_{x_i} + f_0$$

in the weak sense with $h(x,0)=0$. Then it is easy to see that the following energy inequality holds:

$$\int_{B_r}h^2dx + \iint_{Q_r}h^2_xdxdt \leq \|f_0\|_{L^2(Q_r)}\|h\|_{L^2(Q_r)} + \sum_{i=1}^n\|f_i\|_{L^2(Q_r)}\|h_{x_i}\|_{L^2(Q_r)}$$

By ε-Cauchy's inequality and Poincare's inequality (note that $h(x,t)$ vanishes on the boundary of $B_r(0)\times(0,r^2)$), we obtain

$$\int_{B_r}h^2dx + \iint_{Q_r}h^2_xdxdt \leq C\bigg[r^2\|f_0\|^2_{L^2(Q_r)} + \sum_{i=1}^n\|f_i\|^2_{L^2(Q_r)}\bigg]$$

Now let $w(x,t) = u(x,t) - h(x,t)$ on Q_r. Then $w(x,t)$ satisfies

$$\iint_{Q_r}[-wv_t + a_{ij}w_{x_i}v_{x_j}]dxdt = 0$$

for any $v(x,t)\in H^1(0,T;H^1_0(\Omega))$ with $v(x,T)=v(x,0)=0$. Lemma 2.4 implies

$$\|\nabla w\|^2_{L^2(Q_\rho)} \leq C\Big(\frac{\rho}{r}\Big)^{\mu_0}\|\nabla w\|^2_{L^2(Q_{2r})}$$

Hence,

$$\begin{aligned}|\nabla u\|^2_{L^2(Q_\rho)} &\le 2[\|\nabla w\|^2_{L^2(Q_\rho)} + \|\nabla h\|^2_{L^2(Q_\rho)}]\\ &\le C\Big(\frac{\rho}{r}\Big)^{\mu_0}\|\nabla w\|^2_{L^2(Q_{2r})} + 2\|\nabla h\|^2_{L^2(Q_\rho)}\\ &\le C\Big(\frac{\rho}{r}\Big)^{\mu_0}\|\nabla w\|^2_{L^2(Q_{2r})} + C\left[r^2\|f_0\|^2_{L^2(Q_r)} + \sum_{i=1}^n \|f_i\|^2_{L^2(Q_r)}\right]\end{aligned}$$

Next lemma is essential in order to apply the embedding theorem later.

Lemma 2.6 *Under the assumption $H(A)$,*

$$\begin{aligned}\int_{Q_r}(u-u_{0,r})^2dz \le C_1r^2\int_{Q_{2r}}|\nabla u|^2dz + C_2r^2\int_{Q_{2r}}\left[u^2+\sum_{i=1}^n f_i^2\right]dz\\ + C_3r^4\int_{Q_{2r}}[u^2+f_0^2]dz \qquad (2.6)\end{aligned}$$

where the constants C_1, C_2 and C_3 depend only on a_0, A_0, the L^∞-norm of a_i, b_i, $i = 1, 2, \cdots, n$, and c.

Proof the argument is based on Lemmas 3 and 4 in [8]. We choose a cutoff function $\sigma(x)$ as follows: $\sigma(x) = 1$, if $|x| \le r$, $\sigma(x)$ is zero, if $|x| \ge 2r$. Moreover, $0 \le \sigma(x) \le 1$ is smooth and $|\nabla\sigma(x)| \le \frac{2}{r}$.

For any $s \le t \in Q_r$, we denote by $\chi_{[s,t]}$ the characteristic function on $[s, t]$. For each $t \in (0, T]$, we define

$$u_r^\sigma = \frac{\int_{Q_{2r}} u\sigma dz}{\int_{Q_{2r}}\sigma(x)dx}, \quad u_{r,t}^\sigma = \frac{\int_{Q_{2r}(t)} u\sigma dx}{\int_{Q_{2r}(t)}\sigma(x)dx}$$

where $Q_r(t) = \{(x,t) : |x| \le r\}$.

We use

$$\phi(x,t) = \sigma(x)\chi_{[s,t]}(u_{r,t}^\sigma - u_{r,s}^\sigma)$$

as a test function (otherwise, we use the Steklov averaging to approximate ϕ and then take the limit), we have

$$\int_{Q_T}[-u\phi_t + (a_{ij}u_{x_j} + a_iu)\phi_{x_i}]dxdt = \int_{Q_T}[(b_iu_{x_i} + cu + f_0)\phi - f_i\phi_{x_i}]dxdt$$

By the definition of $u_{r,t}^\sigma$ and the choice of $\phi(x,t)$, we have

$$\begin{aligned}\int_{Q_T} u\phi_t dx &= \int_s^t\int_\Omega u_t\sigma(u_{r,t}^\sigma - u_{r,s}^\sigma)dz = \left[\int_{Q_r(t)} u\sigma dx - \int_{Q_r(s)} u\sigma dx\right](u_{r,t}^\sigma - u_{r,s}^\sigma)\\ &\ge c_0r^n(u_{r,t}^\sigma - u_{r,s}^\sigma)^2\end{aligned}$$

where c_0 is a positive constant depending only on Ω.

Using Cauchy-Schwarz's inequality and the assumption, we have

$$\begin{aligned}
&\int_s^t \int_{B_r} a_{ij} u_{x_j} \sigma_{x_i} (u_{r,t}^\sigma - u_{r,s}^\sigma) dz \\
&\le \varepsilon (u_{r,t}^\sigma - u_{r,s}^\sigma)^2 \int_s^t \int_{B_r} |\nabla \sigma|^2 dz + C(\varepsilon) \int_s^t \int_{B_{2r}} |\nabla u|^2 dz \\
&\le C\varepsilon r^n (u_{r,t}^\sigma - u_{r,s}^\sigma)^2 + C(\varepsilon) \int_s^t \int_{B_{2r}} \nabla u^2 dz
\end{aligned}$$

where at the final step the facts $|t-s| \le r^2$ and $|\nabla \sigma| \le \frac{2}{r}$ are used.

Similarly, we see

$$\int_{Q_T} f_i \phi_{x_i} dz \le \varepsilon r^n (u_{r,t}^\sigma - u_{r,s}^\sigma)^2 + C(\varepsilon) \int_s^t \int_{B_{2r}} f_i^2 dz$$

$$\int_{Q_T} a_i u \phi_{x_i} dz \le \varepsilon r^n (u_{r,t}^\sigma - u_{r,s}^\sigma)^2 + C(\varepsilon) \int_s^t \int_{B_{2r}} u^2 dz$$

We again use the fact $|t-s| \le r^2$ to have by Cauchy-Schwarz's inequality that

$$\int_{Q_T} b_i u_{x_i} \phi dz \le \varepsilon r^n (u_{r,t}^\sigma - u_{r,s}^\sigma)^2 + C(\varepsilon) r^2 \int_s^t \int_{B_{2r}} \nabla u^2 dz$$

$$\int_{Q_T} c u \phi dz \le \varepsilon r^n (u_{r,t}^\sigma - u_{r,s}^\sigma)^2 + C(\varepsilon) r^2 \int_s^t \int_{B_{2r}} u^2 dz$$

$$\int_{Q_T} f_0 \phi dz \le \varepsilon r^n (u_{r,t}^\sigma - u_{r,s}^\sigma)^2 + C(\varepsilon) r^2 \int_s^t \int_{B_{2r}} f_0^2 dz$$

We sum up the above estimates and take ε to be small enough to conclude

$$\begin{aligned}
(u_{r,t}^\sigma - u_{r,s}^\sigma)^2 \le Cr^{-n} \int_s^t \int_{B_{2r}} \nabla u^2 dz + Cr^{-n} \int_s^t \int_{B_{2r}} \left[u^2 + \sum_{i=1}^n f_i^2 \right] dz \\
+ Cr^{-n+2} \int_s^t \int_{B_{2r}} (u^2 + f_0^2) dz \qquad (2.7)
\end{aligned}$$

It is clear that $F(h) = \int_{Q_r} |u-h|^2 dz$ takes the minimum at $h = u_{0,r}$. It follows that

$$\begin{aligned}
\int_{Q_r} |u - u_{0,r}|^2 dz &\le \int_{Q_r} |u - u_r^\sigma|^2 dz \\
&\le 2 \int_{Q_r} |u - u_{r,t}^\sigma|^2 dz + 2 \int_{Q_r} |u_{r,t}^\sigma - u_r^\sigma|^2 dz \\
&\le Cr^2 \int_{Q_{2r}} |\nabla u|^2 dz + 2 \int_{Q_r} |u_{r,t}^\sigma - u_r^\sigma|^2 dz \qquad (2.8)
\end{aligned}$$

where at the final step we have used (elliptic version) Poincare's inequality. To estimate the final term in (2.8), we observe from the definition that

$$u_r^\sigma = \frac{1}{|Q_r|}\int_{Q_r} u_{r,s}^\sigma dz$$

Using the estimate (2.7) and the fact $|Q_r| = Cr^{n+2}$, we obtain

$$\int_{Q_r} |u_{r,t}^\sigma - u_r^\sigma|^2 dz$$

$$\le Cr^2\int_s^t\int_{B_{2r}} \nabla u^2 dz + Cr^2\int_s^t\int_{B_{2r}}\left[u^2 + \sum_{i=1}^n f_i^2\right]dz + Cr^4\int_s^t\int_{B_{2r}}(u^2+f_0^2)dz \tag{2.8a}$$

This completes the proof of Lemma 2.6.

Now we can show

Theorem 1 *Let the assumption H(A) hold and let $u(x,t)$ be a weak solution of Eq.(1.1), then for any $0<\mu<\mu_0 = n+2\delta_0$,*

$$\|\nabla u\|_{2,\mu,Q_T}^2 \le C\left[\|f_0\|_{2,(\mu-2)^+,Q_T}^2 + \sum_{i=1}^n \|f_i\|_{2,\mu,Q_T}^2 + \|u\|_{L^2(0,T;H^1(\Omega))}^2\right]$$

in particular, if $\mu > n$, then

$$u(x,t)\in C^{\alpha,\frac{\alpha}{2}}(Q_T), \text{ where } \alpha = \frac{\mu-n}{2}$$

where C depends only on $a_0, A_0, \|b_i\|_{L^\infty(Q_T)}, \|a_i\|_{L^\infty(Q_T)}, \|c\|_{L^\infty(Q_T)}$ and Ω.

Proof We write the integral equality (2.1) into the following form:

$$\iint_{Q_r}[-uv_t + a_{ij}u_{x_i}u_{x_j}]dxdt = \iint_{Q_r}[f_0^* v - f_i^* v_{x_i}]dxdt$$

for any $v(x,t)\in H^1(0,T;H_0^1(B_r(z_0)))$ with $v(x,T)=v(x,0)=0$, where

$$f_0^* = f_0 - b_i u_{x_i} - cu, \quad f_i^* = f_i - a_i u$$

Using Lemma 2.5, we have

$$\|\nabla u\|_{L^2(Q_\rho)}^2 \le C\left[\left(\frac{\rho}{r}\right)^{\mu_0}\|\nabla u\|_{2,Q_{2r}}^2 + r^2\|f_0^*\|_{2,Q_r}^2 + \sum_{i=1}^n \|f_i^*\|_{2,Q_r}^2\right] \tag{2.9}$$

It is clear $\mathcal{L}^{2,0}(Q_T) = L^2(Q_T)$. By the definition of $\mathcal{L}^{2,\mu}(Q_T)$ and Proposition B we see that

$$r^2\|f_0^*\|_{2,r}^2 \le C[r^\mu\|f_0\|_{2,(\mu-2)^+,Q_r}^2 + r^{\mu+2}\|u\|_{2,\mu,Q_r}^2 + r^2\|\nabla u\|_{2,Q_r}^2]$$

Similarly,

$$\sum_{i=1}^{n}\|f_i^*\|_{2,Q_r}^2 \le Cr^\mu\left[\sum_{i=0}^{n}\|f_i\|_{2,\mu,Q_r}^2+\|u\|_{2,\mu,Q_r}^2\right]$$

Assume that there exists a number μ such that

$$\|u\|_{2,Q_r}^2 \le C^* S_\mu \tag{2.10}$$

where

$$S_\mu=\|f_0\|_{2,(\mu-2)^+,Q_T}^2+\sum_{i=1}^{n}\|f_i\|_{2,\mu,Q_T}^2+\|u\|_{L^2(0,T;H^1(\Omega))}^2$$

and C^* is a constant depending only on the known data. The existence of such a μ is obvious (see Lemma 2.6) since

$$L^2(0,T;H^1(\Omega))\hookrightarrow \mathcal{L}^{2,2}(Q_T)$$

Hence we have

$$\|\nabla u\|_{2,Q_\rho}^2 \le C\Big[\Big(\Big(\frac{\rho}{r}\Big)^{\mu_0}+r^2\Big)\|\nabla u\|_{2,Q_{2r}}^2+r^\mu S_\mu\Big]$$

By applying the same iteration technique (see Lemma 1.18 in [2]) as the inequality (2.44) of [2], we have the estimate

$$\|\nabla u\|_{2,\mu,Q_\rho}^2 \le CS_\mu$$

where $2\le\mu<\mu_0$.

Lemma 2.6 yields $u\in\mathcal{L}^{2,\mu+2}(Q_\rho)$ and hence the estimate (2.10) holds for $\mu+2$. By repeating the above process, after a finite number of steps we have the desired estimate for any μ which satisfies $2<\mu<\mu_0$. Finally, as $\mu_0=n+2\delta_0$ and

$$u(x,t)\in\mathcal{L}^{2,\mu+2}(Q_T)$$

Proposition A yields that $u(x,t)\in C^{\alpha,\frac{\alpha}{2}}(Q_T)$ with $\alpha=\dfrac{\mu-n}{2}$.

Now we are going to derive the global $\mathcal{L}^{2,\mu}$-estimates. Let $S=\partial\Omega$ be of class C^1.

Theorem 2 *Let $u(x,t)\in L^2(0,T;H^1(\Omega))$ be a weak solution of the equation (1.1) subject to the following initial and boundary conditions:*

$$u(x,t)=g(x,t)\quad on\ S_T$$
$$u(x,0)=u_0(x)\quad on\ \Omega$$

Moreover, assume there exists a function $\psi(x,t)$ such that

$$\psi(x,t)=g(x,t)\quad\text{on } S_T,\ \text{and } \psi(x,0)=u_0(x)\quad\text{on }\Omega$$

If $F(x,t) = (f_1, \cdots, f_n; f_0)$ *and* $\psi_t \in \mathcal{L}^{(\mu-2)^+}(Q_T), \nabla\psi \in \mathcal{L}^{2,\mu}(Q_T)^n$ *then for any* $0 < \mu < \mu_0 = n + 2\delta_0$,

$$\|\nabla u\|_{2,\mu,Q_T} \leq C\left[\sum_{i=1}^{n}[\|f_i\|_{2,\mu,Q_T} + \|\psi_{x_i}\|_{2,\mu,Q_T}] + \|f_0\|_{2,(\mu-2)^+,Q_T}\right] + C[\|\psi_t\|_{2,(\mu-2)^+,Q_T} + \|u\|_{L^2(0,T;H^1(\Omega))}]$$

In particular, if $n < \mu < \mu_0$,

$$u(x,t) \in C^{\alpha,\frac{\alpha}{2}}(\overline{Q}_T)$$

where $\alpha = \dfrac{\mu - n}{2} \in (0, \delta_0)$.

Proof Let $w(x,t) = u(x,t) - \psi(x,t)$ on Q_T. Then $w(x,t)$ solves the equation (2.1) weakly, where f_i $(1 \leq i \leq n)$ and f_0 are replaced, respectively, by

$$f_i^* = f_i - a_{ij}\psi_{x_j}$$

and

$$f_0^* = f_0 + \psi_t - b_i\psi_{x_i} - c\psi$$

Hence we need only to show the result for homogeneous initial and boundary data. In this case, as $S = \partial\Omega$ is of class C^1, if Q_{2r} is replaced by $Q_{2r} \cap Q_T$ (see [2]), Lemma 2.1 to Lemma 2.5 still hold. Lemma 2.6 also holds with some mild modification (cf. [8], Lemma 3 and Lemma 4). All of the rest can be carried over if one uses $Q_r \cap Q_T$ to replace Q_r. We omit the details here.

Remark If the boundary condition on S_T is replaced by the following:

$$[a_{ij}u_{x_j} + a_i u]\cos(\vec{n}, x_i) = g(x,t)$$

where $\vec{n}$ is the outward normal on S, the result of Theorem 2 still holds. Indeed, we may assume $g(x,t) = 0$ (otherwise, we choose a function $G(x,t)$ such that

$$[a_{ij}G_{x_j} + a_i G]\cos(\vec{n}, x_i) = g(x,t)$$

and set $v(x,t) = u(x,t) - G(x,t)$). As $S \in C^1$, in a neighborhood Q_r of $z_0 \in S_T$, we can introduce a transformation to flat the lateral boundary and then extend all of the coefficients as well as the inhomogeneous terms in (1.1) into Q_r^* (the image of Q_T) by a simple reflection. Then the desired result follows from the interior estimate.

3. Applications

To illustrate some applications of the preceding theory, we consider the following strongly coupled parabolic system:

$$\psi_t - \nabla[\sigma(u)\nabla\psi] = 0, \quad (x,t) \in Q_T \tag{3.1}$$

$$u_t - \nabla[k(u)\nabla u] = \sigma(u)|\nabla\psi|^2, \quad (x,t) \in Q_T \tag{3.2}$$

$$\psi(x,t) = g(x,t), \quad u(x,t) = f(x,t) \quad \text{on } S_T \tag{3.3}$$

$$\psi(x,0) = \psi_0(x), u(x,0) = u_0(x) \quad \text{on } \Omega \tag{3.4}$$

where $0 < \sigma_0 \le \sigma(u) \le \sigma_1$ and $0 < k_0 \le k(u) \le k_1$.

The above system can be used as a model for an incompressible, unidirectional flow with a temperature-dependent viscosity (cf [9]). It is also a special case for Maxwell's system with the effect of temperature (cf. [10]). Other applications can be found in [11] and [12], where the derivative of ψ with respect to t is assumed to be zero. There are two major difficulties for the system. The first one is that the system is coupled in the coefficient of the leading term. The second is that the growth order with respect to the gradient of the solution is critical. Therefore, the general regularity theory is not applicable (cf. [8], [13], [14], etc.). We start with the following definition of a weak solution.

Definition *A pair of functions $(u(x,t), \psi(x,t))$ defined on Q_T is said to be a weak solution to the problem (3.1)–(3.4) if*

$$\psi(x,t) - g(x,t) \in L^2(0,T; H_0^1(\Omega)); \quad u(x,t) - f(x,t) \in L^2(0,T; H_0^1(\Omega))$$

and (u,ψ) satisfy

$$\iint_{Q_T} [-\psi w_t + \sigma(u)\nabla\psi\nabla w]dxdt = \int_\Omega w(x,0)\psi_0(x)dx$$

$$\iint_{Q_T} [-u w_t + k(u)\nabla u \nabla w]dxdt = \iint_{Q_T} \sigma(u)|\nabla\psi|^2 dxdt + \int_\Omega w(x,0)u_0(x)dx$$

where $w(x,t) \in H^1(0,T; H_0^1(\Omega))$ is arbitrary with $w(x,T) = 0$.

It has been shown in [9] that under suitable assumption on the known data, the problem (3.1)–(3.4) has at least one weak solution. By applying the $\mathcal{L}^{2,\mu}$-estimate, we can show that a weak solution is also classical.

Theorem 3 *There exists a $\alpha \in (0,1)$ such that*

$$u \in C^{2+\alpha, 1+\frac{\alpha}{2}}(Q_T), \quad \psi \in C^{2+\alpha,1+\frac{\alpha}{2}}(Q_T)$$

Moreover, if $u_0(x), \psi_0(x) \in C^{2+\alpha,1+\frac{\alpha}{2}}(\overline{\Omega})$ and $g(x,t), f(x,t) \in C^{2+\alpha,1+\frac{\alpha}{2}}(S_T)$, and satisfy the consistency conditions on $S \times \{t=0\}$:

$$u_0(x) = f(x,0)$$

$$\psi_0(x) = g(x,0)$$

$$f_t(x,0) - \nabla[k(u_0)\nabla u_0] = \sigma(u_0)|\nabla\psi_0|^2$$

$$g_t(x,0) = \nabla[\sigma(u_0)\nabla\psi_0]$$

then, $u(x,t) \in C^{2+\alpha,1+\frac{\alpha}{2}}(\overline{Q}_T)$ *and* $\psi(x,t) \in C^{2+\alpha,1+\frac{\alpha}{2}}(\overline{Q}_T)$.

Proof To show the local regularity, we first note that for any region Q with $\mathrm{dist}\,(Q, S_T) > 0$, by Lemma 2.4

$$\nabla\psi(x,t) \in \mathcal{L}^{2,\mu}(Q)$$

where $\mu = n + \delta$ for some $\delta \in (0,1)$.

Let $U(x,t) = u + \frac{1}{2}\psi^2$. Now we rewrite the equation (3.2) as follows:

$$U_t - \Delta U = \nabla[(\sigma(u) - 1)\psi\nabla\psi]$$

By the maximum principle, we know that ψ is uniformly bounded. Clearly, by Proposition B, any function in $L^\infty(Q_T)$ is a multiplier for $\mathcal{L}^{2,\mu}(Q_T)$ if $0 \le \mu < n+2$. Hence it follows that

$$f_i(x,t) = (\sigma(u) - 1)\psi\psi_{x_i} \in \mathcal{L}^{2,\mu}(Q)$$

for $i = 1, 2, \cdots, n$. It follows by Theorem 1 that

$$\nabla u \in \mathcal{L}^{2,\mu}(Q)$$

where $0 < \mu < n + \delta$ for some $\delta \in (0,1)$.

By using the same technique as Lemma 2.6, one can easily obtain $u \in \mathcal{L}^{2,2+\mu}(Q)$ for $0 < \mu < \mu_0$. Proposition A yields

$$u(x,t) \in C^{\alpha,\frac{\alpha}{2}}(Q_T)$$

As $u \in C^{\alpha,\frac{\alpha}{2}}(Q_T)$, we apply the result obtained in [15] to have

$$\psi(x,t) \in C^{1+\alpha,\frac{1+\alpha}{2}}(Q_T)$$

Thus, by $W_p^{2,1}(Q)$-estimate (cf. [5]), we have

$$u(x,t) \in C^{1+\alpha,\frac{1+\alpha}{2}}(Q_T)$$

Finally, applying the Schauder theory, we obtain the desired regularity. By the same procedure, we can apply the global $\mathcal{L}^{2,\mu}$-estimates to obtain

$$u(x,t) \in C^{2+\alpha,1+\frac{\alpha}{2}}(\overline{Q}_T); \quad \psi(x,t) \in C^{2+\alpha,1+\frac{\alpha}{2}}(\overline{Q}_T)$$

As an immediate consequence, we have

Corollary *The solution of* (3.1)–(3.4) *is unique.*

Remark In practice, the electric potential $\psi(x,t)$ is often assumed to be time-independent (cf. [11], [12], etc.). In this case the equation (3.1) is reduced into

$$-\nabla[\sigma(u)\nabla\psi] = 0$$

Using the $\mathcal{L}^{2,\mu}$-estimates for elliptic equations (cf. [2]) and Theorem 1, we have the same regularity. This result was obtained recently for $n = 2$ and some partial answer for $n > 2$ in [11].

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