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WELL-POSEDNESS OF CAUCHY PROBLEM FOR COUPLED SYSTEM OF LONG-SHORT WAVE EQUATIONS*

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Dedicated to Professor Ding Xiaxi on the occasion of his 70th birthday

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Abstract In this paper we study the Cauchy problem for a class of coupled equations which describe the resonant interaction between long wave and short wave. The global well-posedness of the problem is established in space $H^{\frac{1}{2}+k} \times H^k$ ($k \in Z^+ \cup \{0\}$), the first and second components of which correspond to the short and long wave respectively.

Key Words Cauchy problem, long-short wave equation, well-posedness
Classification 35Q30, 35G25.

1. Introduction and Main Results

In this note we study Cauchy problem for the following long-short wave equation

$$iu_t + u_{xx} = uv + a|u|^{p-1}u, \quad (t,x) \in \mathbf{R} \times \mathbf{R} \tag{1.1}$$

$$v_t = (|u|^2)_x, \quad (t,x) \in \mathbf{R} \times \mathbf{R}, \tag{1.2}$$

$$u(0) = u_0(x), \quad x \in \mathbf{R} \tag{1.3}$$

$$v(0) = v_0(x), \quad x \in \mathbf{R} \tag{1.4}$$

where $a \in \mathbf{R}$, $u(t,x)$ and $v(t,x)$ represent the envelope of the short wave and the amplitude of the long wave respectively. The equations (1.1) (1.2) arise in the study of surface waves with both gravity and capillary modes present and also in plasma physics[1,2]. For $a = 0$, Ma[3] studied (1.1)–(1.4) by inverse scattering method under

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suitable smooth conditions on initial functions. Concerning the Cauchy problem (1.1)–(1.4) in usual Sobolev spaces for (u,v), Guo[4] first proved the global solvability of (1.1)–(1.4) in space $L^\infty(0,T;H^m)\times L^\infty(0,T;H^m)$ for all integer $m\geq 2$ by means of integral estimation method and the fixed point theorem when $a=0$. Recently Tsutsumi & Hatano[5] proved the following results:

(i) When $a=0$, $u_0\in H^{\frac{1}{2}}$, $v_0\in L^2\cap L^\infty$, they prove global solvability of (1.1)–(1.4) in the Space $H^{\frac{1}{2}}\times L^2$.

(ii) When $a\neq 0$, $p=3$, they proved the global well-posedness in space $H^{\frac{1}{2}+k}\times H^k$ for all integers $k\geq 1$.

One natural problem is whether (1.1)–(1.4) generates global flow in the space $H^{\frac{1}{2}}\times L^2$ (or $H^{\frac{1}{2}+k}\times H^k$ for all integer $k\geq 1$) for general p. Our purpose here is to study the global well-posedness of (1.1)–(1.4) in the space $H^{\frac{1}{2}+k}\times H^k$ $(k\in Z^+\cup\{0\})$ for general $p\geq 2$. Our main tools are so called Strichartz type estimates which were established in [6–8] and contraction mapping principle.

Before we state our results we first introduce several notations. For $1\leq p\leq\infty$, we denote by $L^p(\mathbf{R})$ usual Lebegue space of complex and real value functions. $J_{-s}=(I-\Delta)^{-\frac{s}{2}}$ denotes usual Bessel potential, we denote by $W^{s,p}(\mathbf{R})=J_{-s}L^p$ Bessel potential space. When s is an integer, $W^{s,p}$ is just usual Sobolev space. In particular, we simplify write $W^{s,2}=H^s$. Let $D_x^s=(-\Delta)^{\frac{s}{2}}$, then D^{-s} denotes Riesz potential, we denote by $\dot{W}^{s,p}(\mathbf{R})=D^{-s}L^p$ Riesz potential space. For a Banach space X and a time interval $I\in\mathbf{R}$, we denote by $C(I,X)$ the space of strong continuous function from I to X and by $L^p(I,X)$ the space of measurable functions u from I to X such that $\|u(\cdot)\|_X\in L^p(I)$. For the sake of convenience we usually write $L_t^qL_x^p=L^q(I,L^p(\mathbf{R}))$ and $L_x^pL_t^q=L^p(\mathbf{R},L^q(I))$ when this causes no confusion. Different positive constant in the estimates blows might be denoted by the same letter C and if necessary by $C(*,\cdots,*)$ in order to indicate the dependence on the quantities appearing in parentheses.

As is standard practice, we study (1.1)–(1.4) via the corresponding integral equations

$$u(t)=S(t)u_0(x)+\int_0^t S(t-\tau)(uv+a|u|^{p-1}u)d\tau, \tag{1.5}$$

$$v(t)=v_0(x)+\int_0^t \partial_x|u|^2ds \tag{1.6}$$

where $S(t)=\exp(it\Delta)$ is the free propagator which solves free Schrödinger equation.

Naturally, if we put (1.6) into (1.5) it follows

$$u(t) = S(t)u_0(x) + \int_0^t S(t-\tau)F(u)d\tau, \tag{1.7}$$

where

$$F(u) = uv_0(x) + \int_0^t \partial_x |u|^2 d\tau u + a|u|^{p-1}u \tag{1.8}$$

This process makes the equations (1.5) (1.6) become a single equation (1.7) of u with data $(u_0(x), v_0(x))$ which can be solved by a contraction method in the space $X_k(I)$ over the time interval I, defined by

$$\begin{aligned} X_0(I) = \{\, & u \in C(I; H^{\frac{1}{2}}(\mathbf{R})) | \partial_x u \in L_x^\infty(\mathbf{R}; L_t^2(I)), \\ & u \in L_t^4(I; L_x^\infty(\mathbf{R})) \cap L_x^4(\mathbf{R}; L_t^\infty(I))\} \end{aligned} \tag{1.9}$$

with norm

$$\begin{aligned} \|u\|_{X_0} = & \|u\|_{L_t^\infty(I;H^{\frac{1}{2}}(\mathbf{R}))} + \|u\|_{L_t^4(I;L_x^\infty(\mathbf{R}))} \\ & + \|u\|_{L_x^4(\mathbf{R};L_t^\infty(I))} + \|\partial_x u\|_{L_x^\infty(\mathbf{R};L_t^2(I))} \end{aligned} \tag{1.10}$$

and for $k \geq 1$

$$\begin{aligned} X_k(I) = \{\, & u \in (I; H^{\frac{1}{2}+k}(\mathbf{R})) | \partial_x J_k u \in L_x^\infty(\mathbf{R}; L_t^2(I)), \\ & J_k u \in L_t^4(I; L_x^\infty(\mathbf{R})) \cap L_x^4(\mathbf{R}; L_t^\infty(I)), \\ & J_{k-1} u \in L_x^2(\mathbf{R}; L_t^\infty(I))\} \end{aligned} \tag{1.11}$$

with norm

$$\begin{aligned} \|u\|_{X_k} = & \|J_k u\|_{L_t^\infty(I;H^{\frac{1}{2}})} + \|J_k u\|_{L_t^4(I;L_x^\infty(\mathbf{R}))} \\ & + (1+T)^{-\rho} \|J_{k-1} u\|_{L_x^2(\mathbf{R};L_t^\infty(I))} + \|J_k u\|_{L_x^4(\mathbf{R};L_t^\infty(I))} \\ & + \|\partial_x J_k u\|_{L_x^\infty(\mathbf{R};L_t^2(I))} \quad k \in Z^+,\ \rho \geq \frac{1}{2} \end{aligned} \tag{1.12}$$

where $\rho \geq \frac{1}{2}$ is an arbitrary fixed constant, and then we define $v(t,x)$ by the equation (1.6).

Now we are in position to state our results:

Theorem 1 (i) *Let $3 \leq p < 5$, $(u_0(x), v_0(x)) \in H^{\frac{1}{2}} \times L^2$, there exists $T \geq 0$ and a unique pairs of functions $(u,v) \in C([-T,T]; H^{\frac{1}{2}}) \times C([-T,T]; L^2)$ satisfying (1.5) (1.6) with $u \in X_0([-T,T])$.*

(ii) *If $a = 0$, the solutions which were obtained in* (i) *can be extended to infinity, that is* $(u,v) \in C(\mathbf{R}; H^{\frac{1}{2}}) \times C(\mathbf{R}; L^2)$ *with* $u \in X_0([-T,T])$ *for any* $T > 0$ *and*

$$\|u(t)\|_{L^2} = \|u_0(x)\|_{L^2} \quad \text{for } t \in \mathbf{R} \tag{1.13}$$

Theorem 2 (i) *Let* $2 \le p < \infty$, $(u_0(x), v_0(x)) \in H^{\frac{1}{2}+1} \times H^1$, (1.5) (1.6) *have a unique pairs of solutions* $(u,v) \in C(\mathbf{R}; H^{\frac{1}{2}+1}) \times C(\mathbf{R}; H^1)$ *such that* $u \in X_1([-T,T])$ *for any* $T > 0$, (1.13) *and*

$$E(t) = \int_{\mathbf{R}} \left(v(t)|u(t)|^2 + |u_x|^2 + \frac{2a}{p+1}|u(t)|^{p+1} \right) dx = E(0), \tag{1.14}$$

$$\int_{\mathbf{R}} (v^2(t) + 2Im(u(t)\overline{u_x(t)}))dx = \int_{\mathbf{R}} (v_0^2(x) + 2Im(u_0(x)\overline{u_{0x}(x)}))dx \tag{1.15}$$

(ii) *Let* $p = 2l+1$, $(u_0(x), v_0(x)) \in H^{\frac{1}{2}+k} \times H^k$, $l, k \in Z^+$. *Then* (1.5) (1.6) *have a unique pairs of solutions* $(u,v) \in C(\mathbf{R}; H^{\frac{1}{2}+k}) \times C(\mathbf{R}; H^k)$ *such that* $u \in X_k([-T,T])$ *for any* $T > 0$.

Remark 3 From Theorem 2 we easily show that (1.1)–(1.4) generate nonlinear flow $W(t)$ on $H^{k+\frac{1}{2}} \times H^k$ $(k \in Z^+)$ under composition, in other words, the mapping $(u_0(x), v_0(x)) \to W(t)(u_0(x), v_0(x))$ is well-defined from $H^{k+\frac{1}{2}} \times H^k \to C(\mathbf{R}; H^{k+\frac{1}{2}}) \times C(\mathbf{R}; H^k)$. But Theorem 1 implies that (1.1)–(1.4) generate nonlinear flow $W(t)$ on $H^{\frac{1}{2}} \times L^2$ under the composition only in the case $a = 0$. When $a \ne 0$, we only prove that (1.1)–(1.4) is local posedness in X_0, hence it is also the open problem that whether (1.1)–(1.4) generate global flow in $H^{\frac{1}{2}} \times L^2$.

Remark 4 When $a = 0$ or $a \ne 0$ and $p = 3$, Theorem 1 and Theorem 2 imply the results in [4, 5]. On the other hand, here we give some simple and uniform methods of estimation which suit to some similar problems.

As a direct result of the global posedness in Theorem 1 and Theorem 2 and the process of their proof, we also have

Remark 5 Under the conditions of Theorem 1 and Theorem 2, for any $T > 0$ and $(u_0(x), v_0(x)) \in H^{k+\frac{1}{2}} \times H^k$ $(k \in Z^+ \cup \{0\})$ there exists $\varepsilon > 0$ such that the mapping $(u_0(x), v_0(x)) \to W(t)(u_0(x), v_0(x))$ is Lipschitz continuous from $B_\varepsilon^k(u_0(x), v_0(x)) \equiv \{(\varphi, \psi) \in H^{\frac{1}{2}+k} \times H^k); \|\varphi - u_0(x)\|_{H^{\frac{1}{2}+k}} < \varepsilon, \|\psi - v_0(x)\|_{H^k} < \varepsilon\}$ to $X_k([-T,T]) \times L^\infty([-T,T]; H^k)\}$.

Our plan in this paper is the following. In Section 2 we give some basic preliminary estimates. Section 3 is devoted to proving the local version of Theorem 1 and Theorem 2. In Section 4 we complete the proof of Theorem 1 and Theorem 2. For simplicity we restrict ourselves to positive time. For any r with $1 \le r \le \infty$, we denote by r' the exponent dual to r.

2. Preliminary Estimates

In this section we collect some basic estimates related to Schrödinger free propagator $S(t)$ and give some further estimates in X_k which are required in the sequence.

Lemma 2.1 [6 − 9] *$S(t)$ satisfies the following estimates*

(i) *For any (q,r) with $0 \le \frac{2}{q} = \frac{1}{2} - \frac{1}{r} \le \frac{1}{2}$,*

$$\|S(t)\varphi\|_{L_t^q L_x^r} \le C\|\varphi\|_{L^2} \tag{2.1}$$

(ii) *For any (q_j,r_j) with $0 \le \frac{2}{q_j} = \frac{1}{2} - \frac{1}{r_j} \le \frac{1}{2}$, $j = 1,2$ and for any time interval $I \in \mathbf{R}$ with $0 \in \overline{I}$, the operator G defined by*

$$Gf(t,x) = \int_0^t S(t-\tau)f(\tau,x)d\tau \tag{2.2}$$

satisfies the estimates

$$\|Gf\|_{L_t^{q_1} L_x^{r_1}} \le C\|f\|_{L_t^{q_2'} L_x^{r_2'}} \tag{2.3}$$

where C is independent of I.

Lemma 2.2 [6, 8, 10] (i) *$S(t)$ satisfies the following estimates*

$$\|S(t)\varphi\|_{L_x^\infty L_t^2} \le C\|(-\partial_x^2)^{-\frac{1}{4}}\varphi\|_{L^2}, \tag{2.4}$$

$$\|S(t)\varphi\|_{L_x^2 L_t^\infty} \le C(1+T)^\rho\|J_s\varphi\|_{L^2}, \quad \rho \ge \frac{1}{2},\ s > \frac{1}{2} \tag{2.5}$$

$$\|S(t)\varphi\|_{L_x^4 L_t^\infty} \le \|(-\Delta)^{\frac{1}{8}}\varphi\|_{L^2} \tag{2.6}$$

(ii) *For any time interval $I = [-T,T] \subset \mathbf{R}$, the operator G defined by (2.2) in Lemma 2.1 satisfies the estimates*

$$\|(-\partial_x^2)^{\frac{1}{4}}Gf\|_{L_t^\infty L_x^2} \le C\|f\|_{L_x^1 L_t^2}, \tag{2.7}$$

$$\|\partial_x Gf\|_{L_x^\infty L_t^2} \le C\|f\|_{L_x^1 L_t^2} \tag{2.8}$$

and

$$(1+T)^{-\rho}\|Gf\|_{L_x^2 L_t^\infty} \le C\|J_s f\|_{L_t^1 L_x^2} \quad \rho \ge \frac{1}{2},\ s > \frac{1}{2}. \tag{2.9}$$

where C is independent of I and $\rho \ge \frac{1}{2}$ is an arbitrary constant.

As a direct result of Lemma 2.1 and Lemma 2.2 we have

Corollary 2.3 *Let* $u_0(x) \in H^{\frac{1}{2}+k}$, $X_k(I)$ *is defined as* (1.9) *and* (1.10), *then*

$$\|S(t)u_0(x)\|_{X_k} \le C\|u_0(x)\|_{H^{\frac{1}{2}+k}} \tag{2.10}$$

Lemma 2.4 *For any time interval* $I = [-T, T] \subset \mathbf{R}$, *let* $f(x,t) \in L_x^1 L_t^2 \cap L_t^1 L_x^2$, *then the operator* Gf *defined by* (2.2) *in Lemma* 2.1 *satisfies the estimates*

$$\|Gf\|_{X_k} \le C\|J_k f\|_{L_x^1 L_t^2} + C\|J_k f\|_{L_t^1 L_x^2} \tag{2.11}$$

where C *is independent of* I.

Proof As a direct result of Lemma 2.1 and Lemma 2.2 we have

$$\left\|\partial_x J_k \int_0^t S(t-\tau)f(x,\tau)\right\|_{L_x^\infty L_t^2} \le C\|J_k f\|_{L_x^1 L_t^2}, \tag{2.12}$$

$$\left\|J_k \int_0^t S(t-\tau)f(x,\tau)d\tau\right\|_{L_t^4 L_x^\infty} \le C\|J_k f\|_{L_t^1 L_x^2}, \tag{2.13}$$

$$\left\|(1+T)^{-\rho} J_{k-1} \int_0^t S(t-\tau)f(x,\tau)\right\|_{L_x^\infty L_t^2} \le C\|J_k f\|_{L_t^1 L_x^2}, \quad k \ge 1 \tag{2.14}$$

Due to the concept of equivalent norm, it follows

$$\begin{aligned}
&\left\|\int_0^t S(t-\tau)f(x,\tau)d\tau\right\|_{H^{k+\frac{1}{2}}} \\
&\le C\left\|J_k \int_0^t S(t-\tau)f(x,\tau)d\tau\right\|_{L_x^2} + C\left\|D^{\frac{1}{2}} J_k \int_0^t S(t-\tau)f(x,\tau)d\tau\right\|_{L_x^2} \\
&\le C\|J_k f\|_{L_t^1 L_x^2} + C\|J_k f\|_{L_x^1 L_t^2}
\end{aligned} \tag{2.15}$$

by Lemma 2.1 and Lemma 2.2. At last we consider

$$\begin{aligned}
&\left\|J_k \int_0^t S(t-\tau)f(x,\tau)d\tau\right\|_{L_t^4 L_x^\infty} = \left\|S(t)S(-t)J_k \int_0^t S(t-\tau)f(x,\tau)d\tau\right\|_{L_t^4 L_x^\infty} \\
&\le C\left\|D_x^{\frac{1}{4}} S(-t) J_k \int_0^t S(t-\tau)f(x,\tau)d\tau\right\|_{L_t^\infty L_x^2} \\
&\le C\left\|D_x^{\frac{1}{2}} J_k \int_0^t S(t-\tau)f(x,\tau)d\tau\right\|_{L_t^\infty L_x^2}^{\frac{1}{2}} \left\|J_k \int_0^t S(t-\tau)f(x,\tau)d\tau\right\|_{L_t^\infty L_x^2}^{\frac{1}{2}} \\
&\le C\|J_k f\|_{L_x^1 L_t^2} + C\|J_k f\|_{L_t^1 L_x^2}
\end{aligned} \tag{2.16}$$

by Lemma 2.1, Lemma 2.2 and the interpolation theorem. Collecting (2.12)–(2.16) implies (2.11).

3. Local Posedness

This section is devoted to proving the local version of Theorem 1 and Theorem 2. Throughout this section we put $I=[0,T]$ with $T>0$ for simplicity. For $v_0(x)\in H^k(\mathbf{R})$ and $u\in X_k$ with $k\in Z^+\cup\{0\}$, we define

$$F(u)=uv_0(x)+\int_0^t \partial_x|u|^2 d\tau u+a|u|^{p-1}u=F_1+F_2+F_3 \tag{3.1}$$

In view of Lemma 2.1–Lemma 2.4 we have

$$\|GF(u)\|_{X_k}\le C\|J_kF(u)\|_{L_x^1L_t^2}+C\|J_kF(u)\|_{L_t^1L_x^2} \tag{3.2}$$

where C is independent of I. In order to prove the local posedness precisely, we first do a series of nonlinear estimates in X_k.

Lemma 3.1 *Let $v_0(x)\in L^2(\mathbf{R})$ and $u,\widetilde{u}\in X_0(I)$, $3\le p<5$. Then we have*

$$\|F(u)\|_{L_t^1L_x^2}\le CT^{\frac34}\|v_0(x)\|_{L_x^2}\|u\|_{X_0}+CT^{\frac54}\|u\|_{X_0}^3+CT^{\frac{5-p}{4}}\|u\|_{X_0}^p \tag{3.3}$$

$$\|F(u)\|_{L_x^1L_t^2}\le CT^{\frac12}\|v_0(x)\|_{L_x^2}\|u\|_{X_0}+CT\|u\|_{X_0}^3+CT^{\frac{7-p}{8}}\|u\|_{X_0}^p \tag{3.4}$$

$$\begin{aligned}\|F(u)-F(\widetilde{u})\|_{L_t^1L_x^2}\le{}& CT^{\frac34}\|v_0(x)\|_{L_x^2}\|u-\widetilde{u}\|_{X_0}+CT^{\frac54}(\|u\|_{X_0}^2+\|\widetilde{u}\|_{X_0}^2)\\ &\cdot\|u-\widetilde{u}\|_{X_0}+CT^{\frac{5-p}{4}}(\|u\|_{X_0}^{p-1}+\|\widetilde{u}\|_{X_0}^{p-1})\|u-\widetilde{u}\|_{X_0}\end{aligned} \tag{3.5}$$

$$\begin{aligned}\|F(u)-F(\widetilde{u})\|_{L_x^1L_t^2}\le{}& CT^{\frac12}\|v_0(x)\|_{L_x^2}\|u-\widetilde{u}\|_{X_0}+CT(\|u\|_{X_0}^2+\|\widetilde{u}\|_{X_0}^2)\\ &\cdot\|u-\widetilde{u}\|_{X_0}+CT^{\frac{7-p}{8}}(\|u\|_{X_0}^{p-1}+\|\widetilde{u}\|_{X_0}^{p-1})\|u-\widetilde{u}\|_{X_0}\end{aligned} \tag{3.6}$$

Proof We first estimate (3.3) and (3.4), by noting that Hölder inequality, it follows

$$\|F_1(u)\|_{L_t^1L_x^2}\le\int_0^T\|v_0(x)\|_{L_x^2}\|u\|_{L_x^\infty}d\tau\le T^{\frac34}\|v_0(x)\|_{L_x^2}\|u\|_{L_t^4L_x^\infty} \tag{3.7}$$

$$\begin{aligned}\|F_1(u)\|_{L_x^1L_t^2}&\le\int_{\mathbf{R}}|v_0(x)|\cdot\|u\|_{L_t^2}dx\le\|v_0(x)\|_{L_x^2}\|u\|_{L_x^2L_t^2}\\ &\le T^{\frac12}\|v_0(x)\|_{L_x^2}\|u\|_{L_t^\infty L_x^2}\end{aligned} \tag{3.8}$$

$$\begin{aligned}\|F_2(u)\|_{L_t^1L_x^2}&\le\left\|\int_0^t(u_x\overline{u}+\overline{u}_xu)d\tau u\right\|_{L_t^1L_x^2}\le 2\|(\|u_x\|_{L_t^2}\|u\|_{L_t^2}u)\|_{L_t^1L_x^2}\\ &\le 2T^{\frac34}\|u_x\|_{L_x^\infty L_t^2}\|u\|_{L_x^2L_t^2}\|u\|_{L_t^4L_x^\infty}\\ &\le 2T^{\frac54}\|u_x\|_{L_x^\infty L_t^2}\|u\|_{L_t^\infty L_x^2}\|u\|_{L_t^4L_x^\infty}\end{aligned} \tag{3.9}$$

$$\|F_2(u)\|_{L_x^1L_t^2} \le 2\|(\|u_x\|_{L_t^2}\|u\|_{L_t^2}u)\|_{L_x^1L_t^2} \le 2\|(\|u_x\|_{L_t^2}\|u\|_{L_t^2}^2)\|_{L_x^1}$$

$$\le 2\|u_x\|_{L_x^\infty L_t^2}\|u\|_{L_x^2L_t^2}^2 \le 2t\|u_x\|_{L_x^\infty L_t^2}\|u\|_{L_t^\infty L_x^2}^2 \tag{3.10}$$

$$\|F_3(u)\|_{L_t^1L_x^2} \le a\int_0^T \|u\|_{L_x^\infty}^{p-1}\|u\|_{L_x^2}d\tau \le aT^{\frac{5-p}{4}}\|u\|_{L_t^4L_x^\infty}^{p-1}\|u\|_{L_t^\infty L_x^2} \tag{3.11}$$

At last, we come to estimate $\|F_3(u)\|_{L_x^1L_t^2}$,

$$\|F_3(u)\|_{L_x^1L_t^2} \le a\|(|u|^{p-1}u)\|_{L_x^1L_t^2} \le a\int_{\mathbf{R}} \|u\|_{L_t^\infty}^{p-1}\|u\|_{L_t^2}dx$$

$$\le a\|u\|_{L_x^4L_t^\infty}^{p-1}\left(\int_{\mathbf{R}} \|u\|_{L_t^2}^{\frac{4}{5-p}}dx\right)^{\frac{5-p}{4}} \tag{3.12}$$

noting that $3 \le p < 5$ and Minkowski inequality, we have

$$\left(\int_{\mathbf{R}}\left(\int_0^T |u|^2d\tau\right)^{\frac{2}{5-p}}dx\right)^{\frac{5-p}{4}} \le \left(\int_0^T\left(\int_{\mathbf{R}} |u|^{\frac{4}{5-p}}dx\right)^{\frac{5-p}{2}}dt\right)^{\frac{1}{2}}$$

$$\le \left(\int_0^T \|u\|_{L_x^\infty}^{p-3}\|u\|_{L_x^2}^{5-p}dt\right) \le T^{\frac{7-p}{8}}\|u\|_{L_t^4L_x^\infty}^{\frac{p-3}{2}}\|u\|_{L_t^\infty L_x^2}^{\frac{5-p}{2}} \tag{3.13}$$

Putting (3.12) together with (3.13) yields

$$\|F_3(u)\|_{L_x^1L_t^2} \le aT^{\frac{7-p}{8}}\|u\|_{L_x^4L_t^\infty}^{p-1}\|u\|_{L_t^4L_x^\infty}^{\frac{p-3}{2}}\|u\|_{L_t^\infty L_x^2}^{\frac{5-p}{2}} \tag{3.14}$$

Collecting (3.7)–(3.14), we obtain (3.3), (3.4). In the exactly same way as leading to (3.3) and (3.4), we easily obtain (3.5) and (3.6).

Lemma 3.2 *Let $2 \le p < \infty$, $v_0(x) \in H^1(\mathbf{R})$, $u, \tilde{u} \in X_1(I)$. Then*

$$\|J_1F(u)\|_{L_t^1L_x^2} \le CT^{\frac{3}{4}}\|v_0(x)\|_{H_x^1}\|u\|_{X_1} + CT^{\frac{3}{2}}\|u\|_{X_1}^3 + CT\|u\|_{X_1}^p \tag{3.15}$$

$$\|J_1F(u)\|_{L_x^1L_t^2} \le CT^{\frac{1}{2}}\|v_0(x)\|_{H_x^1}\|u\|_{X_1} + CT\|u\|_{X_1}^3 + CT^{\frac{1}{2}}(1+T^\rho)\|u\|_{X_1}^p \tag{3.16}$$

$$\|J_1(F(u)-F(\tilde{u}))\|_{L_t^1L_x^2} \le CT^{\frac{3}{4}}\|v_0(x)\|_{H_x^1}\|u-\tilde{u}\|_{X_1} + CT^{\frac{3}{2}}(\|u\|_{X_1}^2 + \|\tilde{u}\|_{X_1}^2)$$

$$\cdot\|u-\tilde{u}\|_{X_1} + CT(\|u\|_{X_1}^{p-1} + \|\tilde{u}\|_{X_1}^{p-1})\|u-\tilde{u}\|_{X_1} \tag{3.17}$$

$$\|J_1(F(u)-F(\tilde{u}))\|_{L_x^1L_t^2} \le CT^{\frac{1}{2}}\|v_0(x)\|_{H_x^1}\|u-\tilde{u}\|_{X_1} + CT(\|u\|_{X_1}^2 + \|\tilde{u}\|_{X_1}^2)$$

$$\cdot\|u-\tilde{u}\|_{X_1} + CT^{\frac{1}{2}}(1+T^\rho)(\|u\|_{X_1}^{p-1} + \|\tilde{u}\|_{X_1}^{p-1})\|u-\tilde{u}\|_{X_1} \tag{3.18}$$

where ρ is the same as in (1.12).

Proof We first estimate nonlinear term corresponding to $F_1(u)$, putting (3.7) together with

$$\|\partial_xF_1(u)\|_{L_t^1L_x^2} \le \int_0^T(\|\partial_xv_0(x)\|_{L_x^2}\|u\|_{L_x^\infty} + \|v_0(x)\|_{L_x^2}\|\partial_xu\|_{L_x^\infty})d\tau$$

$$\leq 2T^{\frac{3}{4}}\|v_0(x)\|_{H_x^{\frac{1}{2}}}\|J_1u\|_{L_t^4L_x^\infty} \tag{3.19}$$

implies

$$\|J_1F_1(u)\|_{L_t^1L_x^2} \leq CT^{\frac{3}{4}}\|v_0(x)\|_{H_x^{\frac{1}{2}}}\|u\|_{X_1} \tag{3.20}$$

in exactly similar way as leading to (3.20) we have

$$\|J_1F_1(u)\|_{L_x^1L_t^2} \leq CT^{\frac{1}{2}}\|v_0(x)\|_{H_x^{\frac{1}{2}}}\|u\|_{X_1} \tag{3.21}$$

by (3.8). We now consider the estimates which correspond to F_2. It follows

$$\begin{aligned}
&\|\partial_x F_2(u)\|_{L_t^1L_x^2}\\
&\leq \left\|\int_0^t (u_{xx}\tilde{u} + 2\tilde{u}_x u_x + u\tilde{u}_{xx})d\tau u\right\|_{L_t^1L_x^2} + 2\left\|\int_0^t (u_x\tilde{u} + u\tilde{u}_x)d\tau u_x\right\|_{L_t^1L_x^2}\\
&\leq 2\|(\|u_{xx}\|_{L_t^2}\|u\|_{L_t^2} + \|u_x\|_{L_t^2}^2)u\|_{L_t^1L_x^2} + 4\|(\|u\|_{L_t^2}\|u_x\|_{L_t^2})u_x\|_{L_t^1L_x^2}
\end{aligned} \tag{3.22}$$

$$\begin{aligned}
&\|\partial_x F_2(u)\|_{L_x^1L_t^2}\\
&\leq 2\|(\|u_{xx}\|_{L_t^2}\|u\|_{L_t^2} + \|u_x\|_{L_t^2}^2)u\|_{L_x^1L_t^2} + 4\|(\|u\|_{L_t^2}\|u_x\|_{L_t^2})u_x\|_{L_x^1L_t^2}
\end{aligned} \tag{3.23}$$

In view of the technique of the proof in (3.9) and (3.10) we easily obtain

$$\|J_1F_2(u)\|_{L_t^1L_x^2} \leq 4T^{\frac{5}{4}}\|J_1u_x\|_{L_x^\infty L_t^2}\|J_1u\|_{L_t^\infty L_x^2}\|J_1u\|_{L_t^4L_x^\infty} \tag{3.24}$$

$$\|J_1F_2(u)\|_{L_x^1L_t^2} \leq 4T\|J_1u_x\|_{L_x^\infty L_t^2}\|J_1u\|_{L_t^\infty L_x^2}^2 \tag{3.25}$$

On the other hand, $\|J_1F_2(u)\|_{L_t^1L_x^2}$ can be estimated as

$$\begin{aligned}
&\|\partial_x F_2(u)\|_{L_t^1L_x^2}\\
&\leq 2\|(\|u_{xx}\|_{L_x^\infty L_t^2}\|u\|_{L_x^2L_t^2}\|u\|_{L_x^\infty})\|_{L_t^1} + 2\|(\|u_x\|_{L_x^2L_t^2}\|u_x\|_{L_x^\infty L_t^2}\|u\|_{L_x^\infty})\|_{L_t^1}\\
&\quad + 4\|(\|u_x\|_{L_x^\infty L_t^2}\|u\|_{L_x^\infty L_t^2}\|u_x\|_{L_x^2})\|_{L_t^1}\\
&\leq CT^{\frac{3}{2}}\|u_{xx}\|_{L_x^\infty L_t^2}\|u\|_{L_t^\infty L_x^2}\|J_1u\|_{L_t^\infty L_x^2} + CT^{\frac{3}{2}}\|u_x\|_{L_x^\infty L_t^2}\|J_1u\|_{L_t^\infty L_x^2}^2
\end{aligned} \tag{3.24$'$}$$

by the following inequality

$$\|u\|_{L_x^\infty L_t^2} \leq \sup_{x\in\mathbf{R}}\|u\|_{L_t^2L_x^\infty} \leq C\|J_1u\|_{L_t^2L_x^2} \leq CT^{\frac{1}{2}}\|J_1u\|_{L_t^\infty L_x^2}$$

At last, we come to do the estimates corresponding to $F_3(u)$. In virtue of the concept of equivalent norm, it follows

$$\|J_1F_3(u)\|_{L_t^1L_x^2} \leq a\int_0^T \|u\|_{L_x^\infty}^{p-1}(\|u\|_{L_x^2} + \|u_x\|_{L_x^2})d\tau$$

$$\leq CT\|J_1u\|^{p-1}_{L^\infty_t L^2_x}\|J_1u\|_{L^\infty_t L^2_x} \tag{3.26}$$

$$\begin{aligned}\|J_1F_3(u)\|_{L^1_xL^2_t} &\leq a\int_{\mathbf{R}}\|u\|^{p-1}_{L^\infty_t}(\|u\|_{L^2_t}+\|u_x\|_{L^2_t})dx\\ &\leq a\|u\|^{p-2}_{L^\infty_t L^\infty_x}\|u\|_{L^2_xL^\infty_t}\|J_1u\|_{L^2_tL^2_x}\\ &\leq CT^{\frac12}\|J_1u\|^{p-2}_{L^\infty_t L^2_x}\|u\|^2_{L^2_xL^\infty_t}\|J_1u\|_{L^\infty_t L^2_x}\end{aligned} \tag{3.27}$$

by Sobolev imbedding theorem $H^1 \hookrightarrow L^\infty$. Collecting (3.20), (3.21), (3.24)′, (3.25), (3.26) and (3.27) yields (3.15) and (3.16) by the definition of X_1 in (1.12). In the exactly same way as leading to (3.15) (3.16) we also have (3.17) and (3.18). Thus this completes the proof of Lemma 3.2.

As much similar as the proof of (3.24) (3.25) in Lemma 3.2, we have

Lemma 3.3 *Let $p = 2l+1$ ($l \in Z^+$), $k \geq 2$, $v_0(x) \in H^k(\mathbf{R})$, $u, \tilde{u} \in X_k(I)$. Then*

$$\|J_kF(u)\|_{L^1_tL^2_x} \leq CT^{\frac34}\|v_0(x)\|_{H^k_x}\|u\|_{X_k} + CT^{\frac54}\|u\|^3_{X_k} + CT\|u\|^p_{X_k} \tag{3.28}$$

$$\|J_kF(u)\|_{L^1_xL^2_t} \leq CT^{\frac12}\|v_0(x)\|_{H^k_x}\|u\|_{X_k} + CT\|u\|^3_{X_k} + CT^{\frac12}\|u\|^p_{X_k} \tag{3.29}$$

$$\begin{aligned}\|J_k(F(u)-F(\tilde{u}))\|_{L^1_tL^2_x} \leq\ & CT^{\frac34}\|v_0(x)\|_{H^k_x}\|u-\tilde{u}\|_{X_k}\\ &+ CT^{\frac54}(\|u\|^2_{X_k}+\|\tilde{u}\|^2_{X_k})\|u-\tilde{u}\|_{X_k}\\ &+ CT(\|u\|^{p-1}_{X_k}+\|\tilde{u}\|^{p-1}_{X_k})\|u-\tilde{u}\|_{X_k}\end{aligned} \tag{3.30}$$

$$\begin{aligned}\|J_k(F(u)-F(\tilde{u}))\|_{L^1_xL^2_t} \leq\ & CT^{\frac12}\|v_0(x)\|_{H^k_x}\|u-\tilde{u}\|_{X_k}\\ &+ CT(\|u\|^2_{X_k}+\|\tilde{u}\|^2_{X_k})\|u-\tilde{u}\|_{X_k}\\ &+ CT^{\frac12}(\|u\|^{p-1}_{X_k}+\|\tilde{u}\|^{p-1}_{X_k})\|u-\tilde{u}\|_{X_k}\end{aligned} \tag{3.31}$$

We now in position to prove the local version of Theorem 1 and Theorem 2. For $k \in Z^+ \cup \{0\}$ and $Q_k > 0$, let

$$B^k_{Q_k} = \{w \in X_k(I) \mid \|w\|_{X_k} \leq Q_k\} \tag{3.32}$$

For any $v_0(x) \in H^k$, $u_0(x) \in H^{\frac12+k}$, we define

$$\Psi(w) = S(t)u_0(x) + \int_0^t S(t-\tau)F(w)d\tau \tag{3.33}$$

where $F(w)$ is defined as (3.1). We shall prove that for appropriate Q_k and T, the operator Ψ is a contraction on $B^k_{Q_k}$. In view of (3.2), Lemma 1, Lemma 2 and Lemma 3, we always have

$$\|\Psi(w)\|_{X_k} \leq C\|u_0(x)\|_{H^{\frac12+k}} + C_k(T)(\|w\|^3_{X_k} + \|w\|_{X_k} + \|w\|^p_{X_k}) \tag{3.34}$$

$$\|\Psi(w)-\Psi(\widetilde{w})\|_{X_k} \le \widetilde{C}_k(T)(\|w\|^2_{X_k}+\|\widetilde{w}\|^2_{X_k}+\|v_0\|_{H^k}+\|w\|^{p-1}_{X_k} \\ +\|\widetilde{w}\|^{p-1}_{X_k})\|w-\widetilde{w}\|_{X_k} \tag{3.35}$$

where $C_k(T)$ and $\widetilde{C}_k(T)$ are positive constants only dependent on k and T such that

$$C_k(T)\to 0 \quad \text{as } T\to 0$$

and

$$\widetilde{C}_k(T)\to 0 \quad \text{as } T\to 0$$

respectively. We now choose $Q_k>0$ as that

$$C\|u_0(x)\|_{H^{\frac12+k}} \le \frac{Q_k}{2} \tag{3.36}$$

For fixed $Q_k>0$, we take $T>0$ so small that

$$\widetilde{C}_k(T)(Q_k^3+Q_k+Q_k^p) < \frac{Q_k}{2} \tag{3.37}$$

Then we easily see that $\Psi: B^k_{Q_k}\to B^k_{Q_k}$ by (3.34), moreover we take $T>0$ so small such that

$$\widetilde{C}_k(T)(2Q_k^2+2Q_k^{p-1}+\|v_0\|_{H^k})<1 \tag{3.38}$$

by (3.35), we conclude that Ψ is a contraction on $B^k_{Q_k}$. Consequently, there exists a unique function $u\in X_k(I)$ with $\Psi(u)=u$ and we define v by (1.6). We prove that $v\in C(I;H^k_x)$. Note that for $s,t\in I$,

$$\Big\|\int_s^t J_{-k}\partial_x u^2(\tau)d\tau\Big\|_{L^2_x} \le C\Big\|\Big(\sum_{|\alpha|\le k}\int_s^t |\partial_x^\alpha u|^2 d\tau\Big)^{\frac12}\sum_{|\alpha|\le k}\|\partial_x^\alpha u_x\|_{L^2_t}\Big\|_{L^2_x} \\ \le C|t-s|^{\frac12}\|J_{-k}u\|_{L^\infty_t L^2_x}\|J_{-k}\partial_x u\|_{L^\infty_x L^2_t} \tag{3.39}$$

by Hölder inequality. So (3.39) implies $v\in C(I;H^k)$. Thus we complete the proof of (i) in Theorem 1 and the local version of Theorem 2.

4. The Global Posedness

In this section we come to prove the global posedness of (1.5) (1.6) in space X_k. We begin with the conservation laws. By direct calculation and making use of the equation (1.1) (1.2) we get

Lemma 4.1 *Let* (u, v) *be a smooth solution of* (1.5), (1.6). *Then*

$$\int_{\mathbf{R}} |u(t,x)|^2 dx = \int_{\mathbf{R}} |u_0(x)|^2 dx \tag{4.1}$$

$$E(t) = \int_{\mathbf{R}} \Big(v(t,x)|u(t,x)|^2 + |u_x(t,x)|^2 + \frac{2a}{p+1}|u(t,x)|^{p+1}\Big)dx + E(0) \tag{4.2}$$

$$\int_{\mathbf{R}} (v^2(t,x) + 2Im(u(t,x)\overline{u_x(t,x)}))dx = \int_{\mathbf{R}} (v_0^2(x) + 2Im(u_0(x)\overline{u_{0x}(x)}))dx \tag{4.3}$$

We now in position to prove Theorems. In essence, the proof of (ii) of Theorem 1 is exactly the same as [5], so we only prove Theorem 2. We first consider the case $k = 1$. From Lemma 4.1 the local solution $u(t,x)$ obtained in Section 3 satisfies

$$\|u(t,x)\|_{L_t^\infty L_x^2} \le \|u_0(x)\|_{L^2}, \quad t \in [0,T] \tag{4.4}$$

$$\|J_1 u(t,x)\|_{L_t^\infty L_x^2} \le M_1, \quad t \in [0,T] \tag{4.5}$$

In view of (3.19)–(3.21), (3.24)′ and (3.25)–(3.27) we have by (4.4) and (4.5)

$$\begin{aligned}\|u\|_{X_1} \le & C\|u_0(x)\|_{H^{\frac{3}{2}}} + C(T^{\frac{1}{2}} + T^{\frac{3}{4}})\|v_0(x)\|_{H_x^{\frac{1}{2}}}\|u\|_{X_1} \\ & + C(T + T^{\frac{3}{2}})\|u\|_{H_x^{\frac{1}{2}}}^2\|u\|_{X_1} + C(T + T^{\frac{1}{2}} + T^{\frac{1}{2}+\rho})\|u\|_{H_x^{\frac{1}{2}}}^{p-1}\|u\|_{X_1} \\ \le & C\|u_0(x)\|_{H^{\frac{3}{2}}} + C(T^{\frac{1}{2}} + T^{\frac{3}{2}} + T^{\frac{1}{2}+\rho})\|u\|_{X_1}\end{aligned} \tag{4.6}$$

which implies the (i) of Theorem 2 by local posedness results in Section 3.

Now we come to prove (ii) of Theorem 2. It is easy to show for $k \ge 1$

$$\|J_{k+1}F_1(u)\|_{L_t^1 L_x^2} \le CT^{\frac{3}{4}}\|v_0(x)\|_{H_x^{k+1}}\|u\|_{X_{k+1}} \tag{4.7}$$

$$\|J_{k+1}F_1(u)\|_{L_x^1 L_t^2} \le CT^{\frac{1}{2}}\|v_0(x)\|_{H_x^{k+1}}\|u\|_{X_{k+1}} \tag{4.8}$$

$$\begin{aligned}\|J_{k+1}F_2(u)\|_{L_t^1 L_x^2} \le & CT^{\frac{5}{4}}\|J_{k+1}u_x\|_{L_x^\infty L_t^2}\|J_k u\|_{L_t^\infty L_x^2}\|J_k u\|_{L_t^4 L_x^\infty} \\ & + CT^{\frac{5}{4}}\|J_k u_x\|_{L_x^\infty L_t^2}\|J_{k+1}u\|_{L_t^\infty L_x^2}\|J_k u\|_{L_t^4 L_x^\infty} \\ & + CT^{\frac{5}{4}}\|J_k u_x\|_{L_x^\infty L_t^2}\|J_k u\|_{L_t^\infty L_x^2}\|J_{k+1}u\|_{L_t^4 L_x^\infty}\end{aligned} \tag{4.9}$$

$$\begin{aligned}\|J_{k+1}F_2(u)\|_{L_x^1 L_t^2} \le & CT\|J_{k+1}u_x\|_{L_x^\infty L_t^2}\|J_k u\|_{L_t^\infty L_x^2}^2 \\ & + CT\|J_k u_x\|_{L_x^\infty L_t^2}\|J_k u\|_{L_t^\infty L_x^2}\|J_{k+1}u\|_{L_t^\infty L_x^2}\end{aligned} \tag{4.10}$$

and

$$\|J_{k+1}F_3(u)\|_{L_t^1 L_x^2} \le CT\|J_k u\|_{L_t^\infty L_x^2}^{2l}\|J_{k+1}u\|_{L_t^\infty L_x^2} \tag{4.11}$$

At last, we consider the estimate of $\|J_{k+1}F_3(u)\|_{L^1_xL^2_t}$. In view of (3.27) we only consider

$$\|\partial_x^{k+1}F_3(u)\|_{L^1_xL^2_t} \le C \sum_{\alpha_1+\cdots+\alpha_{2j+1}=k+1} \Big\| \prod_{i=1}^{2j+1} \partial_x^{\alpha_i} u \Big\|_{L^1_xL^2_t} \tag{4.12}$$

When $k \ge 2$, there is at almost one $|\alpha_i| \ge k$ $(1 \le i \le 2j+1)$. No loss of generality we assume $|\alpha_{2j+1}| \ge k$, therefore

$$\begin{aligned}\|\partial_x^{k+1}F_3(u)\|_{L^1_xL^2_t} &\le C \sum_{\alpha_1+\cdots+\alpha_{2j+1}=k+1} \prod_{i=1}^{2j-2} \|J_1\partial_x^{\alpha_i}u\|_{L^\infty_t L^2_x} \|\partial_x^{\alpha_{2j-1}}u\|_{L^4_xL^\infty_t} \\ &\quad \times \|\partial_x^{\alpha_{2j}}u\|_{L^4_xL^\infty_t} \|\partial_x^{\alpha_{2j+1}}u\|_{L^2_xL^2_t} \\ &\le CT^{\frac12}\|J_ku\|^{2l-2}_{L^\infty_tL^2_x}\|J_ku\|^2_{L^4_xL^\infty_t}\|J_{(-k+1)}u\|_{L^\infty_tL^2_x}\end{aligned} \tag{4.13}$$

and when $k=1$

$$\|\partial_x^2F_3(u)\|_{L^1_xL^2_t} \le C\|\partial_x^2u\cdot|u|^{2l}\|_{L^1_xL^2_t} + \|(|\partial_xu|^2\cdot|u|^{2l-1})\|_{L^1_xL^2_t} = I_1+I_2 \tag{4.14}$$

Note that

$$I_1 \le CT^{\frac12}\|J_1u\|^{2l-2}_{L^\infty_tL^2_x}\|u\|^2_{L^4_xL^\infty_t}\|J_2u\|_{L^\infty_tL^2_x} \tag{4.15}$$

$$\begin{aligned} I_2 &\le C\int_{\mathbf{R}} \|u\|^{2l-1}_{L^\infty_t}\|\partial_xu\|_{L^\infty_t}\|\partial_xu\|_{L^2_t}dx \\ &\le CT^{\frac12}\|J_1u\|^{2l-2}_{L^\infty_tL^2_x}\|u\|_{L^4_xL^\infty_t}\|\partial_xu\|_{L^4_xL^\infty_t}\|\partial_xu\|_{L^\infty_tL^2_x}\end{aligned} \tag{4.16}$$

Collecting (4.7)–(4.16) with (3.27) we have

$$\begin{aligned}\|u\|_{X_{k+1}} &\le \|u_0\|_{H^{\frac12+k+1}} + C(T^{\frac12}+T^{\frac34})\|v_0\|_{H^{k+1}}\|u\|_{X_{k+1}} \\ &\quad + C(T^{\frac54}+T)\|u\|^2_{X_k}\|u\|_{X_{k+1}} \\ &\quad + C(T^{\frac12}+T)\|u\|^{p-1}_{X_k}\|u\|_{X_{k+1}}, \quad p=2l+1\end{aligned} \tag{4.17}$$

which together with (ii) of Theorem 2 implies (ii) of Theorem 2.

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