J.Partial Diff. Eqs. **10**(1997), 19–30
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REMARKS ON LOCAL REGULARITY FOR TWO SPACE DIMENSIONAL WAVE MAPS

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Dedicated to Professor Gu Chaohao on the occasion of his 70th birthday and his 50th year of educational work

(Received June 18, 1996)

Abstract In this paper, we continue to study the equation

$$\Box\phi^I + f^I(\phi, \partial\phi) = 0$$

where $\Box = -\partial_t^2 + \Delta$ denotes the standard D'Alembertian in R^{2+1} and the nonlinear terms f have the form

$$f^I = \sum_{JK} \Gamma^I_{JK}(\phi) Q_0(\phi^J, \phi^K)$$

with

$$Q_0(\phi, \varphi) = -\partial_t\phi\partial_t\varphi + \sum_{i=1}^{2} \partial_i\phi\partial_i\varphi$$

and $\Gamma^I_{JK}(\phi)$ being C^∞ function of ϕ. In Y. Zhou [1], we showed that the initial value problem

$$\phi(0, x) = \phi_0(x), \quad \partial_t\phi(0, x) = \phi_1(x)$$

is locally well posed for

$$\phi_0 \in H^{s+1}, \quad \phi_1 \in H^s$$

with $s = \dfrac{1}{8}$. Here, we shall further prove that the initial value problem is locally well posed for any $s > 0$.

Key Words Wave equation; local well-posedness.
Classification 35L.

1. Introduction

In this paper, we continue to study the equation

$$\Box\phi^I + f^I(\phi, \partial\phi) = 0 \tag{1.1}$$

where $\Box = -\partial_t^2 + \Delta$ denotes the standard D'Alembertian in R^{2+1} and the nonlinear terms f have the form

$$f^I = \sum_{JK} \Gamma^I_{JK}(\phi) Q_0(\phi^J, \phi^K) \tag{1.2}$$

with

$$Q_0(\phi, \varphi) = -\partial_t \phi \partial_t \varphi + \sum_{i=1}^{2} \partial_i \phi \partial_i \varphi \tag{1.3}$$

and $\Gamma^I_{JK}(\phi)$ being C^∞ function of ϕ. We call it the equations of wave maps type.

We are interested in the problem of minimal regularity of initial conditions for which the initial value problem

$$\phi(0,x) = \phi_0(x), \quad \partial_t \phi(0,x) = \phi_1(x) \tag{1.4}$$

is locally well posed. In Y. Zhou [1], we showed that the problem is locally well posed for

$$\phi_0 \in H^{s+1}, \quad \phi_1 \in H^s \tag{1.5}$$

with $s = \frac{1}{8}$. Here, we shall improve it to allow $s > 0$.

Theorem 1.1 *The initial value problem* (1.4) *for the equation* (1.1) *is locally well posed for* $\phi_0 \in H^{s+1}$ *and* $\phi_1 \in H^s$ *for any* $s > 0$.

In Section 2, we will state and prove a more precise version of Theorem 1.1.

2. Proof of Theorem 1.1

We begin with introducing a space-time norm similar to that in our previous paper [2]. We rewrite (1.1) as a first order system by letting

$$\phi_\pm = (\partial_t \mp \sqrt{-1}|D_x|)\phi \tag{2.1}$$

where

$$|D_x| = \sqrt{-\Delta} \tag{2.2}$$

then

$$(\partial_t \pm \sqrt{-1}|D_x|)\phi_\pm = f \tag{2.3}$$

Introduce the Fourier integral operators $F_\pm$ by

$$F_\pm \phi(t,x) = (2\pi)^{-2} \int e^{\sqrt{-1}(x\cdot\xi \pm t|\xi|)} \hat{\phi}(t,\xi) d\xi \tag{2.4}$$

Here and hereafter, $\hat{\phi}$ denotes the space Fourier transform of ϕ, then it follows from (2.3) that

$$\partial_t F_\pm \phi_\pm = F_\pm f \tag{2.5}$$

Let

$$\psi_\pm = F_\pm \phi_\pm \tag{2.6}$$

Then

$$\partial_t \psi_\pm = F_\pm f \tag{2.7}$$

Therefore

$$\psi_\pm(t,x) = \phi_\pm(0,x) + \int_0^t (F_\pm f)(s)ds \tag{2.8}$$

For a local solution on the time interval $[0,T]$, (2.8) is equivalent to

$$\psi_\pm(t,x) = \chi\Big(\frac{t}{T}\Big)\Big(\phi_\pm(0,x) + \int_0^t (F_\pm f)(s)ds\Big)$$

where $\chi \in C_0^\infty(R)$ and $\chi(s) \equiv 1$, if $|s| \le 1$, $\chi(s) \equiv 0$, if $|s| \ge 2$.

We Introduce the norms

$$M_{a,b}(\psi) = \Big(\iint (1+|\tau|+|\xi|)^{2a}(1+|\tau|)^{2b}\widetilde{\psi}^2(\tau,\xi)d\tau d\xi\Big)^{\frac{1}{2}} \tag{2.9}$$

where $\widetilde{\psi}$ denotes the space-time Fourier transform of ψ. It follows from Lemma 5.3 in [2] that

$$\sum_\pm M_{s,s+\frac{1}{2}}(\psi_\pm) \le CT^{\frac{1}{2}-s}(\|\phi_0\|_{H^{s+1}} + \|\phi_1\|_{H^s}) + C_\sigma T^{-\sigma}\sum_\pm M_{s,s-\frac{1}{2}}(F_\pm f) \tag{2.10}$$

for any $\sigma > 0$. We define

$$N_s(\phi) = \sum_\pm N_s^\pm(\phi) \tag{2.11}$$

where

$$N_s^\pm(\phi) = M_{s,s+\frac{1}{2}}(\psi_\pm) \tag{2.12}$$

it is easy to see that

$$N_s^\pm(\phi)^2 = \iint (1+|\tau|+|\xi|)^{2s}(1+|\tau \pm |\xi||)^{1+2s}(\tau \mp |\xi|)^2\widetilde{\phi}^2(\tau,\xi)d\tau d\xi \tag{2.13}$$

The main result of this paper is the following theorem which can be used to estimate $M_{s,s-\frac{1}{2}}(F_\pm f)$:

Theorem 2.1 *Suppose that both ϕ and φ vanish when $|t| > 2T$, then for any $0 < s < \frac{1}{4}$, we have*

(i)

$$M_{s,s-\frac{1}{2}}(F_\pm Q_0(\phi,\varphi)) \le C_\sigma T^{2s-\sigma} N_s(\phi)N_s(\varphi) \tag{2.14}$$

where $0 < \sigma < s$.

(ii)

$$M_{s,s-\frac{1}{2}}(F_{\pm}(\Gamma\cdot\varphi)) \le C_\sigma T^{s-\sigma} N_s(\Gamma) \sum_{\pm} M_{s,s-\frac{1}{2}}(F_{\pm}\varphi) \tag{2.15}$$

where $0<\sigma<s$.

(iii) *If* $T^s N_s(\phi) \le M$, *then for any* C^2 *function* Γ *with* $\Gamma(0)=0$, *there holds*

$$T^s N_s(\Gamma(\phi)) \le C(M) \tag{2.16}$$

Theorem 2.1 will be proved in the next section. Writing

$$\Gamma^I_{J,K}(\phi) = \Gamma^I_{J,K}(0) + (\Gamma^I_{J,K}(\phi) - \Gamma^I_{J,K}(0))$$

we apply Theorem 2.1 to get

$$M_{s,s-\frac{1}{2}}(F_{\pm}f) \le T^{-2\sigma} C(M) M^2 \tag{2.17}$$

where

$$M = T^s \sum_J N_s(\phi^J) \tag{2.18}$$

Thus, it follows from (2.10) that

$$M \le C_0(\|\phi_0\|_{H^{s+1}} + \|\phi_1\|_{H^s}) + T^{s-3\sigma} C(M) M^2 \tag{2.19}$$

If we take $\sigma = \dfrac{s}{4}$, then there exists a $T_0>0$ depending only on

$$\delta = \|\phi_0\|_{H^{s+1}} + \|\phi_1\|_{H^s} \tag{2.20}$$

such that for any $0<T\le T_0$, there holds

$$M \le 2C_0\delta \tag{2.21}$$

Thus, it follows

Proposition 2.2 *Let* ϕ *be a smooth solution of* (1.1) (1.4), *with initial data satisfying*

$$\|\phi_0\|_{H^{s+1}} + \|\phi_1\|_{H^s} \le \delta$$

for some positive number δ, *where* $0<s<\dfrac{1}{4}$, *then there exists a positive constant* T_0 *depending only on* δ *such that if* $T\le T_0$, *then there holds*

$$\sup_{|t|\le T} \|D\phi(t,\cdot)\|_{H^s} \le C\delta \tag{2.22}$$

and

$$\sup_{|t|\le T} \|D^2\phi(t,\cdot)\|_{H^s} \le C(\delta)(\|\phi_0\|_{H^{s+2}} + \|\phi_1\|_{H^{s+1}}) \tag{2.23}$$

moreover, if $\phi^{(1)}$, $\phi^{(2)}$ *are two such solutions with initial data* $\phi_0^{(1)}$, $\phi_0^{(2)}$ *and* $\phi_1^{(1)}$, $\phi_1^{(2)}$, *then*

$$\sup_{|t|\leq T} \|D(\phi^{(1)}(t,\cdot)-\phi^{(2)}(t,\cdot))\|_{H^s} \leq C(\delta)(\|\phi_0^{(1)}-\phi_0^{(2)}\|_{H^{s+1}}+\|\phi_1^{(1)}-\phi_1^{(2)}\|_{H^s}) \quad (2.24)$$

Proof We only prove (2.22), (2.23) and (2.24) can be proved in a similar way. We extend ϕ beyond time T such that it vanishes when $t > 2T$ as above. By our previous argument, there exists a $T_0 > 0$ such that for any $0 < T \leq T_0$, there holds

$$N_s(\phi) \leq CT^{-s}\delta \quad (2.25)$$

Noting that $F_\pm$ are unitary operators, we get

$$\|D\phi(t,\cdot)\|_{H^s} \leq \sum_{\pm} \|\phi_\pm(t,\cdot)\|_{H^s} = \sum_{\pm} \|\psi_\pm(t,\cdot)\|_{H^s} \quad (2.26)$$

it follows from Sobolev inequality for the time variable that

$$\|\psi_\pm(t,\cdot)\|_{H^s} \leq CT^s \|\psi_\pm\|_{H_t^{\frac{1}{2}+s} H_x^s} \leq CT^s M_{s,s+\frac{1}{2}}(\psi_\pm) \leq C\delta$$

This completes the proof of the proposition.

Remark 2.3 By (2.23), it follows from classical local existence theorem that the smooth solution can actually be continued to the time T_0.

In order to state more precisely our local well posedness result, we introduce

Definition 2.4 *ϕ is said to be a strong H^{1+s} solution to the Cauchy problem* (1.1) (1.4) *on the time interval* $[0,T]$, *if*

$$D\phi \in C([0,T], H^s) \quad (2.27)$$

and there exists a sequence of smooth solutions ϕ_n of (1.1) *on the time interval* $[0,T]$ *such that*

$$\sup_{0\leq t\leq T} \|D\phi_n(t,\cdot) - D\phi(t,\cdot)\|_{H^s} \to 0 \quad \text{as } n\to\infty$$

We shall prove

Theorem 2.5 *Suppose that the initial data ϕ_0, ϕ_1 satisfy*

$$\delta = \|\phi_0\|_{H^{s+1}} + \|\phi_1\|_{H^s} < \infty$$

with $0 < s < \frac{1}{4}$. *Then there exists a positive constant T_0 depending only on δ and a unique strong H^{s+1} solution to the Cauchy problem* (1.1) (1.4) *on the time interval* $[0,T_0]$ *satisfying*

$$\sup_{0\leq t\leq T_0} \|D\phi(t,\cdot)\|_{H^s} \leq C\delta \quad (2.28)$$

Proof Indentical with the proof of Theorem 5.2 in [2].

3. Smoothing Estimates

It remains to prove Theorem 2.1. We first prove the part (i). This would follow from Lemma 3.1 and poincare's inequality.

Lemma 3.1 *Consider the space time norms* (2.9), *null form* (1.3) *and Fourier Integral operator* (2.4). *We have*

$$M_{s,s-\frac{1}{2}}(F_{\pm}Q_0(\phi,\varphi)) \le C_\delta\Big(\sum_{\pm} M_{s,\sigma+\frac{1}{2}}(F_{\pm}\phi_{\pm})\Big)\Big(\sum_{\pm} M_{s,\sigma+\frac{1}{2}}(F_{\pm}\varphi_{\pm})\Big) \tag{3.1}$$

where $0 < s < \frac{1}{4}$, $0 < \sigma < s$ *and*

$$\phi_{\pm} = (\partial_t \mp \sqrt{-1}|D_x|)\phi \tag{3.2}$$

$$\varphi_{\pm} = (\partial_t \mp \sqrt{-1}|D_x|)\varphi \tag{3.3}$$

Proof We only estimate $M_{s,s-\frac{1}{2}}(F_{-}Q_0(\phi,\varphi))$. The estimate of $M_{s,s-\frac{1}{2}}(F_{+}Q_0(\phi,\varphi))$ is identical. Let $\widetilde{\phi}(\tau,\xi), \widetilde{\varphi}(\lambda,\eta)$ be the space-time Fourier transform of ϕ,φ, we may assume $\tau \le 0$ and $\lambda \le 0$ in the support of $\widetilde{\phi}$, $\widetilde{\varphi}$, otherwise, we decompose them such that each of them is the sum of two functions, one has support in $\tau \ge 0$ and the other has support in $\tau \le 0$. Thus, we have to consider four cases. However, all the four cases are similar, so we only consider one case. Under our assumption, we shall prove

$$M_{s,s-\frac{1}{2}}(F_{-}Q_0(\phi,\varphi)) \le C_\sigma M_{s,\sigma+\frac{1}{2}}(F_{+}\phi_{+})M_{s,\sigma+\frac{1}{2}}(F_{+}\varphi_{+}) \tag{3.4}$$

By duality, it is enough to prove

$$I = \iint (F_{-}h)Q_0(\phi,\varphi)dxdt \le CM_{-s,\frac{1}{2}-s}(h)M_{s,\sigma+\frac{1}{2}}(F_{+}\phi_{+})M_{s,\sigma+\frac{1}{2}}(F_{+}\varphi_{+}) \tag{3.5}$$

Let

$$H(u,\zeta) = \frac{\widetilde{h}(u,\zeta)(1+|u|)^{\frac{1}{2}-s}}{(1+|u|+|\zeta|)^s} \tag{3.6}$$

$$F(\tau,\xi) = \widetilde{\phi}(\tau-|\xi|,\xi)(1+|\tau|+|\xi|)^s(2|\xi|-\tau)(1+|\tau|)^{\sigma+\frac{1}{2}} \tag{3.7}$$

$$G(\lambda,\eta) = \widetilde{\varphi}(\lambda-|\eta|,\eta)(1+|\lambda|+|\eta|)^s(2|\eta|-\lambda)(1+|\lambda|)^{\sigma+\frac{1}{2}} \tag{3.8}$$

Then, it follows from Parseval's identity that we only need to prove

$$\begin{aligned} I = & \\ \iint & \frac{b(\tau,\lambda,\xi,\eta)(1+|u|+|\xi+\eta|)^s H(u,\xi+\eta)F(\tau,\xi)G(\lambda,\eta)d\tau d\lambda d\xi d\eta}{(1+|u|)^{\frac{1}{2}-s}(1+|\tau|)^{\sigma+\frac{1}{2}}(1+|\lambda|)^{\sigma+\frac{1}{2}}(2|\xi|-\tau)(2|\eta|-\lambda)(1+|\tau|+|\xi|)^s(1+|\lambda|+|\eta|)^s} \\ \le & C\|F\|\,\|G\|\,\|H\| \end{aligned} \tag{3.9}$$

where

$$b(\tau,\lambda,\xi,\eta)=(\tau-|\xi|)(\lambda-|\eta|)-\xi\cdot\eta \tag{3.10}$$

$$u=\lambda+\tau-(|\xi|+|\eta|-|\xi+\eta|) \tag{3.11}$$

and $\|F\|$ denotes the L^2 norm of F etc.

We have

$$2b=[(u-|\xi+\eta|)^2-(\xi+\eta)^2]-[(\tau-|\xi|)^2-\xi^2]-[(\lambda-\eta)^2-\eta^2] \tag{3.12}$$

Thus

$$2b\le|(u-|\xi+\eta|)^2-(\xi+\eta)^2|+|(\tau-|\xi|)^2-\xi^2|+|(\lambda-\eta)^2-\eta^2| \tag{3.13}$$

We shall consider three cases

Case 1. the first term is maximum among the right hand side of (3.13).

Case 2. the second term is maximum among the right hand side of (3.13).

Case 3. the third term is maximum among the right hand side of (3.13).

Case 3. is similar to Case 2, so we shall only deal with Case 1 and Case 2.

We first consider Case 1. By (3.13), we get

$$I\le C\iint\frac{|b|^{\frac12+s}(1+|u|+|\xi+\eta|)^s(|u|+|\xi+\eta|)^{\frac12-s}H(u,\xi+\eta)F(\tau,\xi)G(\lambda,\eta)d\tau d\lambda d\xi d\eta}{(1+|\tau|)^{\sigma+\frac12}(1+|\lambda|)^{\sigma+\frac12}(|\xi|+|\tau|)(|\eta|+|\lambda|)(1+|\tau|+|\xi|)^s(1+|\lambda|+|\eta|)^s} \tag{3.14}$$

We have

$$|u|+|\xi+\eta|\le 2(|\tau|+|\xi|)+2(|\lambda|+|\eta|) \tag{3.15}$$

Without loss of generality, we assume

$$|u|+|\xi+\eta|\le 4(|\tau|+|\xi|) \tag{3.16}$$

Then

$$I\le C\iint\frac{|b|^{\frac12+s}H(u,\xi+\eta)F(\tau,\xi)G(\lambda,\eta)d\tau d\lambda d\xi d\eta}{(1+|\tau|)^{\sigma+\frac12}(1+|\lambda|)^{\sigma+\frac12}(|\xi|+|\eta|)^{\frac12+s}(|\eta|+|\lambda|)(1+|\lambda|+|\eta|)^s} \tag{3.17}$$

We have

$$|b|\le(|\xi|\,|\eta|-\xi\cdot\eta)+|\tau|(|\lambda|+|\eta|)+|\lambda|\,|\eta|$$

so

$$|b|^{\frac12+s}\le(|\xi|\,|\eta|-\xi\cdot\eta)^{\frac12+s}+[|\tau|(|\lambda|+|\eta|)]^{\frac12+s}+|\lambda|^{\frac12+s}|\eta|^{\frac12+s} \tag{3.18}$$

Thus, we have

$$I\le I_1+I_2+I_3$$

where I_1, I_2 and I_3 are defined in an obvious way. I_2 and I_3 can be estimated by Lemma A of [1], we only estimate I_1, we have

$$\begin{aligned} I_1 &\le C \iint \frac{(|\xi|\,|\eta| - \xi\cdot\eta)^{\frac{1}{2}+s} H(u,\xi+\eta)F(\tau,\xi)G(\lambda,\eta)d\tau d\lambda d\xi d\eta}{(1+|\tau|)^{\sigma+\frac{1}{2}}(1+|\lambda|)^{\sigma+\frac{1}{2}}|\xi|^{\frac{1}{2}+s}|\eta|^{1+s}} \\ &\le C \iint \frac{\|F(\tau,\cdot)\|_{L^2}\|G(\lambda,\cdot)\|_{L^2} J(\tau+\lambda)d\tau d\lambda}{(1+|\tau|)^{\sigma+\frac{1}{2}}(1+|\lambda|)^{\sigma+\frac{1}{2}}} \end{aligned} \tag{3.19}$$

where

$$J^2(s) = \iint \frac{(|\xi|\,|\eta| - \xi\cdot\eta)^{1+2s} F^2(s - |\xi| - |\eta| + |\xi+\eta|)d\xi d\eta}{|\xi|^{2s+1}|\eta|^{2s+2}} \tag{3.20}$$

It follows from Lemma 1 of [1] that

$$J \le C\|H\|$$

so

$$I_1 \le C\|H\| \iint \frac{\|F(\tau,\cdot)\|_{L^2}\|G(\lambda,\cdot)\|_{L^2} d\tau d\lambda}{(1+|\tau|)^{\sigma+\frac{1}{2}}(1+|\lambda|)^{\sigma+\frac{1}{2}}} \tag{3.21}$$

Therefore, the desired estimate follows from Holder's inequality.

We now consider Case 2. We may assume $2|u| \le |\tau|$ otherwise, we have

$$|b| \le C(|\tau| + |\xi|)|\tau|^{\frac{1}{2}+s}|u|^{\frac{1}{2}-s}$$

so I can be easily estimated by Lemma A of [1]. By our assumption, we have

$$\frac{1}{(1+|u|)^{\frac{1}{2}-s}(1+|\tau|)^{\sigma+\frac{1}{2}}} \le \frac{1}{(1+|\tau|)^{\frac{1}{2}-s}(1+|u|)^{\sigma+\frac{1}{2}}}$$

so it follows that

$$I \le \iint \frac{|b|^{\frac{1}{2}+s}(1+|u|+|\xi+\eta|)^s H(u,\xi+\eta)F(\tau,\xi)G(\lambda,\eta)dud\lambda d\xi d\eta}{(1+|u|)^{\sigma+\frac{1}{2}}(1+|\lambda|)^{\sigma+\frac{1}{2}}(|\xi|+|\tau|)^{\frac{1}{2}+s}(|\eta|+|\lambda|)(1+|\tau|+|\xi|)^s(1+|\lambda|+|\eta|)^s} \tag{3.22}$$

Making a change of variables ξ to $\xi-\eta$ followed by η to $-\eta$, we get

$$I \le \iint \frac{|b|^{\frac{1}{2}+s}(1+|u|+|\xi|)^s H(u,\xi)F(\tau,\xi+\eta)G(\lambda,\eta)dud\lambda d\xi d\eta}{(1+|u|)^{\sigma+\frac{1}{2}}(1+|\lambda|)^{\sigma+\frac{1}{2}}(|\xi+\eta|+|\tau|)^{\frac{1}{2}+s}(|\eta|+|\lambda|)(1+|\tau|+|\xi+\eta|)^s(1+|\lambda|+|\eta|)^s} \tag{3.23}$$

and

$$b = (\lambda - |\eta|)(u - |\xi|) - (\lambda - |\eta|)^2 + (\xi+\eta)\cdot\eta$$

$$= (\lambda - |\eta|)(u - |\xi|) + \xi \cdot \eta - [(\lambda - |\eta|)^2 - \eta^2]$$

Thus,

$$|b| \leq (|\xi|\,|\eta| + \xi \cdot \eta) + |u|(|\lambda| + |\eta|) + |\lambda|\,|\xi| + |\lambda|(|\lambda| + |\eta|) \tag{3.24}$$

Therefore, we have

$$I \leq J_1 + J_2 + J_3 + J_4$$

where J_1, J_2, J_3 and J_4 are defined in an obvious way. Recall that $2|u| \leq |\tau|$, J_2, J_3 and J_4 can be easily estimated by Lemma A of [1], we only deal with J_1

$$J_1 = \iint \frac{(|\xi|\,|\eta| + \xi \cdot \eta)^{\frac{1}{2}+s}(1 + |u| + |\xi|)^s H(u,\xi)F(\tau, \xi+\eta)G(\lambda,\eta)dud\lambda d\xi d\eta}{(1+|u|)^{\sigma+\frac{1}{2}}(1+|\lambda|)^{\sigma+\frac{1}{2}}(|\xi+\eta| + |\tau|)^{\frac{1}{2}+s}(|\eta| + |\lambda|)(1 + |\tau| + |\xi+\eta|)^s(1+|\lambda| + |\eta|)^s} \tag{3.25}$$

If $2|\eta| \leq |\xi|$, then $|\xi| \leq 2|\xi + \eta|$, so we get

$$J_1 \leq C \iint \frac{(|\xi|\,|\eta| + \xi \cdot \eta)^{\frac{1}{2}+s} H(u,\xi)F(\tau,\xi+\eta)G(\lambda,\eta)dud\lambda d\xi d\eta}{(1+|u|)^{\sigma+\frac{1}{2}}(1+|\lambda|)^{\sigma+\frac{1}{2}}|\xi|^{\frac{1}{2}+s}|\eta|^{1+s}} \tag{3.26}$$

this can be estimated in the same way as I_1 in (3.19). If $2|\eta| \geq |\xi|$, then by

$$|\xi||\eta| + \xi \cdot \eta = (\xi + \eta)^2 - (|\xi| - |\eta|)^2 \leq 2|\xi + \eta|^2$$

we get

$$J_1 \leq$$

$$C \iint \frac{H(u,\xi)(|\xi+\eta| + |\eta| - |\xi|)^{\frac{1}{4}} F(\lambda - u + |\xi+\eta| + |\eta| - |\xi|, \xi+\eta)G(\lambda,\eta)dud\lambda d\xi d\eta}{(1+|u|)^{\sigma+\frac{1}{2}}(1+|\lambda|)^{\sigma+\frac{1}{2}}|\xi|^{\frac{3}{8}}|\eta|^{\frac{3}{8}}}$$

$$= C \iint \frac{dud\lambda}{(1+|u|)^{\sigma+\frac{1}{2}}(1+|\lambda|)^{\sigma+\frac{1}{2}}}$$

$$\iint F(\lambda - u + \tau + |\xi|, \xi)(\tau + |\xi|)^{\frac{1}{4}}(\widetilde{\phi}_u * \widetilde{\phi}_\lambda)(\tau,\xi)d\tau d\xi \tag{3.27}$$

where $*$ denotes the convolution

$$\widetilde{\phi}_u(v,\xi) = \delta(v + |\xi|)H(u,\xi)|\xi|^{-\frac{3}{8}}$$

and

$$\widetilde{\phi}_\lambda(v,\xi) = \delta(v - |\xi|)G(\lambda,\xi)|\xi|^{-\frac{3}{8}}$$

so we get

$$J_1 \leq C\|F\| \iint \frac{L(u,\lambda)dud\lambda}{(1+|u|)^{\sigma+\frac{1}{2}}(1+|\lambda|)^{\sigma+\frac{1}{2}}} \tag{3.28}$$

where

$$L^2(u,\lambda) = \iint (\tau + |\xi|)^{\frac{1}{2}} (\widetilde{\phi}_u * \widetilde{\phi}_\lambda)^2(\tau,\xi) d\tau d\xi \tag{3.29}$$

It follows from the part (ii) of Theorem 1 of [3] that

$$L(u,\lambda) \le C\|H(u,\cdot)\|_{L^2}\|G(\lambda,\cdot)\|_{L^2} \tag{3.30}$$

Therefore, the desired conclusion follows from Holder's inequality.

We now prove the part (ii) of Theorem 2.1. This would follow from Lemma 3.2 and Poincare's inequality.

Lemma 3.2 *Consider the space time norms* (2.9) *and Fourier Integral operators* (2.4), *we have*

$$M_{s,s-\frac{1}{2}}(F_\pm(\Gamma,\varphi)) \le C_\sigma \sum_\pm M_{s,\sigma+\frac{1}{2}}(F_\pm\Gamma_\pm) \sum_\pm M_{s,s-\frac{1}{2}}(F_\pm\varphi) \tag{3.31}$$

where

$$\Gamma_\pm = (\partial_t \mp \sqrt{-1}|D_x|)\Gamma \tag{3.32}$$

Proof As in the proof of Lemma 3.1, we only estimate $M_{s,s-\frac{1}{2}}(F_+(\Gamma\cdot\varphi))$ and we may assume $u \le 0$ and $\tau \ge 0$ in the support of $\widetilde{\Gamma}(u,\xi)$ and $\widetilde{\varphi}(\tau,\zeta)$. We shall prove

$$M_{s,s-\frac{1}{2}}(F_+(\Gamma\cdot\varphi)) \le C_\sigma M_{s,\sigma+\frac{1}{2}}(F_+\Gamma_+) M_{s,s-\frac{1}{2}}(F_-\varphi) \tag{3.33}$$

By duality, we only need to prove that

$$\iint (F_+\phi)\Gamma\varphi dx dt \le M_{-s,\frac{1}{2}-s}(\phi) M_{s,\sigma+\frac{1}{2}}(F_+\Gamma_+) M_{s,s-\frac{1}{2}}(F_-\varphi) \tag{3.34}$$

Let

$$F(\tau,\zeta) = \frac{\widetilde{\varphi}(\tau+|\zeta|,\zeta)(1+|\tau|+|\zeta|)^s}{(1+\tau)^{\frac{1}{2}-s}} \tag{3.35}$$

$$G(\lambda,\eta) = \frac{\widetilde{\phi}(\lambda,\eta)(1+|\lambda|)^{\frac{1}{2}-s}}{(1+|\lambda|+|\eta|)^s} \tag{3.36}$$

$$H(u,\xi) = \widetilde{\Gamma}(u-|\xi|,\xi)(1+|u|)^{\sigma+\frac{1}{2}}(2|\xi|-u)(1+|u|+|\xi|)^s \tag{3.37}$$

Then it follows from Parseval identity that we only have to prove

$$\begin{aligned} K = &\iint \frac{(1+|\tau|)^{\frac{1}{2}-s}(1+|\lambda|+|\eta|)^s F(\tau,\xi-\eta)G(\lambda,\eta)H(u,\xi)dud\lambda d\xi d\eta}{(1+|\lambda|)^{\frac{1}{2}-s}(1+|u|)^{\sigma+\frac{1}{2}}(2|\xi|-u)(1+|u|+|\xi|)^s(1+|\tau|+|\xi-\eta|)^s} \\ &\le C\|F\|\,\|G\|\,\|H\| \end{aligned} \tag{3.38}$$

where

$$\tau + |\xi-\eta| = u - |\xi| - (\lambda - |\eta|) \ge 0 \tag{3.39}$$

without loss of generality, we assume $|\lambda| \le |\tau|$, otherwise, if $|\lambda| \ge |\tau|$, then

$$K \le \iint \frac{(1+|\lambda|+|\eta|)^s F(\tau,\xi-\eta)G(\lambda,\eta)H(u,\xi)dud\lambda d\xi d\eta}{(1+|u|)^{\sigma+\frac{1}{2}}(|\xi|+|u|)(1+|u|+|\xi|)^s(1+|\tau|+|\xi-\eta|)^s} \tag{3.40}$$

so the desired conclusion follows from Lemma A of [1]. Therefore

$$K \le \iint \frac{(1+|\tau|)^{\frac{1}{2}+s}(1+|\lambda|+|\eta|)^s F(\tau,\xi-\eta)G(\lambda,\eta)H(u,\xi)dud\lambda d\xi d\eta}{(1+|\lambda|)^{\frac{1}{2}+s}(1+|u|)^{\sigma+\frac{1}{2}}(|\xi|+|u|)(1+|u|+|\xi|)^s(1+|\tau|+|\xi-\eta|)^s} \tag{3.41}$$

By (3.39), we have

$$\begin{aligned}|\tau||\xi-\eta| &\le |\tau(2|\xi-\eta|+\tau)| = (\tau+|\xi-\eta|)^2 - |\xi-\eta|^2 \\ &= -2[(u-|\xi|)(\lambda-|\eta|)+\xi\cdot\eta] + [(u-|\xi|)^2-|\xi|^2] + [(\lambda-|\eta|)^2-|\eta|^2]\end{aligned} \tag{3.42}$$

Thus,

$$(1+|\tau|)(1+|\xi-\eta|) \le [2(1+|\tau|)+|\xi-\eta|] + 2|b| + [(|u|+2|\xi|)|u|] + [|\lambda|(|\lambda|+2|\eta|)]$$

where

$$b = (u-|\xi|)(\lambda-|\eta|)+\xi\cdot\eta$$

so we get

$$\begin{aligned}(1+|\tau|)^{\frac{1}{2}+s} \le & \frac{(2|b|)^{\frac{1}{2}+s}}{(1+|\xi-\eta|)^{\frac{1}{2}+s}} + \frac{[2(1+|\tau|)+|\xi-\eta|]^{\frac{1}{2}+s}}{(1+|\xi-\eta|)^{\frac{1}{2}+s}} \\ &+ \frac{[(|u|+2|\xi|)|u|]^{\frac{1}{2}+s}}{(1+|\xi-\eta|)^{\frac{1}{2}+s}} + \frac{[|\lambda|(|\lambda|+2|\eta|)]^{\frac{1}{2}+s}}{(1+|\xi-\eta|)^{\frac{1}{2}+s}}\end{aligned}$$

Therefore

$$K \le K_1 + K_2 + K_3 + K_4$$

where K_1, K_2, K_3 and K_4 are defined in an obvious way. It is easy to estimate K_2, K_3 and K_4 by Lemma A of [1], so we only need to estimate

$$K_1 = C\iint \frac{|b|^{\frac{1}{2}+s}(1+|\lambda|+|\eta|)^s F(\tau,\xi-\eta)G(\lambda,\eta)H(u,\xi)dud\lambda d\xi d\eta}{(1+|\lambda|)^{\frac{1}{2}+s}(1+|u|)^{\sigma+\frac{1}{2}}(|\xi|+|u|)(1+|u|+|\xi|)^s(1+|\xi-\eta|)^{\frac{1}{2}+s}(1+|\tau|+|\xi-\eta|)^s}$$

This can be estimated in the same way as I in (3.22).

Finally, the part (iii) of Theorem 2.1 can be proved in the same way as that in Lemma 5.4 of [2]. This completes the proof of Theorem 2.1.

Acknowledgements

The author would like to thank S. Klainerman for his helpful discussion.

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