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GLOBAL EXISTENCE OF SUPERSONIC FLOW PAST A CURVED CONVEX WEDGE *

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Dedicated to Professor Ding Xiaxi on the occasion of his 70th birthday

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Abstract In this paper we discuss the supersonic flow past a curved convex wedge. Our conclusion is that if the vertex angle of the wedge is less than a critical angle, the shock attached the head of the wedge is weak, and if the wedge is formed by a smooth convex curve, monotonically increasing, then the global solution of such a boundary value problem exists.

Key Words Global existence; supersonic flow; quasilinear hyperbolic system.
Classification 35L60, 35L67, 35L50, 76N15.

1. Introduction

It is well known that when a supersonic flow hits a wedge with small vertex angle, there will appear an oblique shock attached on the edge of the wedge. If the surface of the wedge is a smooth curved surface, then by its influence the shock front will also be a curved surface. It is shown in [1] that if the wedge has constant section and the vertex angle is less than a critical value, then the shock front and the flow behind the shock can be determined locally. An interesting and important question is then whether we can determine the flow with the shock front globally? When the surface of the wedge is composed of two straight lines combined by a smooth curve, the global solution can be obtained by constructing infinite reflection of rarefaction waves (See [2]). In this paper we will use a different way to discuss such a problem. Our conclusion is that if the shock is weak, the vertex angle of the wedge is less than the critical angle, and if the surface of the wedge is formed by a smooth convex curve, monotonically increasing, then the global solution does exist.

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2. Description of the Problem and Its Reduction

Let us first give a precise description of the problem. Assume that the wedge is symmetric with respect to a centre plane, then we always consider the upper half of the wedge. Assume also that the surface of the wedge has constant section on any plane perpendicular to its edge, and the equation of the surface is $y = f(x)$ satisfying $f'(x) > 0$, $f''(x) \leq 0$. Besides, the flow is assumed to be isentropic and irrotational, this assumption is acceptable if the possible shock is weak. In this case the system to describe the flow can be written as

$$\begin{cases} (u^2 - a^2)u_x + uv(u_y + v_x) + (v^2 - a^2)v_y = 0 \\ u_y = v_x \end{cases} \tag{1}$$

where a represents the sonic speed. These unknown functions also satisfy Bernoulli relation

$$\frac{1}{2}(u^2 + v^2) + \frac{a^2}{\gamma - 1} = const \tag{2}$$

Ahead of the shock the flow is constant with its parameters $u = u_0$, $v = v_0$, $\rho = \rho_0$, satisfying $u_0 > a_0$, $v_0 = 0$. So the constant in (2) equals $\frac{1}{2}u_0^2 + \frac{1}{\gamma - 1}a_0^2$.

Denote the location of the unknown shock by $y = s(x)$, we consider a boundary value problem of (1) in the domain

$$\Omega : x > 0, \quad f(x) \leq y \leq s(x) \tag{3}$$

while its boundary is denoted by $B : y = f(x)$ and $S : y = g(x)$. And the boundary conditions are

$$v = uf'(x) \quad \text{on } B \tag{4}$$

$$u + vs'(x) = u_0, \quad \rho(us'(x) - v) = \rho_0 u_0 s'(x) \quad \text{on } S \tag{5}$$

where the condition (5) is called Rankine-Hugoniot condition.

The system can be diagonalized by introducing Riemann invariants. Denote

$$\lambda_{\pm} = \frac{uv \pm a\sqrt{u^2 + v^2 - a^2}}{u^2 - a^2} \tag{6}$$

which represents the characteristic directions of the system (1). Then by introducing suitable integral factor $k_{\pm}$ we can define functions r and s by

$$\begin{cases} ds = k_-(du + \lambda_- dv) \\ dr = k_+(du + \lambda_+ dv) \end{cases} \tag{7}$$

That is

$$\begin{cases} \dfrac{\partial s}{\partial x} + \lambda_+ \dfrac{\partial s}{\partial y} = 0 \\ \dfrac{\partial r}{\partial x} + \lambda_- \dfrac{\partial r}{\partial y} = 0 \end{cases} \tag{8}$$

Here s and r are called Riemann invariants. Since any smooth function of s or smooth function of r also satisfies the equation in (8) respectively, these functions are also called Riemann invariants.

Regard u, v as independent variables, and x, y as unknown functions, then the system is equivalent to

$$\begin{cases} (a^2 - u^2)y_v + uv(y_u + x_v) + (a^2 - v^2)x_u = 0 \\ x_v = y_u \end{cases} \tag{9}$$

Its characteristic directions are $\mu_\pm = -(\lambda_\mp)^{-1}$. Let

$$q = (u^2 + v^2)^{1/2}, \quad \theta = \arctan \frac{v}{u},$$

then

$$\frac{dv}{du} = -(\lambda_\pm)^{-1} \Longleftrightarrow \frac{d\theta}{dq} = \frac{\mp\sqrt{q^2 - a^2}}{aq} \tag{10}$$

Because

$$ds = du + \lambda_- dv, \quad ds = 0 \Longleftrightarrow \frac{dv}{du} = -(\lambda_-)^{-1}.$$

Hence by letting

$$F(q) = \int \frac{\sqrt{q^2 - a^2}}{aq} dq,$$

we have $d(\theta - F(q)) = 0$. Correspondingly, the equation of r in (8) is equivalent to $d(\theta + F(q)) = 0$. Therefore, we can simply take

$$s = \theta - F(q), \quad r = \theta + F(q) \tag{11}$$

as Riemann invariants, where $F(q)$ is a given monotonne increasing function of q.

The system (1) or (8) is strictly hyperbolic system with respect to x direction. Our aim in this paper is to prove the existence of global solution of the boundary value problem (1), (4), (5). The result is

Theorem 1 *For the boundary value problem described as above, the C^1 smooth solution of the problem (1), (4), (5) globally exists, provided $f'(x_0)$ is small, and $f'(x) \geq 0$, $f''(x) \leq 0$ is satisfied for any $x > 0$.*

According to the local existence and uniqueness of C^1 solution to the typical boundary value problems (See [1]), the problem is solvable locally in $x < x_0$ with small x_0, and then we may consider an initial boundary value problem in

$$\Omega_1 : x > x_0, \quad f(x) \le y \le s(x) \tag{12}$$

with boundary conditions (4), (5) and initial data given on $\Gamma_0 : x = x_0,\ f(x_0) < y < s(x_0)$ instead. Since the data on $x = x_0$ are taken from the local solution to the problem (1), (4), (5), the consistency conditions at the points $(x_0, f(x_0))$ and $(x_0, s(x_0))$ are automatically satisfied. Still by using the local theory of boundary value problems for quasilinear hyperbolic system we can extend the domain of existence of C^1 solution, and the crucial point to obtain a global solution is to establish a uniform estimates of the solution and their derivatives of first order. This will be done in the following two sections.

3. Boundedness of Solution

Consider the boundary value problem (1), (4), (5) with initial data given on $x = x_0$ in Ω_1. If we prove that the solution can be extend to $x_0 \le x \le x_0 + h$ with uniform h independent of x_0, then the theorem stated in previous section is obtained. To simplify our discussion we assume that h is small such that through each point (x, y) in the domain $x_0 < x < x_0 + h$, $f(x) < y < s(x)$, at least one of the two leftward characteristics intersects with the initial line $x = x_0$ in the interval $f(x_0) \le y \le s(x_0)$.

Lemma 1 *If the problem* (1), (4), (5) *admits a C^1 solution (r, s) in the domain $x_0 \le x \le x_0 + h$, $f(x) \le y \le s(x)$, then the C^0 norm of r, s has an upper bound independent of h.*

Proof Assume that $P(x, y)$ is an arbitrary point in the domain. Let us consider three cases. First, assume that both λ_+ and λ_- backward characteristic lines intersect the initial line $\Gamma_0 : x = x_0$. Denote the intersections by Q_+ and Q_-, then by the system (8) we have

$$r(P) = r(Q_-), \quad s(P) = s(Q_+),$$

which implies

$$|r(P), s(P)| \le \sup_y \max(|r(x_0, y)|, |s(x_0, y)|) \tag{13}$$

In the second case, we assume that the λ_+ characteristics intersects the surface of the wedge at Q_+, and λ_- characteristics intersects Γ_0 at Q_-. By using the system (8) we have $s(P) = s(Q_+)$. We know $\theta = \arctan f'(x)$, because on the surface of the wedge, the direction of the velocity is tangential to the surface. Then (u, v) is located in the domain $0 \le \theta \le \arctan f'(x_0)$, $a_* < q < q_0$, where a_* is the critical sonic speed (See [3]). Correspondingly, $s(Q_+)$ is bounded uniformly. Combining it with the fact $r(P) = r(Q_-)$, we have the uniform boundedness of $r(P)$ and $s(P)$.

In the last case, we assume that the λ_- characteristics intersects the shock front at Q_-, and λ_+ characteristics intersects Γ_0 at Q_+. By the system (8) we have $r(P) = r(Q_-)$ and $s(P) = s(Q_+)$. Since $(u(Q_-), v(Q_-))$ must located on the shock polar, it is located in a fixed bounded domain on u, v plane. Hence $r(Q_-)$ has its bound independent of h. Combining the fact $s(P) = s(Q_+)$ we again obtain the uniform boundedness of $r(P)$ and $s(P)$.

Next let us indicate the data on $x = x_0$ are monotonne with respect to y.

Lemma 2 *Assume that (r, s) is the local smooth solution of the problem* (1), (4), (5), *x_0 is small, then $r'_y(x_0, y) \ge 0$, $s'_y(x_0, y) \ge 0$.*

Proof The equation of shock polar on u, v plane is (See [4])

$$\rho u^2 + \rho_0 u_0^2 - (\rho - \rho_0) u u_0 + \rho v^2 = 0 \tag{14}$$

which can be written as $G(u, v) = 0$. Therefore, along the tangential direction ℓ of the shock $y = s(x)$ we have

$$G_u u_\ell + G_v v_\ell = 0$$

$$G_u(u_r r_\ell + u_s s_\ell) + G_v(v_r r_\ell + v_s s_\ell) = 0$$

Multiplying it by $\dfrac{\partial(r, s)}{\partial(u, v)}$, we obtain

$$(G_u s_v - G_v s_u) r_\ell - (G_u r_v - G_v r_u) s_\ell = 0 \tag{15}$$

Notice that two families of characteristics of the system (9) on u, v plane are epicycloid defined outside the circle $q = a_*$. The shock polar intersects epicycloid transversally except at $(u_0, 0)$. Moreover, if we take the direction of q-increasing as the direction for each curves, then at any point satisfying $a_* < q < q_0$ the shock polar is in between two epicycloids passing through this point (See [4]). Therefore, considering the coefficients in (15) are proportional to sine of the angle between corresponding curves: $G = const.$, $s = const.$ and $r = const.$, we know that $(G_u s_v - G_v s_u)$ and $(G_u r_v - G_v r_u)$ always take different sign. It certainly implies that $k = r_\ell / s_\ell < 0$.

Denote $a_1 = f'(0)$, $a_2 = s'(0)$, we have

$$a_2 > \lambda_+ > a_1 \ge 0 > \lambda_- \tag{16}$$

Notice that $(1, a_1)$ is the tangential direction ℓ_1 of the surface of the wedge at the origin, and $(1, a_2)$ is the tangential direction of the shock front at the origin, we can write four equalities for the derivatives of r and s at the origin. That is

$$\begin{cases} r_x + a_1 r_y + s_x + a_1 s_y = d \\ r_x + a_2 r_y - k s_x - k a_2 s_y = 0 \\ r_x + \lambda_- r_y = 0 \\ s_x + \lambda_+ s_y = 0 \end{cases} \tag{17}$$

where $d = \dfrac{\partial \theta}{\partial \ell_1} = f''(x)(1 + f'(x))^{-3/2} \le 0$.

Direct calculation gives us

$$(-k(a_2 - \lambda_+)(a_1 - \lambda_-) + (\lambda_+ - a_1)(a_2 - \lambda_-))r_y = k(\lambda_+ - a_2)d \tag{18}$$

in view of $\lambda_- < a_1 < a_2 < \lambda_+$, $|k| < 1$, we have

$$(\lambda_+ - a_1)(a_2 - \lambda_-) > |k(a_2 - \lambda_+)(a_1 - \lambda_-)|$$

then $\dfrac{\partial r}{\partial y} \ge 0$.

Furthermore, by using the first, second and fourth equalities in (17) we have

$$s_y = \frac{d - (a_1 - \lambda_-) r_y}{a_1 - \lambda_+} \ge 0$$

Then by using the continuity we obtain the conclusion of the lemma.

Lemma 3 *For the functions* $\lambda_\pm(r, s)$ *we have* $\dfrac{\partial \lambda_-}{\partial r} > 0$ *and* $\dfrac{\partial \lambda_+}{\partial s} > 0$.

Proof By using (11) we have

$$\frac{\partial(r, s)}{\partial(\theta, q)} = \begin{pmatrix} 1 & F' \\ 1 & -F' \end{pmatrix}$$

Then

$$\begin{aligned} \frac{\partial(\lambda_-, \lambda_+)}{\partial(r, s)} &= \frac{\partial(\lambda_-, \lambda_+)}{\partial(\theta, q)} \cdot \frac{\partial(\theta, q)}{\partial(r, s)} \\ &= \frac{\partial(\lambda_-, \lambda_+)}{\partial(\theta, q)} \cdot \left(\frac{\partial(r, s)}{\partial(\theta, q)} \right)^{-1} \\ &= \frac{\partial(\lambda_-, \lambda_+)}{\partial(\theta, q)} \cdot \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2F'} & -\frac{1}{2F'} \end{pmatrix} \end{aligned}$$

Namely

$$\frac{\partial\lambda_-}{\partial r}=\frac{1}{2}\frac{\partial\lambda_-}{\partial\theta}+\frac{1}{2F'}\frac{\partial\lambda_-}{\partial q},\quad \frac{\partial\lambda_+}{\partial s}=\frac{1}{2}\frac{\partial\lambda_+}{\partial\theta}-\frac{1}{2F'}\frac{\partial\lambda_+}{\partial q} \tag{19}$$

Notice $\lambda_\pm=-(\mu_\mp)^{-1}$, where μ is the slope $\frac{dv}{du}$ of characteristics of (9) on u,v plane, then

$$\frac{\partial\lambda_-}{\partial\theta}=\frac{1}{\mu_+^2}\frac{\partial\mu_+}{\partial\theta}=\frac{1}{\mu_+^2}\frac{\partial}{\partial\theta}\Big(\frac{dv}{du}\Big)_+ \tag{20}$$

Denote the angle in between the radius and the μ_+ characteristics by α, then $\tan\alpha=q\frac{d\theta}{dq}=qF'(q)$. On the other hand, $\tan(\alpha+\theta)=\frac{dv}{du}$. Hence

$$\frac{\partial\lambda_-}{\partial\theta}=\frac{1}{\mu_+^2}\sec^2(\alpha+\theta)\Big(\frac{d\alpha}{d\theta}+1\Big)>0$$

by virtue of $\frac{d\alpha}{d\theta}>0$.

Similarly, we have $\frac{\partial\lambda_-}{\partial q}=\frac{1}{\mu_+^2}\frac{\partial}{\partial q}\Big(\frac{dv}{du}\Big)_+$. Direct calculation implies

$$\begin{aligned}\frac{d}{dq}(qF'(q))&=\frac{d}{dq}\Big(\frac{(q^2-a^2)^{1/2}}{a}\Big)\\&=\frac{1}{a^2}(a(q^2-a^2)^{-1/2}(q-aa_q)-(q^2-a^2)^{1/2}a_q)\\&=\frac{1}{a^2}\Big(\frac{\gamma+1}{2}aq(q^2-a^2)^{-1/2}+\frac{(\gamma-1)q}{2a}(q^2-a^2)^{1/2}\Big)>0\end{aligned}$$

hence $\frac{\partial\lambda_-}{\partial q}>0$. And then $\frac{\partial\lambda_-}{\partial r}>0$ is obtained from (19).

On the other hand, notice that $\mu_-=\Big(\frac{dv}{du}\Big)_-$ is the slope of the characteristics $\theta=-F(q)$. Then by the same method we have $\frac{\partial\lambda_+}{\partial\theta}>0$, $\frac{\partial\lambda_+}{\partial q}<0$. Substituting it into (19) we obtain $\frac{\partial\lambda_+}{\partial s}>0$.

Lemma 4 *If the problem (1), (4), (5) admits a C^1 solution (r,s) in the domain $x_0\le x\le x_0+h$, $f(x)\le y\le s(x)$, and $r(x_0,y),s(x_0,y)$ are nondecreasing functions of y, then the C^1 norm of the solution is independent of h. Moreover, $r_y'(x,y)\ge 0$, $s_y'(x,y)\ge 0$ holds for any $x>x_0$.*

Proof We still consider three cases as we did in the proof of Lemma 1. First, if two characteristics of (1) through $P(x,y)$ intersect Γ_0, then the proof is the same as that in corresponding theorem of Cauchy problem for quasilinear hyperbolic system with one space variables in [5, 6].

In the second case, the backward λ_+ characteristics intersects the surface of the wedge. By the method in [6], we know $\frac{\partial r}{\partial y}$ and then $\frac{\partial r}{\partial x}$ can be controlled by $r'_y(x_0, y)$, $s'_y(x_0, y)$. Besides, $\frac{\partial r}{\partial y} \geq 0$, $\frac{\partial r}{\partial x} = -\lambda_- \frac{\partial r}{\partial y} \geq 0$.

From the above fact and $f'(x) > 0$ we have

$$\frac{\partial r}{\partial \ell_1} = (1 + f'^2)^{-1/2} \frac{\partial r}{\partial x} + f'(1 + f'^2)^{-1/2} \frac{\partial r}{\partial y} \geq 0$$

In view of $f''(x) \leq 0$, we have

$$\frac{\partial r}{\partial \ell_1} + \frac{\partial s}{\partial \ell_1} \leq 0$$

which implies $\frac{\partial s}{\partial \ell_1} \leq 0$. Then notice that

$$\frac{\partial s}{\partial \ell_1} = \frac{\partial s}{\partial x} + a_1 \frac{\partial s}{\partial y} = (a_1 - \lambda_+) \frac{\partial s}{\partial y}$$

we have $\frac{\partial s}{\partial y} \geq 0$ because $a_1 < \lambda_+$.

In order to estimate $\frac{\partial s}{\partial y}$ inside Ω_1, we introduce a function $w = e^{h(r,s)} \frac{\partial s}{\partial y}$, where $h(r, s)$ satisfies

$$\frac{\partial h}{\partial r} = \frac{1}{\lambda_+ - \lambda_-} \frac{\partial \lambda_+}{\partial r}$$

By differentiating the first equality of (8) with respect to y, we obtain

$$\frac{\partial^2 s}{\partial x \partial y} + \lambda_+ \frac{\partial^2 s}{\partial y^2} + \frac{\partial \lambda_+}{\partial r} \frac{\partial r}{\partial y} \frac{\partial s}{\partial y} + \frac{\partial \lambda_+}{\partial s} \left(\frac{\partial s}{\partial y} \right)^2 = 0 \tag{21}$$

From the definition of w we have

$$\frac{\partial w}{\partial x} = e^{h(r,s)} \left(\frac{\partial^2 s}{\partial x \partial y} + \frac{\partial h}{\partial r} \frac{\partial r}{\partial x} \frac{\partial s}{\partial y} + \frac{\partial h}{\partial s} \frac{\partial s}{\partial x} \frac{\partial s}{\partial y} \right)$$

$$\frac{\partial w}{\partial y} = e^{h(r,s)} \left(\frac{\partial^2 s}{\partial y^2} + \frac{\partial h}{\partial r} \frac{\partial r}{\partial y} \frac{\partial s}{\partial y} + \frac{\partial h}{\partial s} \left(\frac{\partial s}{\partial y} \right)^2 \right)$$

by using (21) and the definition of h we can obtain

$$\frac{\partial w}{\partial x} + \lambda_+ \frac{\partial w}{\partial y} = -\frac{\partial \lambda_+}{\partial s} e^{-h(r,s)} w^2 \tag{22}$$

Denote the λ_+-characteristics through $(\beta, f(\beta))$ by $y(x, \beta)$, then it satisfies

$$\begin{cases} \frac{d}{dx} y(x, \beta) = \lambda_+(r(x, y(x, \beta)), s_1(\beta)) \\ y(\beta, \beta) = f(\beta) \end{cases} \tag{23}$$

where $s_1(\beta)$ is the value of s on the characteristic line. Correspondingly, the value of w on the line is $w(x, y(x, \beta))$, which satisfies

$$\begin{cases} \dfrac{dw}{dx} = -\dfrac{\partial\lambda_+}{\partial s} e^{-h(r,s)} w^2 \\ w(\beta, f(\beta)) = w_0(\beta) \geq 0 \end{cases} \tag{24}$$

The solution of (24) is

$$w = \frac{w_0}{1 + w_0 \displaystyle\int_\beta^x \frac{\partial\lambda_+}{\partial s} e^{-h(r,s)} dx} \tag{25}$$

Since $\dfrac{\partial s}{\partial y} \geq 0$ on B implies $w_0 \geq 0$, we have $0 \leq w \leq w_0$ from the expression (25), and then $\dfrac{\partial s}{\partial y} \geq 0$ on the characteristics starting from $(\beta, f(\beta))$ is its direct conclusion. Moreover, the C^0 norm of $\dfrac{\partial s}{\partial y}$ (and then $\dfrac{\partial s}{\partial x}$) can be controlled by $r'_y(x_0, y)$, $s'_y(x_0, y)$.

The last case is that the leftward λ_--characteristics through P intersects the shock front $y = s(x)$. Obviously, through each point on the shock the λ_+-characteristics intersects Γ_0, then the boundedness of $\dfrac{\partial s}{\partial y}, \dfrac{\partial s}{\partial x}$ can be obtained as in Case 1. Moreover, we have $\dfrac{\partial s}{\partial y} \geq 0$ on S.

Now the point is to prove that $\dfrac{\partial r}{\partial y} \geq 0$ also holds on S. Since

$$\frac{\partial s}{\partial \ell} = \frac{\partial s}{\partial x} + a_2 \frac{\partial s}{\partial y} = (a_2 - \lambda_+) \frac{\partial s}{\partial y} \leq 0$$

by the equality (15) we have $\dfrac{\partial r}{\partial \ell} \geq 0$. Therefore, $\dfrac{\partial r}{\partial y} = (a_2 - \lambda_-)^{-1} \dfrac{\partial r}{\partial \ell} \geq 0$. The next step is to prove $\dfrac{\partial r}{\partial y} \geq 0$ inside Ω_1, since the method is quite similar as we did in the second case to discuss $\dfrac{\partial s}{\partial y}$, we omit it here.

Proof of Theorem 1 Summarizing the above four lammas we can readily prove Theorem 1 now. By using the theorem on existence of local solution to boundary value problems of quasilinear hyperbolic system, we can find $h > 0$, such that C^1 solution exists in $x_0 \leq x \leq x_0 + h$. Since the C^1 norm of r, s is uniformlly bounded, and their monotomicity on $x_0 = x_0 + h$ still holds according to Lemma 1, 4, the same method can be applied when x_0 is replaced by $x_0 + h$. Then the solution r, s can be extended to whole Ω_1, and the theorem on global existence in Section 2 is proved.

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