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EXISTENCE OF TIME PERIODIC SOLUTIONS TO BOUNDARY VALUE PROBLEM OF ONE-DIMENSIONAL SEMILINEAR VISCOELASTIC DYNAMIC EQUATION WITH MEMORY*

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Dedicated to Professor Gu Chaohao on the occasion of his 70th birthday and his 50th year of educational work

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Abstract In this paper, we prove the existence of time-periodic solutions to the boundary value problem of semilinear one-dimensional dynamical equation for viscoelastic materials.

Key Words Nonlinear viscoelasticity; integrodifferential equation; periodic solution.

Classification 45K05, 73K15, 35B10.

1. Introduction

This paper concerns the study of the existence of time periodic classical solutions to boundary value problem for one dimensional viscoelastic equation

$$u_{tt}(x,t) = \frac{\partial}{\partial x}\sigma + f(x,t), \quad 0 < x < 1, \quad t \in \mathbf{R} \tag{1.1}$$

$$u(0,t) = u(1,t) = 0 \tag{1.2}$$

where the constitutive relation is given by

$$\sigma(t) = p(u_x(x,t)) - \int_0^\infty a(\tau)q(u_x(x,t-\tau))d\tau \tag{1.3}$$

and $p(\xi)$ is a linear function

$$p(\xi) = c^2\xi \tag{1.4}$$

In the past twenty years, there were a lot of works on the initial-boundary value problems of Equation (1.1), (1.3) (See, for instance, [1] and [2] for the existence of global

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classical solutions for small data; [3] for the existence of global weak solutions for large data). Nevertheless, there were only a few works dealing with periodic solutions to the boundary problem (1.1)–(1.3).

In the special case $p(\xi) \equiv q(\xi)$ and

$$p'(\xi) - \hat{a}(0)q'(\xi) > 0 \tag{1.5}$$

where $\hat{a}(0) = \int_0^\infty a(t)dt$, Freireisl [4] proved the existence of periodic weak solutions to the problem (1.1)–(1.3). Condition (1.5) implies that the material is viscoelastic solid. In linear viscoelasticity, we showed that for viscoelastic solid, the problem (1.1)–(1.3) has a T-periodic solution for any T-periodic function $f(x,t)$; and for a viscoelastic liquid like material, the problem admits a T-periodic solution for a T-periodic function $f(x,t)$ if and only if

$$\int_0^T f(x,t)dt = 0 \tag{1.6}$$

(See [5]).

In the present paper, we discuss the semilinear case where the material is viscoelastic solid, i.e. $p(\xi)$ has the form (1.4) and

$$c^2 - \hat{a}(0)q'(\xi) > 0 \tag{1.7}$$

In [6], Hrusa showed the global existence of classical solution to the initial-boundary value problem with large data and gave an estimation of the solution to the problem with kernel $a(t) = \exp(-\lambda t)$. Under some assumptions, we prove the existence of T-periodic classical solution to the problem (1.1)–(1.4) with a more general kernel $a(t)$.

2. Main Results

Throughout this paper, we assume that q and the kernel a satisfy the following hypotheses:

(H_1) $q \in C^1(\mathbf{R})$

with

$$q(0) = 0 \tag{2.1}$$

and there exist positive constants λ and μ such that

$$\mu \le \hat{a}(0)q'(\xi) \le c^2 - \lambda \tag{2.2}$$

(H_2) $a \in L^1(0,\infty) \cap C^\infty([0,\infty))$, $a(t) \not\equiv 0$ and there exists a positive constant δ_0 such that $b(t) = e^{\delta_0 t}a(t)$ is completely monotone (See, for example, [7]), that is

$$(-1)^j b^{(j)}(t) \ge 0, \quad j = 1, 2, \cdots$$

Remark It is easy to see that the kernel of the form

$$a(t)=\sum_{l=1}^{N}k_l e^{-\lambda_l t},\quad k_l,\lambda_l>0 \tag{2.3}$$

satisfies the assumption (H_2). Such kernels are commonly used in rheology.

Let $T>0$ and let B be a Banach space. A measurable function $u:\mathbf{R}\to B$ is called T-periodic if $u(t+T)=u(t)$ for almost all $t\in\mathbf{R}$. By $C_T^k(\mathbf{R};B)$, $L_T^p(\mathbf{R};B)$ and $H_T^s(\mathbf{R};B)$ we denote the subspaces of $C^k(\mathbf{R};B)$, $L^p(\mathbf{R};B)$ and Sobolev space $H_{\rm loc}^s(\mathbf{R};B)$, respectively, of all T-periodic functions.

Our main results may be stated as

Theorem A *Let the assumptions (H_1) and (H_2) hold. Then for any*

$$f\in C_T(\mathbf{R};L^2(0,1))\quad \text{with } f_t\in L_T^2(\mathbf{R};L^2(0,1)) \tag{2.4}$$

the problem (1.1)–(1.4) *admits a T-periodic solution*

$$u\in C_T^{2-k}(\mathbf{R};L^2(0,1)),\quad k=0,1,2 \tag{2.5}$$

3. Proof of Main Theorem

Let $\{w^k(x)\}$ be a basis of $H_0^1(0,1)$, which is orthonormal in $L^2(0,1)$. In fact, we have $w^k(x)=\sqrt{2}\sin(k\pi x)$. We construct Galerkin's approximations of the T-periodic solution to the problem (1.1)–(1.4) of the form

$$u^n(x,t)=\sum_{i=1}^{n}y_i(t)w^i(x) \tag{3.1}$$

from the system

$$\begin{aligned}\int_0^1 u_{tt}^n w^k(x)dx+c^2\int_0^1 u_x^n w_x^k(x)dx-\int_0^1\int_0^\infty a(\tau)q(u_x^n(x,t-\tau))w_x^k(x)d\tau dx\\ =\int_0^1 fw^k(x)dx,\quad k=1,\cdots,n\end{aligned} \tag{3.2}$$

The relation (3.2) is a system of semilinear integrodifferential equations in the following form

$$\ddot{\mathbf{y}}(t)+\widetilde{\mathbf{P}}\mathbf{y}-\int_0^\infty a(\tau)\widetilde{\mathbf{q}}(\mathbf{y}(t-\tau))d\tau=\widetilde{\mathbf{f}}(t) \tag{3.3}$$

where $\mathbf{y}=(y_1,\cdots,y_n)^T$, $\widetilde{\mathbf{P}}=(\widetilde{p}_{kl})$, $\widetilde{\mathbf{q}}=(\widetilde{q}_1,\cdots,\widetilde{q}_n)^T$

$$\widetilde{p}_{kl}=c^2\int_0^1 w_x^k w_x^l dx$$

$$\widetilde{q}_k(\mathbf{y}) = \int_0^1 q(u_x^n) w_x^k dx \tag{3.4}$$
$$\widetilde{f}_k(t) = \int_0^1 f w^k dx, \quad k = 1, \cdots, n$$

For the system (3.3), we have

Lemma 3.1 *Assume that hypotheses* (H_1) *and* (H_2) *hold. Then for each* $\widetilde{\mathbf{f}} \in C_T(\mathbf{R}; \mathbf{R}^n)$, *the system* (3.3) *admits a T-periodic solution* $\mathbf{y} \in C_T^2(\mathbf{R}; \mathbf{R}^n)$.

Proof The system (3.3) can be written in the following form

$$\ddot{\mathbf{y}}(t) + \widetilde{\mathbf{g}}(\mathbf{y}) + \int_0^\infty a(\tau) \widetilde{\mathbf{h}}(\mathbf{y}(t), \mathbf{y}(t-\tau)) d\tau = \widetilde{\mathbf{f}}(t) \tag{3.5}$$

where

$$\widetilde{\mathbf{g}}(\mathbf{y}) = \widetilde{\mathbf{P}}\mathbf{y} - \hat{a}(0)\widetilde{\mathbf{q}}(\mathbf{y}), \tag{3.6}$$
$$\widetilde{\mathbf{h}}(\mathbf{y}, \mathbf{z}) = \widetilde{\mathbf{q}}(\mathbf{y}) - \widetilde{\mathbf{q}}(\mathbf{z}) \tag{3.7}$$

Then Lemma 3.1 follows from Theorem A in [8].

Lemma 3.2 *Under the assumptions of Lemma* 3.1, *the following estimate holds*

$$\|u_t^n(\cdot, t)\|_{L^2(0,1)}^2 + \|u_x^n(\cdot, t)\|_{L^2(0,1)}^2 \le C \int_0^T \|\mathbf{f}(\cdot, s)\|_{L^2(0,1)}^2 ds \tag{3.8}$$

Hereafter, C denotes various positive constants independent of n.

For proving the lemma, see Lemma 3.3 in [9].

Now we estimate the second order derivatives of u^n. At first, we assume that the solution $\mathbf{y} \in C_T^3(\mathbf{R}; \mathbf{R}^n)$. Multiplying (3.5) by $e^{\delta t}$, $\delta \in (0, \delta_0]$ and differentiating the resulting equation, we have

$$\frac{d}{dt}(e^{\delta t}(\ddot{\mathbf{y}} + \widetilde{\mathbf{P}}\mathbf{y} - \widetilde{\mathbf{f}}(t))) = a_\delta(0) e^{\delta t} \widetilde{\mathbf{q}}(\mathbf{y}(t)) + \int_{-\infty}^t a_\delta'(t-s) e^{\delta s} \widetilde{\mathbf{q}}(\mathbf{y}(s)) ds \tag{3.9}$$

where $a_\delta = e^{\delta t} a(t)$. Let $r_\delta(t)$ be the resolvent kernel associated with $a_\delta'(t)$, i.e.

$$a_\delta(0) r_\delta(t) + \frac{a_\delta'(t)}{a_\delta(0)} + \int_{-\infty}^t a_\delta'(t-s) r_\delta(s) ds = 0 \tag{3.10}$$

Then (3.9) can be transformed in the form

$$a(0) e^{\delta t} \widetilde{\mathbf{q}}(\mathbf{y}(t)) = \frac{d}{dt}(e^{\delta t}(\ddot{\mathbf{y}} + \widetilde{\mathbf{P}}\mathbf{y} - \widetilde{\mathbf{f}}(t)))$$
$$+ a(0) \int_{-\infty}^t r_\delta(t-s) \frac{d}{ds}(e^{\delta s}(\ddot{\mathbf{y}} + \widetilde{\mathbf{P}}\mathbf{y}(s) - \widetilde{\mathbf{f}}(s))) ds \tag{3.11}$$

Integrating by parts, we can write the above equation as

$$e^{\delta t}(\mathbf{y}^{(3)}+a(0)r_\delta(0)\ddot{\mathbf{y}}+\widetilde{\mathbf{P}}\dot{\mathbf{y}}+a(0)r_\delta(0)\widetilde{\mathbf{P}}\mathbf{y}-a(0)\widetilde{\mathbf{q}}(\mathbf{y}(t)))$$
$$+a(0)\int_{-\infty}^{t} r'_\delta(t-s)e^{\delta s}(\ddot{\mathbf{y}}(s)+\widetilde{\mathbf{P}}\mathbf{y}(s))ds+\delta e^{\delta t}(\ddot{\mathbf{y}}+\widetilde{\mathbf{P}}\mathbf{y})=\mathbf{f}_1(t) \tag{3.12}$$

where

$$\mathbf{f}_1(t)=(a(0)r_\delta(0)+\delta)e^{\delta t}\widetilde{\mathbf{f}}(t)+e^{\delta t}\dot{\widetilde{\mathbf{f}}}(t)+a(0)\int_{-\infty}^{t} r'_\delta(t-s)e^{\delta s}\widetilde{\mathbf{f}}(s)ds \tag{3.13}$$

Taking the scalar product of (3.12) with $e^{\delta t}(\ddot{\mathbf{y}}+(a(0)r_\delta(0)-\varepsilon)\dot{\mathbf{y}})$, we obtain

$$\begin{aligned}
&\frac{d}{dt}\Big(e^{2\delta t}\Big(\frac{1}{2}|\ddot{\mathbf{y}}|^2+\frac{1}{2}\widetilde{\mathbf{P}}\dot{\mathbf{y}}\cdot\dot{\mathbf{y}}+(a(0)r_\delta(0)-\varepsilon)\ddot{\mathbf{y}}\cdot\dot{\mathbf{y}}+\frac{1}{2}a(0)r_\delta(0)(a(0)r_\delta(0)-\varepsilon)|\dot{\mathbf{y}}|^2\Big)\\
&\quad+e^{2\delta t}\Big(\varepsilon|\ddot{\mathbf{y}}|^2-\varepsilon\widetilde{\mathbf{P}}\dot{\mathbf{y}}\cdot\dot{\mathbf{y}}+a(0)\frac{\partial\widetilde{\mathbf{q}}}{\partial\mathbf{y}}(\mathbf{y})\dot{\mathbf{y}}\cdot\dot{\mathbf{y}}\Big)\\
&=-a(0)\frac{d}{dt}(e^{2\delta t}(r_\delta(0)\widetilde{\mathbf{P}}\mathbf{y}-\widetilde{\mathbf{q}}(\mathbf{y}))\cdot\dot{\mathbf{y}})+\delta e^{2\delta t}\widetilde{\mathbf{P}}\dot{\mathbf{y}}\cdot\dot{\mathbf{y}}\\
&\quad+2a(0)\delta e^{2\delta t}(r_\delta(0)\widetilde{\mathbf{P}}\mathbf{y}-\widetilde{\mathbf{q}}(\mathbf{y}))\cdot\dot{\mathbf{y}}-\delta e^{2\delta t}\widetilde{\mathbf{P}}\mathbf{y}\cdot\ddot{\mathbf{y}}\\
&\quad+(a(0)r_\delta(0)-\varepsilon)\delta e^{2\delta t}\ddot{\mathbf{y}}\cdot\dot{\mathbf{y}}+(a(0)r_\delta(0)-\varepsilon)\delta a(0)r_\delta(0)e^{2\delta t}|\dot{\mathbf{y}}|^2\\
&\quad-(a(0)r_\delta(0)-\varepsilon)\delta e^{2\delta t}(\ddot{\mathbf{y}}+\widetilde{\mathbf{P}}\mathbf{y})\cdot\dot{\mathbf{y}}-a(0)r_\delta(0)(a(0)r_\delta(0)-\varepsilon)e^{2\delta t}\widetilde{\mathbf{P}}\dot{\mathbf{y}}\cdot\mathbf{y}\\
&\quad+a(0)(a(0)r_\delta(0)-\varepsilon)e^{2\delta t}\widetilde{\mathbf{q}}(\mathbf{y})\cdot\dot{\mathbf{y}}-a(0)r'_\delta(0)e^{2\delta t}\dot{\mathbf{y}}\cdot(\ddot{\mathbf{y}}+(a(0)r_\delta(0)-\varepsilon)\dot{\mathbf{y}})\\
&\quad-a(0)e^{\delta t}\int_{-\infty}^{t} r''_\delta(t-s)e^{\delta s}\dot{\mathbf{y}}(s)ds\cdot(\ddot{\mathbf{y}}+(a(0)r_\delta(0)-\varepsilon)\dot{\mathbf{y}})\\
&\quad+a(0)\delta e^{\delta t}\int_{-\infty}^{t} r'_\delta(t-s)e^{\delta s}\dot{\mathbf{y}}(s)ds\cdot(\ddot{\mathbf{y}}+(a(0)r_\delta(0)-\varepsilon)\dot{\mathbf{y}})\\
&\quad+e^{\delta t}((a(0)r_\delta(0)+\delta)e^{\delta t}\widetilde{\mathbf{f}}(t)+e^{\delta t}\dot{\widetilde{\mathbf{f}}}(t)\\
&\quad+a(0)\int_{-\infty}^{t} r'_\delta(t-s)e^{\delta s}\widetilde{\mathbf{f}}(s)ds)\cdot(\ddot{\mathbf{y}}+(a(0)r_\delta(0)-\varepsilon)\dot{\mathbf{y}})
\end{aligned} \tag{3.14}$$

Lemma 3.3 *Assume that* (H_2) *holds. Then for* $\delta\in(0,\delta_0]$, *we have* (i) $r_\delta(t)\in C^\infty[0,\infty)$; (ii) $\int_0^\infty|r'_\delta(t)|dt$ *and* $\int_0^\infty|r''_\delta(t)|dt$ *are convergent and bounded independent of* $\delta\in(0,\delta_0]$.

The proof of Lemma 3.3 can be found in [10].

It is easy to verify that $r_\delta(0)\geq r_{\delta_0}(0)>0$. Taking $\varepsilon\leq\frac{1}{2}a(0)r_\delta(0)$, we can get

$$\frac{1}{2}|\ddot{\mathbf{y}}|^2+(a(0)r_\delta(0)-\varepsilon)\ddot{\mathbf{y}}\cdot\dot{\mathbf{y}}+\frac{1}{2}a(0)r_\delta(0)(a(0)r_\delta(0)-\varepsilon)|\dot{\mathbf{y}}|^2\geq\alpha_1(|\ddot{\mathbf{y}}|^2+\dot{\mathbf{y}}|^2) \tag{3.15}$$

where α_1 is a positive constant. Taking $\varepsilon\leq\frac{\mu a(0)}{2\hat{a}(0)}$, we have from assumption (2.2) that

$$\varepsilon|\ddot{\mathbf{y}}|^2-\varepsilon\widetilde{\mathbf{P}}\dot{\mathbf{y}}\cdot\dot{\mathbf{y}}+a(0)\frac{\partial\widetilde{\mathbf{q}}}{\partial\mathbf{y}}(\mathbf{y})\dot{\mathbf{y}}\cdot\dot{\mathbf{y}}\geq\alpha_2(\varepsilon|\ddot{\mathbf{y}}|^2+\widetilde{\mathbf{P}}\dot{\mathbf{y}}\cdot\dot{\mathbf{y}}) \tag{3.16}$$

for certain positive constant α_2.

Integrating (3.14) with respect to t from t to $t+T$, and using (3.15), (3.16), Lemma 3.2 and 3.3, we can obtain that

$$\begin{aligned}
&e^{2\delta t}(e^{2\delta T}-1)(\|u^n_{tt}(\cdot,t)\|^2_{L^2(0,1)}+\|u^n_t(\cdot,t)\|^2_{H^1(0,1)})\\
&\quad+\int_t^{t+T} e^{2\delta s}(\|u^n_{tt}(\cdot,s)\|^2_{L^2(0,1)}+\|u^n_t(\cdot,s)\|^2_{H^1(0,1)})ds\\
&\le C\Big\{\delta\int_t^{t+T} e^{2\delta s}(\|u^n_{tt}(\cdot,s)\|^2_{L^2(0,1)}+\|u^n_t(\cdot,s)\|^2_{H^1(0,1)}+\|u^n_{xx}(\cdot,s)\|^2_{L^2(0,1)})ds\\
&\quad+B(\delta)e^{2\delta t}(e^{2\delta T}-1)\int_t^{t+T}(\|f(\cdot,s)\|^2_{L^2(0,1)}+\|f_t(\cdot,s)\|^2_{L^2(0,1)})ds\Big\} \qquad (3.17)
\end{aligned}$$

where $B(\delta)$ is a constant depending on δ. Here, we omit the details of estimating the terms in the right side of (3.14). For example

$$\begin{aligned}
|\widetilde{\mathbf{P}}\mathbf{y}\cdot\ddot{\mathbf{y}}|&\le c^2\sum_{k,l=1}^n\int_0^1 w^k_x w^l_x dx y_k \ddot{y}_l\\
&=-c^2\sum_{k,l=1}^n\int_0^1 w^k_{xx}w^l dx y_k\ddot{y}_l\le c^2\|u^n_{xx}(\cdot,t)\|_{L^2(0,1)}\|u^n(\cdot,t)\|_{L^2(0,1)}
\end{aligned}$$

Taking the scalar product of (3.12) with $e^{\delta t}\ddot{\mathbf{P}}\mathbf{y}$, we have

$$\begin{aligned}
&\frac{d}{dt}\Big(e^{2\delta t}\Big(\widetilde{\mathbf{P}}\mathbf{y}\cdot\dot{\mathbf{y}}+\frac{1}{2}|\widetilde{\mathbf{P}}\mathbf{y}|^2-\frac{1}{2}\widetilde{\mathbf{P}}\dot{\mathbf{y}}\cdot\dot{\mathbf{y}}\Big)\Big)\\
&\quad+a(0)r_\delta(0)e^{2\delta t}\widetilde{\mathbf{P}}\mathbf{y}\cdot\dot{\mathbf{y}}+a(0)r_\delta(0)e^{2\delta t}|\widetilde{\mathbf{P}}\mathbf{y}|^2\\
&\quad+a(0)e^{\delta t}\widetilde{\mathbf{P}}\mathbf{y}(t)\cdot\int_{-\infty}^t r'_\delta(t-s)e^{\delta s}\widetilde{\mathbf{P}}\mathbf{y}(s)ds-a(0)\widetilde{\mathbf{P}}\mathbf{y}\cdot\widetilde{\mathbf{q}}(\mathbf{y})\\
&\quad+a(0)e^{\delta t}\widetilde{\mathbf{P}}\mathbf{y}(t)\cdot\int_{-\infty}^t r'_\delta(t-s)e^{\delta s}\dot{\mathbf{y}}(s)ds+\delta e^{2\delta t}(\widetilde{\mathbf{P}}\mathbf{y}\cdot\dot{\mathbf{y}}+|\widetilde{\mathbf{P}}\mathbf{y}|^2)\\
&=e^{\delta t}\widetilde{\mathbf{P}}\mathbf{y}\cdot\mathbf{f}_1(t) \qquad (3.18)
\end{aligned}$$

Lemma 3.4 *Let $a(t)$ satisfy assumptions (H_2). Then it holds that*

$$\begin{aligned}
&\int_t^{t+T}\Big(r_\delta(0)e^{2\delta s}|\widetilde{\mathbf{P}}\mathbf{y}(s)|^2+\int_{-\infty}^s r'_\delta(s-\tau)e^{\delta\tau}\widetilde{\mathbf{P}}\mathbf{y}(\tau)d\tau\cdot e^{\delta s}\widetilde{\mathbf{P}}\mathbf{y}(s)\Big)ds\\
&\quad\ge\frac{1}{\hat{a}_\delta(0)}\int_t^{t+T}e^{2\delta s}|\widetilde{\mathbf{P}}\mathbf{y}(s)|^2ds \qquad (3.19)
\end{aligned}$$

For the proof of Lemma 3.4, see [10].

By the assumption (2.2), we have

$$\int_t^{t+T}e^{2\delta s}\widetilde{\mathbf{P}}\mathbf{y}(s)\cdot\widetilde{\mathbf{q}}(s)ds\le\frac{1-\rho}{\hat{a}(0)}\int_t^{t+T}e^{2\delta s}|\widetilde{\mathbf{P}}\mathbf{y}(s)|^2ds, \qquad (3.20)$$

for some $0 < \rho < 1$.

Applying Lemma 3.4, estimation (3.20) and noting that

$$|\tilde{\mathbf{P}}\mathbf{y}(t)|^2 = c^4\|u_{xx}^n(\cdot,t)\|_{L^2(0,1)}^2 \tag{3.21}$$

we get from (3.18) that

$$\begin{aligned} & e^{2\delta t}(e^{2\delta T}-1)\|u_{xx}^n(\cdot,t)\|_{L^2(0,1)}^2 + \int_t^{t+T} e^{2\delta s}\|u_{xx}^n(\cdot,s)\|_{L^2(0,1)}^2 ds \\ \le & C\{e^{2\delta t}(e^{2\delta T}-1)(\|u_{tt}^n(\cdot,t)\|_{L^2(0,1)}^2 + \|u_t^n(\cdot,t)\|_{H^1(0,1)}^2) \\ & + \int_t^{t+T} e^{2\delta s}(\|u_{tt}^n(\cdot,s)\|_{L^2(0,1)}^2 + \|u_t^n(\cdot,s)\|_{H^1(0,1)}^2) ds \\ & + \delta\int_t^{t+T} e^{2\delta s}\|u_{xx}^n(\cdot,s)\|_{L^2(0,1)}^2 ds \\ & + B(\delta)e^{2\delta t}(e^{2\delta T}-1)\int_0^T (\|f(\cdot,s)\|_{L^2(0,1)}^2 + \|f_t(\cdot,s)\|_{L^2(0,1)}^2) ds\} \end{aligned} \tag{3.22}$$

Combining (3.17) with (3.22) and taking δ small enough, we have

Lemma 3.5 *Under the hypotheses of the theorem, the following estimate holds*

$$\begin{aligned} & \|u_{tt}^n(\cdot,t)\|_{L^2(0,1)}^2 + \|u_t^n(\cdot,t)\|_{H^1(0,1)}^2 + \|u_{xx}^n(\cdot,t)\|_{L^2(0,1)}^2 \\ & \le C\int_0^T (\|f(\cdot,s)\|_{L^2(0,1)}^2 + \|f_t(\cdot,s)\|_{L^2(0,1)}^2) ds \end{aligned} \tag{3.23}$$

In fact, we have proved the estimate (3.23) for $\mathbf{y} \in C_T^3(\mathbf{R};\mathbf{R}^n)$. Applying differences instead of derivatives, we can show the lemma for $\mathbf{y} \in C_T^2(\mathbf{R};\mathbf{R}^n)$.

Proof of Theorem By Lemma 3.5, there exists a subsequence of u^n, denoted still by u^n, such that

$$u^n \to u \quad \text{in } C_T(\mathbf{R}; H^1(0,1)) \tag{3.24}$$

$$u_t^n \to u_t \quad \text{in } C_T(\mathbf{R}; L^2(0,1)) \tag{3.25}$$

$$u_{xx}^n \overset{*}{\rightharpoonup} u_{xx} \quad \text{in } L_T^\infty(\mathbf{R}; L^2(0,1)) \tag{3.26}$$

$$u_{xt}^n \overset{*}{\rightharpoonup} u_{xt} \quad \text{in } L_T^\infty(\mathbf{R}; L^2(0,1)) \tag{3.27}$$

$$u_{tt}^n \overset{*}{\rightharpoonup} u_{tt} \quad \text{in } L_T^\infty(\mathbf{R}; L^2(0,1)) \tag{3.28}$$

as $n \to \infty$. It follows from (3.24) and (3.25) that u is a T-periodic weak solution in the sense

$$\int_0^T\int_0^1 u_{tt}v(x,t)dxdt + \int_0^T\int_0^1 c^2u_xv_x(x,t)dxdt$$

$$
\begin{aligned}
&-\int_0^T\int_0^1\int_0^\infty a(\tau)q(u_x(x,t-\tau))v_x(x,t)d\tau dxdt \\
&=\int_0^T\int_0^1 fv(x,t)dxdt, \quad \forall v\in L^2(0,T;H_0^1(0,1)) \qquad (3.29)
\end{aligned}
$$

(See [9]). And (3.26)–(3.28) imply that the solution u is classical. Therefore, u solves the following initial-boundary value problem

$$
u_{tt}=c^2u_{xx}-\int_0^t a(t-\tau)q(u_x(x,\tau))d\tau+\int_{-\infty}^0 a(t-\tau)q(u_x(x,\tau))d\tau+f(x,t) \qquad (3.30)
$$

$$
u(0,t)=u(1,t)=0 \qquad (3.31)
$$

$$
u(x,0)\in H_0^1(0,1)\cap H^2(0,1), \quad u_t(x,0)\in H_0^1(0,1) \qquad (3.32)
$$

Then (2.5) follows from Theorem 2.1 in [5].

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