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EXISTENCE OF TRAVELLING WAVE SOLUTION OF NONLINEAR EQUATIONS WITH NONLOCAL ADVECTION*

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Dedicated to Professor Gu Chaohao on the occasion of his 70th birthday and his 50th year of educational work

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Abstract In this paper, the existence of travelling wave solution for nonlinear equation with nonlocal advection

$$\rho\frac{\partial}{\partial t}\left(\frac{u^m}{m}\right)=\frac{\partial^2 u}{\partial x^2}-\frac{\partial}{\partial x}[\varphi(k*u)u]+u^n f(u)$$

is studied in the case of $m \geq 1$, $n \geq 1$. When $\varepsilon, \varphi, f, m$ and n satisfy some determinate conditions, there exists the travelling wave solution.

Key Words Travelling wave solution; nonlocal advection.
Classification 35K55, 35K27.

1. Introduction

In this paper, we are concerned with the existence of travelling wave solution of nonlinear equation with nonlocal advection

$$\rho\frac{\partial}{\partial t}\left(\frac{u^m}{m}\right)=\frac{\partial^2 u}{\partial x^2}-\frac{\partial}{\partial x}[\varphi(k*u)u]+u^n f(u) \tag{1.1}$$

where $m \geq 1$, $n \geq 1$ and $\rho = \operatorname{sgn} u_x$. Here $u(t,x) \geq 0$ is the population density at the time $t > 0$ and position $x \in R$, $\varphi(s)$ is the velocity of population. We assume that $\varphi(s)$ ($s \in R$) is a strict monotone increased upper convex odd function. The second term on the right-hand side of Eq. (1.1) involved nonlocal advective term. One of the simplest examples of K is

$$K = \varepsilon[1-2H(x)], \quad \varepsilon \geq 0 \tag{1.2}$$

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where $H(x)$ is the Heaviside step function. The nonlocal advective term has been discussed by Mimura [1] and Alt [2] from a view of biological aggregation. Because of the advection, the individuals have the tendency of aggregation. The third term on the right-hand side of (1.1) is a kinetics of the process, which represents the supply due to births and deaths. We assume that $f(s)$ is a function satisfying

$$f \in C^1[0,1],\ f(0) < 0,\ f(1) = 0,\ f'(1) < 0,\ f(x) = \begin{cases} < 0, & u \in (0,a) \\ > 0, & u \in (a,1) \end{cases} \tag{1.3}$$

generally speaking, $f(s) \sim (s-a)(1-s)$. Ecological studies of this nonlinearity are discussed by Okubo[3]. The discrete model for spatially aggregation phenomena for nonlinear degenerate diffusion equation involving a nonlocal advection term is investigated by Ikeda[4]. At present, the studies of travelling wave solution of reaction diffution equation are quite ripe[5].

For $m = n = 1$, the existence of travelling wave solution of (1.1) was investigated by Huang Sixun[6] in the three dimensions, by using the methods to deal with travelling wave solution of reaction diffution equation and techniques to deal with stationary solution. In this paper, we discuss the existence of travelling wave solution of (1.1), by using the techniques of [7] and the techniques to deal with nonlinear degenerate equation.

2. Mathematical Model

In this paper, we consider travelling wave solution of Eq. (1.1), when K takes the form of (1.2). Then $k * u$ can be represented by

$$k * u = \varepsilon\Big[-2\int_{-\infty}^{x} u(t,y)dy + I\Big] \tag{2.1}$$

where $I = \displaystyle\int_{-\infty}^{+\infty} u(t,y)dy$, generally speaking, $I = I(t)$. Set

$$v(t,x) = \int_{-\infty}^{x} u(t,y)dy, \quad u_x(t,x) = w(t,x) \tag{2.2}$$

then $v_x(t,x) = u(t,x)$. Substituting above relation forms into (1.1), we have

$$\rho\frac{\partial}{\partial t}\Big(\frac{u^m}{m}\Big) = \frac{\partial^2 u}{\partial x^2} + 2\varepsilon\dot\varphi[\varepsilon(2v-I)]u^2 + \varphi[\varepsilon(2v-I)]w + u^n f(u) \tag{2.3}$$

where $\dot\varphi = \dfrac{d\varphi}{ds} > 0$. Set

$$u(t,x) = u(\theta), v(t,x) = v(\theta), \quad w(t,x) = w(\theta), \quad \theta = x + ct \tag{2.4}$$

then $I = \int_{-\infty}^{+\infty} u(y-ct)dy = \int_{-\infty}^{+\infty} u(\theta)d\theta$, from this we know that I is an undetermined constant, it is independent of t. Substituting $u(\theta), v(\theta)$ and $w(\theta)$ into (2.3), we get the following nonlinear ODE

$$\begin{cases} \dfrac{dv}{d\theta} = u \\ \dfrac{du}{d\theta} = w \\ \dfrac{dw}{d\theta} = -\{\varphi[\varepsilon(2v-I)]w - cu^{m-1}|w|\} - 2\varepsilon\dot{\varphi}[\varepsilon(2v-I)]u^2 - u^n f(u) \end{cases} \tag{2.5}$$

the boundary value conditions for (2.5) are

$$\begin{aligned} (v,u,w)_{(-\infty)} &= (0,0,0) \\ (v,u,w)_{(+\infty)} &= (I,0,0) \end{aligned} \tag{2.6}$$

where I is an undetermined constant and dependent on u. In order to eliminate the undetermined constant I from (2.5), we make the following transformation

$$\xi = -\theta, \quad V = \varepsilon\Big[v(\theta) - \frac{I}{2}\Big], \quad U = u, \quad W = -w \tag{2.7}$$

Set "$'$"$= \dfrac{d}{d\xi}$, then (2.5) can be reduced to

$$\begin{cases} V' = -\varepsilon U \\ U' = W \\ W' = [\varphi(2V)W + cU^{m-1}|W|] - 2\varepsilon\dot{\varphi}(2V)U^2 - U^n f(U) \end{cases} \tag{2.8}$$

and boundary value conditions are

$$\begin{aligned} (V,U,W)_{(-\infty)} &= (V_0,0,0) \\ (V,U,W)_{(+\infty)} &= (-V_0,0,0) \end{aligned} \tag{2.9}$$

From (2.8), (2.9), it is obvious that the solution (V,U,W) possesses certain symmetry, that is $V(-\theta) = -V(\theta)$, $U(-\theta) = U(\theta)$, $W(-\theta) = -W(\theta)$, now we'll find the solution which is indicated as Fig.1. From Fig.1, we only have to discuss the solution of (2.8) for $\xi < 0$, then $|w| = w$, the boundary value condition is

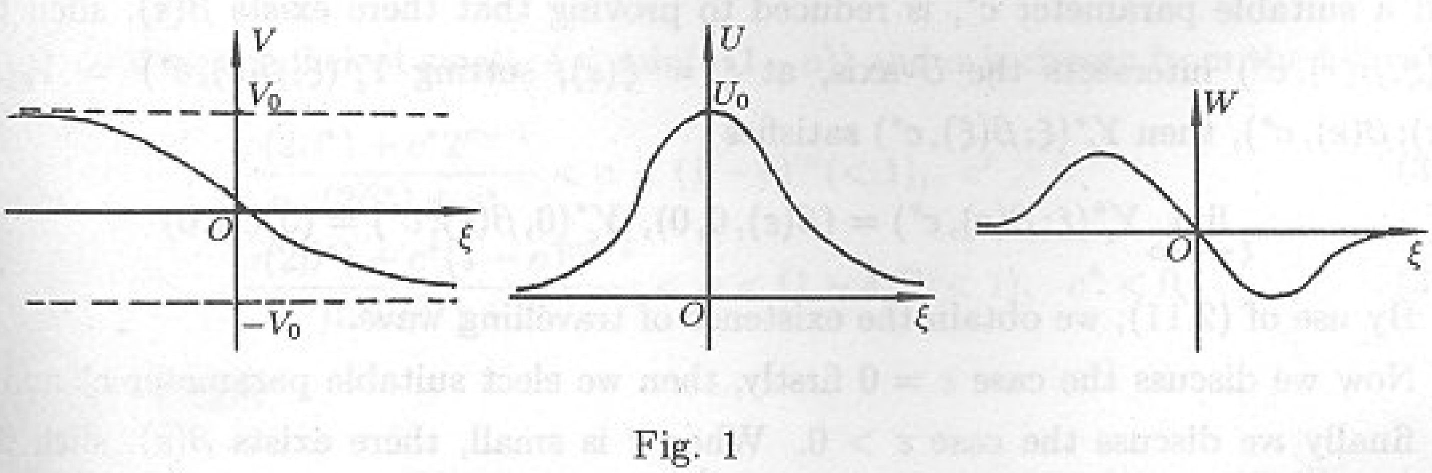

Fig. 1

$$(V,U,W)_{(-\infty)} = (V_0,0,0)$$
$$(V,U,W)_{(0)} = (0,U_0,0) \tag{2.10}$$

for some positive undetermined constants $V_0 = \frac{1}{2}\varepsilon I$ and U_0. Then

$$(v(\theta),u(\theta),w(\theta)) = \begin{cases} \left(\frac{1}{\varepsilon}V(-\theta)+\frac{I}{2}, U(-\theta), -W(-\theta)\right), & \theta \geq 0 \\ \left(\frac{I}{2}-\frac{1}{\varepsilon}V(\theta), U(\theta), W(\theta)\right), & \theta < 0 \end{cases} \tag{2.11}$$

one finds that $(v(\theta),u(\theta),w(\theta))$ satisfies (2.5) (2.6).

Equations (2.8), (2.9) are the problem of three dimensions. When $\varepsilon > 0$, the equilibrium point of (2.8) is $(\beta,0,0)$ $(\beta > 0)$, the linearize (2.8) about the critical point $(V,U,W) = (\beta,0,0)$ is

$$\begin{pmatrix} V \\ U \\ W \end{pmatrix}' = \begin{pmatrix} 0 & -\varepsilon & 0 \\ 0 & 0 & 1 \\ 0 & -nf(0)\bar{r} & \varphi(2\beta)+rc \end{pmatrix} \begin{pmatrix} V-\beta \\ U \\ W \end{pmatrix} \tag{2.12}$$

where $r = \begin{cases} 0, & m > 1 \\ 1, & m = 1 \end{cases}$; $\bar{r} = \begin{cases} 0, & n > 1 \\ 1, & n = 1 \end{cases}$. The characteristic equation of (2.12) is

$$\lambda[\lambda^2 - (\varphi(2\beta)+cr)\lambda + n\bar{r}f(0)] = 0 \tag{2.13}$$

the characteristic values are

$$\lambda_0 = 0,\ \lambda^{\pm} = \frac{1}{2}\{[\varphi(2\beta)+rc] \pm \sqrt{[\varphi(2\beta)+rc]^2 - 4\bar{r}f(0)n}\}$$

Let

$$e^+(\beta) = \left(-\frac{\varepsilon}{\lambda^+(\beta)}, 1, \lambda^+(\beta)\right) \tag{2.14}$$

be an eigenvector associated with the positive eigenvalue. Let $Y_\varepsilon(\xi;\beta,c) = (V_\varepsilon(\xi,\beta,c), U_\varepsilon(\xi,\beta,c), W_\varepsilon(\xi,\beta,c))$ be a trajectory which starts from $(\beta,0,0)$ with the direction $e^+(\beta)$, then the problem of finding the solution of (2.8), (2.10), for a given $\varepsilon > 0$ and a suitable parameter c^*, is reduced to proving that there exists $\beta(\varepsilon)$, such that $Y_\varepsilon(\xi,\beta(\varepsilon),c^*)$ intersects the U-axis, at $\xi = \xi(\varepsilon)$, setting $Y^*_\varepsilon(\xi;\beta(\varepsilon),c^*) = Y_\varepsilon(\xi+\xi(\varepsilon);\beta(\varepsilon),c^*)$, then $Y^*_\varepsilon(\xi;\beta(\xi),c^*)$ satisfies

$$\lim_{\xi\to-\infty} Y^*_\varepsilon(\xi;\beta(\varepsilon),c^*) = (\beta(\varepsilon),0,0),\ Y^*_\varepsilon(0,\beta(\varepsilon),c^*) = (0,U_0,0)$$

By use of (2.11), we obtain the existence of travelling wave.

Now we discuss the case $\varepsilon = 0$ firstly, then we elect suitable parameter c^* and fix c^*, finally we discuss the case $\varepsilon > 0$. When ε is small, there exists $\beta(\varepsilon)$, such that $Y_\varepsilon(\xi;\beta(\varepsilon),c^*)$ intersects the U-axis.

3. The Case $\varepsilon = 0$

Firstly we discuss the case $\varepsilon = 0$. When $\varepsilon = 0$, (2.8) becomes

$$\begin{cases} V' = 0 \\ U' = W \\ W' = [\varphi(2V) + cU^{m-1}]W - U^n f(U) \end{cases} \tag{3.1}$$

Then $V = \beta$ (let β be a positive constant). When $V = \beta$, (U, W) satisfies

$$\begin{cases} U' = W \\ W' = [\varphi(2\beta) + cU^{m-1}]W - U^n f(U) \end{cases} \tag{3.2}$$

For (3.2), let Γ_c be a trajectory which starts from $(0,0)$ with the direction $(1, \lambda^+(\beta))$ and lies in the first quadrant.

Lemma 1 *Fix $\beta > 0$, at least there exists one $c(\beta)$, such that $(U, W)_{(-\infty)} = (0, 0)$, $(U, W)_{(+\infty)} = (1, 0)$.*

Lemma 2 *For any $\beta > 0$, there exists one $c = c(\beta)$, such that Γ_c connects $(0,0)$ to $(1,0)$, i.e. $(U, W)_{(-\infty)} = (0, 0)$, $(U, W)_{(+\infty)} = (1, 0)$.*

By [5] [8], we can prove Lemmas 1–2. Here we omit the detail.

From above, for any $\beta^* > 0$, there exists a unique $c^*(\beta^*)$. Fix β^* and c^*, there exists a unique trajectory $Y_0(\xi; \beta^*, c^*)$ which satisfies

$$\lim_{\xi \to -\infty} Y_0(\xi; \beta^*, c^*) = (\beta^*, 0, 0), \quad \lim_{\xi \to +\infty} Y_0(\xi; \beta^*, c^*) = (\beta^*, 1, 0)$$

Fix $\beta^* > 0$, from Lemma 2, choose c^*, we discuss the properties of the trajectory $Y_0(\xi; \beta, c^*)$ in three dimensions. For convenience, we introduce some numbers:

(1) Let μ be sufficiently large, such that $\mu\varphi(2\beta^*) + c^* > 0$ and

$$(\mu - 1)\varphi(2\beta^*) > c^*(2^{m-1} - 1), \quad c^* \geq 0 \tag{3.3}$$

$$(\mu - 1)\varphi(2\beta^*) > c^*((1 - a)^{m-1} - 1), \quad c^* < 0 \tag{3.4}$$

(2) Let δ be a sufficient small ($\delta < \min(a, 1 - a)$) and α is chosen from the following

$$\frac{\varphi(2\beta^*) + c^* 2^{m-1}}{\mu\varphi(2\beta^*) + c^*} < \alpha < (1 - \delta)^m (< 1), \quad c^* \geq 0 \tag{3.5}$$

$$\frac{\varphi(2\beta^*) + c^*(1 - a)^{m-1}}{\mu\varphi(2\beta^*) + c^*} < \alpha < (1 - \delta)^m (< 1), \quad c^* < 0 \tag{3.6}$$

From (3.6), we get

$$\varphi(2\beta^*) < \mu\varphi(2\beta^*)\alpha + c^*[\alpha - (1 - a)^{m-1}] < \mu\varphi(2\beta^*)\alpha + c^*[\alpha - (1 - \delta)^{m-1}] \tag{3.7}$$

(3) Choose $\overline{\beta}$ as following

$$\beta^* < \overline{\beta} < \begin{cases} \frac{1}{2}\varphi^{-1}\{\mu\varphi(2\beta^*)\alpha + c^*[\alpha - 2^{m-1}]\}, & c^* \geq 0 \\ \min\Big\{\frac{1}{2}\varphi^{-1}[\mu\varphi(2\beta^*)\alpha + c^*(\alpha - (1-a)^{m-1})], \ \frac{1}{2}\varphi^{-1}[\mu\varphi(2\beta^*)]\Big\}, & c^* < 0 \end{cases} \tag{3.8}$$

By use of (3.3)–(3.6) the definition of $\overline{\beta}$ is suitable and $\overline{\beta}$ satisfies $\varphi(2\overline{\beta}) < \mu\varphi(2\beta^*)$.

(4) Consider the function equation

$$F(V,U) = 2\varepsilon\dot{\varphi}(2V)U^2 + U^n f(U) = 0, \quad 0 \leq V \leq \overline{\beta} \tag{3.9}$$

Let $U = U_\varepsilon(V)$ be the smallest root of Equation (3.9) for $U > 1$ ($\varepsilon > 0$). When $\varepsilon = 0$, $U_0(1) = 1$. Because $\varphi(s)$ is a strict monotone increasing upper convex odd function for s, $\dot{\varphi}(2V) > 0$, $\ddot{\varphi}(2V) \leq 0$. Set

$$\sup_{0 \leq V \leq \overline{\beta}} U_\varepsilon(V) = \overline{U}_\varepsilon > 1 \tag{3.10}$$

When ε is sufficient small, we have

$$1 < \overline{U}_\varepsilon < 2 \tag{3.11}$$

$$\frac{dU_\varepsilon(V)}{dV} \leq 0, \quad 0 \leq V \leq \overline{\beta} \tag{3.12}$$

Let $L = \{(V,U), 0 \leq V \leq \overline{\beta}, U = U_\varepsilon(V)\}$, $\Delta = \{(V,U) \mid 0 \leq V \leq \overline{\beta}, 1-\delta < U < U_\varepsilon(V)\}$.

When point (V,U) lies on the curve L, $F(V,U) = 0$; when point (V,U) lies in the region Δ, we have

$$2\varepsilon\dot{\varphi}(2V)U^2 + U^n f(U) > 0 \tag{3.13}$$

Let ε be a sufficient small positive, $0 < \varepsilon < \varepsilon_0$ (ε_0 is a small positive), such that

$$1 < \overline{U}_\varepsilon < \frac{\delta}{(1-\delta)^{m-1}} + (1-\delta) \tag{3.14}$$

By use of (3.5), (3.6), (3.14), we have

$$\alpha < (1-\delta)^{m-1} < \frac{\delta}{\overline{U}_\varepsilon - (1-\delta)} \tag{3.15}$$

Let

$$W^* = [\mu\varphi(2\beta^*) + c^*]\delta \tag{3.16}$$

$$\overline{W}^*(U) = W^*\Big[1 - \frac{U_\varepsilon(0) - U}{U_\varepsilon(0) - (1-\delta)}\Big] \tag{3.17}$$

Define Γ_i^ε $(i=1,2,3)$ and σ_0^ε in two dimensions (U,W) by

$$\Gamma_1^\varepsilon = \{(U,W) \mid U = U_\varepsilon(0), 0 < W < W^*\}$$
$$\Gamma_2^\varepsilon = \{(U,W) \mid 1-\delta < U < U_\varepsilon(0), W = 0\}$$
$$\Gamma_3^\varepsilon = \{(U,W) \mid U < U_\varepsilon(0), W > 0, W = \overline{W}^*(U)\}$$
$$\sigma_0^\varepsilon = \{(U,W) \mid U < U_\varepsilon(0), 0 < W < \overline{W}^*(U)\}$$

and define $E_i^\varepsilon = (i = 1, \cdots, 5)$, l_j $(j=1,2,3)$ and Σ^ε in three dimensions (V,U,W) by

$$E_1^\varepsilon = \{(V,U,W) \mid 0 < V < \overline{\beta}, U = U_\varepsilon(V), 0 < W < \overline{W}^*(U)\}$$
$$E_2^\varepsilon = \{(V,U,W) \mid 0 < V < \overline{\beta}, 1-\delta < U < U_\varepsilon(V), W = 0\}$$
$$E_3^\varepsilon = \{(V,U,W) \mid 0 < V < \overline{\beta}, 1-\delta < U < U_\varepsilon(V), 0 < W = \overline{W}^*(U)\}$$
$$E_4^\varepsilon = \{(V,U,W) \mid V = 0, (U,W) \in \overline{\sigma_0^\varepsilon}\}$$
$$E_5^\varepsilon = \{(V,U,W) \mid V = \overline{\beta}, U \le U_\varepsilon(\overline{\beta}), 0 \le W \le \overline{W}^*(U)\}$$
$$\Sigma^\varepsilon = \{(V,U,W) \mid 0 < V < \overline{\beta}, 1-\delta < U < U_\varepsilon(V), 0 < W < \overline{W}^*(U)\}$$
$$l_1 = \text{int}\{E_3^\varepsilon \cap \overline{E_2^\varepsilon}\}, \quad l_2 = \text{int}\{\overline{E_3^\varepsilon} \cap \overline{E_1^\varepsilon}\}, \quad l_3 = \text{int}\{\overline{E_1^\varepsilon} \cap \overline{E_2^\varepsilon}\}$$

Lemma 3 *When $\varepsilon = 0$, there exists β_0 $(0 < \beta_0 < \overline{\beta} - \beta^*)$, such that for any $\beta \in [\beta^* - \beta_0, \beta^* + \beta_0]$, $Y_0(\xi; \beta, c^*) \in \Sigma^0$ for some ξ; after entering Σ^0, $Y_0(\xi; \beta^* + \beta_0, c^*)$ withdraws Σ^0 from E_1^0 and $Y_0(\xi; \beta^* - \beta_0, c^*)$ withdraws Σ^0 from E_2^0.*

Proof The proof is divided into some steps. **Step one** Project the trajectory $Y_0(\xi;\beta,c^*)$ on the plane (U,W), and let it be $P_0(\xi;\beta,c^*)$. We prove that $P_0(\xi;\beta,c^*)$ enters σ_0^0 from Γ_3^0 only and doesn't enter σ_0^0 from Γ_1^0, Γ_2^0 and points A, B, C (See Fig.2).

Because $U' = W > 0$ on Γ_1^0 and $W' = -U^n f(U) < 0$ on Γ_2^0, the trajectories can't enter σ^0 from Γ_1^0 and Γ_2^0. At the corner point $A = (1, W^*)$,

$$\frac{dW}{dU}\Big|_A = [\varphi(2\beta) + c^*] - \frac{U^n f(U)}{W}\Big|_A < \varphi(2\overline{\beta}) + c^* \tag{3.18}$$

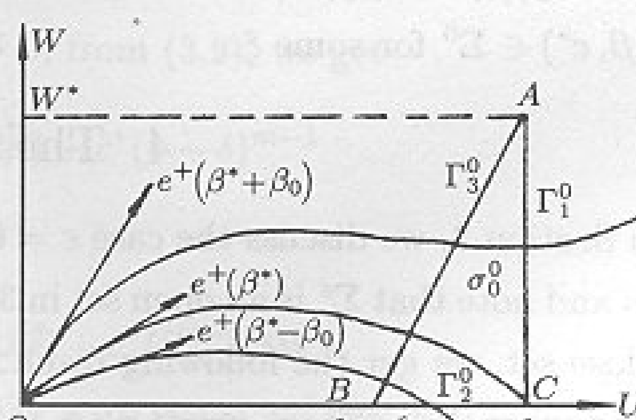

Fig. 2

but from (3.8), we get $\varphi(2\overline{\beta}) < \mu\varphi(2\beta^*)$, then $\frac{dW}{dU}\Big|_A < \varphi(2\overline{\beta}) + c^* < \mu\varphi(2\beta^*) + c^* =$ the slope of the line AB; at point $B = (1-\delta, 0)$, $W' = -(1-\delta)^n f(1-\delta) < 0$; at $C = (1,0)$, C is an equilibrium point of (3.2) (saddle point), then the trajectories can't enter σ_0^0 from corner points A, B and C. Finally we show that the trajectory can't withdraw from σ_0^0 through Γ_3^0. Let $n = (\mu\varphi(2\beta^*) + c^*, -1)$

be a vector normal to Γ_3^0 pointing into σ_0^0, and let $\nu = (W, [\varphi(2\beta)+c^*U^{m-1}]W - U^n f(U))$ be the vector field defined by the right hand side of (3.2), then

$$n \cdot \nu \mid_{\Gamma_3^0} = [\mu\varphi(2\beta^*) + c^*]W - [\varphi(2\beta) + c^* U^{m-1}]W + U^n f(U) \mid_{\Gamma_3^0} \tag{3.19}$$

When $c^* \geq 0$, by $\mu\varphi(2\beta^*) > \varphi(2\beta)$, we get $n \cdot \nu > 0$; when $c^* < 0$, by (3.8), we have

$$\varphi(2\beta^*) < \varphi(2\overline{\beta}) < \mu\varphi(2\beta^*)\alpha + c^*[\alpha - (1-a)^{m-1}] < \mu\varphi(2\beta^*) + c^* - c^*(1-\delta)^{m-1}$$

then $\mu\varphi(2\beta^*) + c^* - \varphi(2\beta) - c^*U^{m-1} > \mu\varphi(2\beta^*) + c^* - \varphi(2\overline{\beta}) - c^*(1-\delta)^{m-1}$ therefore $n \cdot \nu > 0$, which is what needs to show.

Step two We show that there exists a positive constant β_0 $(0 < \beta_0 < \overline{\beta} - \beta^*)$, such that after entering σ_0^0, $P_0(\xi; \beta^* + \beta_0, c^*)$ withdraws σ_0^0 from Γ_1^0; $P_0(\xi; \beta^* - \beta_0, c^*)$ withdraws σ_0^0 from Γ_2^0. From above discussion, we know that $P_0(\xi; \beta^*, c^*)$ starts from the point $(0, 0)$ with the direction $(1, \lambda^+(\beta^*))$ and enters into the point $(1, 0)$. By the continuity with respect to parameter β^* of ODEs, when β is sufficient approximate to β^*, i.e. there exists β_0 $(0 < \beta_0 < \overline{\beta} - \beta^*)$ when $\beta \in [\beta^* - \beta_0, \beta^* + \beta_0]$, we note that σ_0^0 is an open set, then $P_0(\xi; \beta, c^*) \in \sigma_0^0$ for some ξ. Because the trajectory $P_0(\xi; \beta^* + \beta_0, c^*)$ lies above $P_0(\xi; \beta^*, c^*)$, it doesn't intersect with $P_0(\xi; \beta^*, c^*)$. After entering σ_0^0, it doesn't withdraw from A and Γ_3^0 and doesn't enter into C, then after entering σ_0^0, the trajectory $P_0(\xi; \beta^* + \beta_0, c^*)$ starts from $(0, 0)$ with the direction $(1, \lambda^+(\beta^* + \beta_0))$, that must withdraw σ_0^0 from Γ_1^0. In the same way, we know that $P_0(\xi; \beta^* - \beta_0, c^*)$ will withdraw σ_0^0 from Γ_2^0.

Step three. Now we turn to the trajectory $Y_0(\xi; \beta, c^*)$ of 3-dimensional space. Because $V' = 0$, $\beta =$const. At the plane $V = \beta$, $Y_0(\xi; \beta^* + \beta_0, c^*)$ withdraws Σ^0 from E_1^0 and $Y_0(\xi; \beta^* - \beta_0, c^*)$ withdraws Σ^0 from E_2^0, and when $\beta \in [\beta^* - \beta_0, \beta^* + \beta_0]$, $Y_0(\xi; \beta, c^*) \in \Sigma^0$ for some ξ.

4. The Case $\varepsilon > 0$

In Section 3, we discuss the case $\varepsilon = 0$ by continuity with respect to parameter ε of ODEs and note that Σ^ε is an open set in 3-dimensional space (V, U, W), $[\beta^* - \beta_0, \beta^* + \beta_0]$ is a close set, we get the following result.

Theorem 1 *There exists $\varepsilon_1 > 0$, when $\varepsilon \in [0, \varepsilon_1]$ and $\beta \in [\beta^* - \beta_0, \beta^* + \beta_0]$, $Y_\varepsilon(\xi; \beta, c^*) \in \Sigma^\varepsilon$ for some ξ; after entering Σ^ε, $Y_\varepsilon(\xi; \beta^* + \beta_0, c^*)$ withdraws Σ^ε from E_1^ε and $Y_\varepsilon(\xi; \beta^* - \beta_0, c^*)$ withdraws Σ^ε from E_2^ε.*

Let $\overline{\varepsilon} = \min(\varepsilon_1, \varepsilon_2)$, fix $\varepsilon \in (0, \overline{\varepsilon}_1)$, $\beta \in [\beta^* - \beta_0, \beta^* + \beta_0]$ and $c = c^*(\beta^*)$, then define $\overline{\xi}_\beta$ and $\overline{\overline{\xi}}_\beta$ by

$$\overline{\xi}_\beta = \inf\{\xi \mid Y_\varepsilon(\xi; \beta, c^*) \in \Sigma^\varepsilon\}, \quad \overline{\overline{\xi}}_\beta = \inf\{\xi \mid \xi > \overline{\xi}_\beta, Y_\varepsilon(\xi; \beta, c^*) \overline{\in} \Sigma^\varepsilon\}$$

If $Y_\varepsilon(\xi;\beta,c^*) \in \Sigma^\varepsilon$ for all $\xi \in (\overline{\xi}_\beta, +\infty)$, we let $\overline{\overline{\xi}}_\beta = +\infty$. If $\overline{\overline{\xi}}_\beta$ is finite, we let $Y_\varepsilon(\overline{\overline{\xi}}_\beta, \beta, c^*) = (V_\varepsilon(\overline{\overline{\xi}}_\beta;\beta,c^*), U_\varepsilon(\overline{\overline{\xi}}_\beta;\beta,c^*), W_\varepsilon(\overline{\overline{\xi}}_\beta;\beta,c^*)) \triangleq P_\beta$.

Lemma 4 *For any* $\beta \in [\beta^* - \beta_0, \beta^* + \beta_0]$, $P_\beta \in E_1^\varepsilon \cup E_2^\varepsilon \cup E_4^\varepsilon$.

Proof First we indicate that $\overline{\overline{\xi}}_\beta$ is finite, i.e. when the trajectory enters into Σ^ε, it must withdraw from $\overline{\Sigma}^\varepsilon$ at $\xi = \overline{\overline{\xi}}_\beta$. Because $V' = -\varepsilon U < 0$, $U' = W > 0$ in Σ^ε and Σ^ε is bounded, the trajectory $Y_\varepsilon(\xi;\beta,c^*)$ withdraws from Σ^ε for $\xi > \overline{\xi}_\beta$ or approximates to the equilibrium points. Because there are no such equilibrium points in $\overline{\Sigma}^\varepsilon$, it must withdraw from $\overline{\Sigma}^\varepsilon$ at $\xi = \overline{\overline{\xi}}_\beta$, i.e. $\overline{\overline{\xi}}_\beta$ is finite. Now we prove $P_\beta \in E_1^\varepsilon \cup E_2^\varepsilon \cup E_4^\varepsilon$.

Because $V' = -\varepsilon U < 0$ on E_5^ε, the trajectory can't withdraw from Σ^ε through E_5^ε. On E_3^ε, let $n = \left(0, \frac{[\mu\varphi(2\beta^*)+c^*]}{U_\varepsilon(0)-(1-\delta)}\delta, -1\right)$ be a vector normal to E_5^ε pointing into Σ^ε, and let $X = (-\varepsilon U, W, [\varphi(2V) + c^* U^{m-1}]W - 2\varepsilon\dot{\varphi}(2V)U^2 - U^n f(U))$ be vector field defined by (2.8), then

$$n \cdot X = \left\{\frac{[\mu\varphi(2\beta^*) + c^*]\delta}{U_\varepsilon(0) - (1-\delta)} - [\varphi(2V) + c^* U^{m-1}]\right\}W + 2\varepsilon\dot{\varphi}(2V)U^2 + U^n f(U)$$

Because $2\varepsilon\dot{\varphi}(2V)U^2 + U^n f(U) > 0$ for $0 < V < \overline{\beta}, 1-\delta < U < U_\varepsilon(V)$,

$$\begin{aligned} n \cdot X &> \left\{\frac{[\mu\varphi(2\beta^*) + c^*]}{U_\varepsilon(0) - (1-\delta)}\delta - [\varphi(2V) + c^* U^{m-1}]\right\}W \\ &\geq \left\{\frac{[\mu\varphi(2\beta^*) + c^*]\delta}{U_\varepsilon(0) - (1-\delta)} - [\varphi(2\overline{\beta}) + c^* U^{m-1}]\right\}W \end{aligned} \tag{4.1}$$

When $c^* \geq 0$, from (3.11), we know $U < 2$, and from (3.8), we get

$$[\mu\varphi(2\beta^*) + c^*]\alpha > \varphi(2\overline{\beta}) + c^* 2^{m-1} \tag{4.2}$$

by use of (3.15) we get $n \cdot X > 0$. When $c^* > 0$, from (3.9), we get

$$[\mu\varphi(2\beta^*) + c^*]\alpha > \varphi(2\overline{\beta}) + c^*(1-\delta)^{m-1} \tag{4.3}$$

then

$$\begin{aligned} n \cdot X &> \{[\mu\varphi(2\beta^*) + c^*]\alpha - [\varphi(2\overline{\beta}) + c^* U^{m-1}]\}W \\ &> \{\mu\varphi(2\beta^*) + c^*]\alpha - [\varphi(2\overline{\beta}) + c^*(1-\delta)^{m-1}]\}W > 0 \end{aligned}$$

This shows $n \cdot X > 0$, i.e. trajectory can't withdraw from Σ^ε through E_3^ε. Along l_1, because $V' < 0$, $U' = 0$, $W' = -[2\varepsilon\dot{\varphi}(2V)U^2 + U^n f(U)] < 0$; along l_2, $U' > 0$ and by use of $n \cdot X > 0$, we get it can't withdraw from Σ^ε through l_1 and l_2; along l_3, $W' = 0$, $U' = 0$ and $V' < 0$, by use of (3.12) $\dfrac{dV}{dU} \leq 0$, we also know that the trajectory can't withdraw from Σ^ε through l_3. This completes the proof of that lemma.

Theorem 2 *Let any $\beta^* > 0$ and get $c = c^*(\beta^*)$, fix $\varepsilon > 0$, $(0 < \varepsilon < \varepsilon_1)$, then there exists $\beta = \beta(\varepsilon)$, $\beta \in (\beta^* - \beta_0, \beta^* + \beta_0)$ such that $Y_\varepsilon(\xi; \beta(\varepsilon), c^*)$ intersects into U-axis, i.e. there exists the travelling wave solution of the problem* (2.8) (2.9).

Proof When $\varepsilon \in (0, \varepsilon)$, $\beta \in [\beta^* - \beta_0, \beta^* + \beta_0]$, let T be the map from $[\beta^* - \beta_0, \beta^* + \beta_0]$ into $E_1^\varepsilon \cup E_2^\varepsilon \cup E_4^\varepsilon$

$$T(\beta) = P_\beta = Y_\varepsilon(\bar{\bar{\xi}}_\beta; \beta, c^*)$$

From Lemma 4, we know that T is well defined. On E_1^ε, $U' = W > 0$; On E_2^ε, $W' = -2\varepsilon\dot{\varphi}(2V)U^2 - U^n f(U) < 0$, and on E_4^ε, $V' = -\varepsilon U < 0$, then the trajectory$Y_\varepsilon(\xi; \beta, c^*)$ which leaves Σ^ε through E_1^ε, E_2^ε and E_4^ε must do so transversally. By the continuity with respect to the initial value of ODEs, we get T is continuous. Because $[\beta^* - \beta_0, \beta^* + \beta_0]$ is a connected interval, $L = T([\beta^* - \beta_0, \beta^* + \beta_0])$ is a connected curve on $E_1^\varepsilon \cup E_2^\varepsilon \cup E_4^\varepsilon$. From Theorem 1, we know that $T(\beta^* + \beta_0) \in E_1^*$, $T(\beta^* - \beta_0) \in E_2^\varepsilon$, then there exists $\beta(\varepsilon)$

$$\beta(\varepsilon) = \sup\{\beta \mid \beta < \beta^* + \beta_0, T(\beta) \in E_2^\varepsilon\}$$

this $\beta(\varepsilon)$ is the one we wish, i.e. $Y_\varepsilon(\xi; \beta(\varepsilon), c^*)$ crosses the U-axis.

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