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APPROXIMATION OF A TWO-PHASE CONTINUOUS CASTING STEFAN PROBLEM

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Dedicated to Professor Ding Xiaxi on the occasion of his 70th birthday

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Abstract The continuous casting Stefan problem is a mathematical model describing the solidification with convection of a material being cast continuously with a prescribed velocity. We propose a practical piecewise linear finite element scheme motivated by the characteristic finite element method and derive an error estimate for the scheme which is of the same convergence order as that proved for Stefan problem without convection.

Key Words Piecewise linear finite elements, numerical quadrature, error estimates, Stefan problem with convection.

Classification 65N15, 65N30.

1. Introduction

Let Ω be a cylindrical domain $\Omega = \Gamma \times (0, L) \subset \mathbf{R}^d$, $d = 2$ or 3, where $0 < L < +\infty$ and $\Gamma = (0, L_1)$ if $d = 2$ or $\Gamma \subset \mathbf{R}^2$ is a bounded polygonal domain. We write $x = (x', z) \in \Omega$ with $x' \in \Gamma$ and $x_d = z$. Denote by $\Gamma_0 = \Gamma \times \{0\}$, $\Gamma_L = \Gamma \times \{L\}$, $\Gamma_D = \Gamma_0 \cup \Gamma_L$ and $\Gamma_N = \partial\Gamma \times (0, L)$ (cf. Fig.1). For $0 < T < +\infty$, we set $Q_T = \Omega \times (0, T)$.

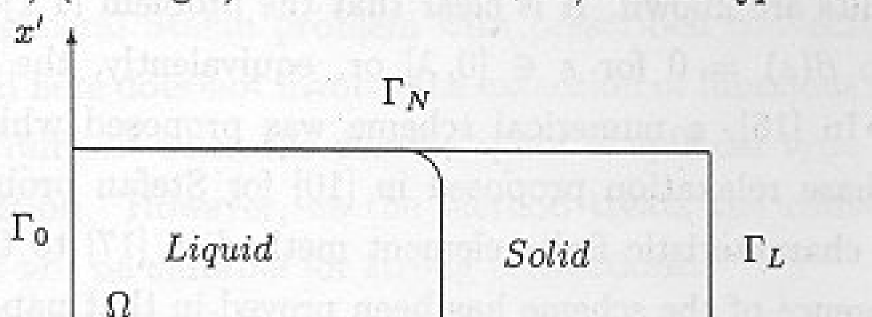

Fig.1 The domain Ω

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We consider the following degenerate nonlinear parabolic problem

$$\frac{\partial u}{\partial t} + b(t)\frac{\partial u}{\partial z} - \Delta\theta = 0 \quad \text{in} \quad Q_T \tag{1.1}$$

$$\theta = \beta(u) \quad \text{in} \quad Q_T \tag{1.2}$$

$$\theta = g_D(x,t) \quad \text{on} \quad \Gamma_D \times (0,T) \tag{1.3}$$

$$-\frac{\partial \theta}{\partial \mathbf{n}} = p(x)\theta + g_N(x,t) \quad \text{on} \quad \Gamma_N \times (0,T) \tag{1.4}$$

$$u(x,0) = u_0(x) \quad \text{in} \quad \Omega \tag{1.5}$$

where θ stands for the temperature, u is the enthalpy, $b(t) \geq 0$ is the extraction velocity of the ingot, and $\mathbf{n}$ is the unit outer normal to $\partial\Omega$. The mapping $\beta : \mathbf{R} \to \mathbf{R}$ is Lipschitz continuous and monotone increasing. It is assumed that $\beta(s) = 0$ for any $s \in [0, \lambda]$ and $0 < \alpha_1 \leq \beta'(s) \leq \alpha_2$ for almost every $s \in \mathbf{R}\backslash[0, \lambda]$, where $\lambda > 0$ is the latent heat. It is clear that the inverse mapping $H = \beta^{-1}$ is a maximal monotone graph in $\mathbf{R} \times \mathbf{R}$ which is Lipschitz continuous in $\mathbf{R}\backslash 0$ and has a jump discontinuity at 0.

The multidimensional two-phase Stefan problem without convection (i.e. when $b = 0$) has been studied by many authors. For the existence and uniqueness of the weak solutions, we refer to[1] and [2]. The convergence of numerical methods for the enthalpy formulation of Stefan problem has been studied in [3], [4], [5] and [6]. The error analysis of the finite element schemes has also been considered in the literature (cf. e.g. [7], [8], [9], [10], [11] and the references therein). The problem (1.1)–(1.5) models a popular industrial solidification process with convection in which a material is cast continuously with prescribed velocity $\mathbf{v} = b(t)\mathbf{e}_d$, where $\mathbf{e}_d = (0, \cdots, 0, 1) \in \mathbf{R}^d$ (cf. e.g. [12] and [13] for the description of the industrial process and mathematical modelling). The existence and uniqueness of the problem (1.1)–(1.5) has been studied in [14] and [15]. Concerning the numerical solutions of the continuous casting problem, relatively few results are known. It is clear that the problem (1.1)–(1.2) is convection dominated due to $\beta(s) = 0$ for $s \in [0, \lambda]$ or, equivalently, the jump discontinuity of the enthalpy. In [16], a numerical scheme was proposed which is based on the nonequilibrium phase relaxation proposed in [10] for Stefan problem to smooth the enthalpy and the characteristic finite element method in [17] to treat the convection term. The convergence of the scheme has been proved in that paper. However, to our best knowledge, there has been no mathematical work concerning the error analysis for the numerical methods solving the continuous casting problem (1.1)–(1.5). In this paper we will propose and study a new scheme which is motivated by the characteristic finite element method in [16] to treat the convection term.

Denote by $\tau > 0$ the time step and $t^n = n\tau$ for any integer $n \geq 0$. Set $\overline{x} = x - b(t)\tau\mathbf{e}_d$

and $\overline{u} = u(\overline{x}, t)$. The characteristic finite element method is based on the observation that (cf. e.g. [17, 16])

$$\frac{\partial u^n}{\partial t} + b(t^n)\frac{\partial u^n}{\partial z} \approx \frac{u^n - \overline{u}^{n-1}}{\tau}$$

where $u^n = u(\cdot, t^n)$, $\overline{x}^{n-1} = \overline{x}(t^{n-1})$, and $\overline{u}^{n-1} = u(\overline{x}^{n-1}, t^{n-1})$. The equations (1.1)–(1.2) then can be discretized in time by

$$\frac{u^n - \overline{u}^{n-1}}{\tau} - \Delta\beta(u^n) = 0 \quad \text{in } \Omega \tag{1.6}$$

However, note that

$$\overline{u}^{n-1}(x) = u(x - b^{n-1}\tau e_d, t^{n-1}) \approx u^{n-1} - b^{n-1}\tau\frac{\partial u^{n-1}}{\partial z}$$

This suggests us to discretize the equations (1.1)–(1.2) in time by

$$\frac{u^n - u^{n-1}}{\tau} + b^{n-1}\frac{\partial u^{n-1}}{\partial z} - \Delta\beta(u^n) = 0 \quad \text{in } \Omega \tag{1.7}$$

which corresponds to an implicit discretization of the diffusion $-\Delta\beta(u)$ and an explicit discretization of the convection $b(t)\partial u/\partial z$. We note that a similar scheme was proposed in [9] but for temperature dependent convections. In this paper we will further approximate (1.7) by piecewise linear finite element method with numerical integration and thus obtain a method which can be implemented easily on computer. Furthermore we will derive an error bound which is of the same convergence order as that proved for Stefan problem without convection in [8] and [9] under the uniqueness condition introduced in [15] for proving the uniqueness of the weak solutions to (1.1)–(1.5). At this position we also remark that unlike the characteristic finite element method in [16] which further discretizes (1.6) by finite elements, the method studied here can be easily extended to solve general Stefan problem with prescribed convection in [18] and [19] because the method here does not involve the extension of functions outside the domain Ω which is usually difficult when the domain is not of cylinder type and the convection is not in one direction. However, as the method treats the convection explicitly, it studied here might not be suitable for strong convections.

The paper is organized as follows. In Section 2 we state assumptions and notations and introduce the approximate problem. In Section 3 we first prove some stability estimates and then use them to obtain the error estimates. The key ingredient is a new sharp boundary estimate for the Green operator (See Lemma 3.2) by using the uniqueness condition in [15], i.e., the free boundary does not touch on the boundary Γ_0. In Section 4 we indicate some possible extensions of the results in this paper.

2. The Finite Element Method

We start by stating the hypotheses concerning the data.

(H1) $\beta : \mathbf{R} \to \mathbf{R}$ is Lipschitz continuous and monotone increasing; $\beta(s) = 0$ for $s \in [0, \lambda]$ and $0 < \alpha_1 \le \beta'(s) \le \alpha_2$ for a.e. $s \in \mathbf{R}\backslash[0, \lambda]$.

(H2) $b \in C^{0,1}[0, T]$; $b \ge 0$.

(H3) $u_0 \in L^\infty(\Omega)$, $\theta_0 := \beta(u_0) \in C(\overline{\Omega})$.

(H4) $p \in C^{0,1}(\overline{\Gamma}_N)$; $p \ge 0$.

(H5) $g_N \in H^1(0, T; C^{0,1}(\overline{\Gamma}_N))$.

(H6) $g_D \in H^1(0, T; C^{0,1}(\overline{\Gamma}_D))$; $g_D(x, 0) = \theta_0(x)$ on Γ_D.

(H7) Uniqueness condition: $g_D(x, t) > 0$ on $\overline{\Gamma}_0 \to [0, T]$.

In view of (H4)–(H6) we may consider p, g_N and g_D extended to Ω in such a way that $p \in C^{0,1}(\overline{\Omega})$, $g_D, g_N \in H^1(0, T; C^{0,1}(\overline{\Omega}))$. It is proved in [15] that under the hypotheses (H1)–(H6), the problem (1.1)–(1.5) has weak solutions (u, θ) satisfying

$$\theta \in L^2(0, T; H^1(\Omega)) \cap C(\overline{Q}_T), \quad u \in H^1(0, T; V^*) \cap L^\infty(Q_T) \tag{2.1}$$

where $V = \{v \in H^1(\Omega) : v = 0 \text{ on } \Gamma_D\}$ and V^* is the dual space of V. Moreover, the weak solution (u, θ) is also unique if (H7) is satisfied.

Let $\{\mathcal{T}_h\}_{h>0}$ be a family of regular triangulations of Ω, where h stands for the mesh-size. Let V_h be the standard piecewise linear finite element space defined over the triangulation $\mathcal{T}_h$ and $\overset{\circ}{V}_h = V_h \cap V$. Denote by $(\cdot, \cdot)$, $\langle \cdot, \cdot \rangle$ the inner products on $L^2(\Omega)$ and $L^2(\Gamma_N)$, respectively. Let Π_h be the local linear interpolant operator, then introduce the following quadrature formulae

$$(\varphi, \chi)_h := \sum_{K \in \mathcal{T}_h} \int_K \Pi_h(\varphi\chi) dx, \quad \langle \varphi, \chi \rangle_h := \sum_{K \in \mathcal{T}_h} \int_{K \cap \Gamma_N} \Pi_h(\varphi\chi) d\sigma$$

for any piecewise uniformly continuous functions φ and χ. It is well-known that (See e.g. [5] and [9])

$$\|\varphi\|^2_{L^2(\Omega)} \le (\varphi, \varphi)_h \le C\|\varphi\|^2_{L^2(\Omega)} \quad \forall \varphi \in V_h \tag{2.2}$$

$$|(\varphi, \chi)_h| \le C\|\varphi\|_{L^2(\Omega)}\|\chi\|_{L^2(\Omega)} \quad \forall \varphi, \chi \in V_h \tag{2.3}$$

$$|\langle \varphi, \chi \rangle_h| \le C\|\varphi\|_{H^1(\Omega)}\|\chi\|_{H^1(\Omega)} \quad \forall \varphi, \chi \in V_h \tag{2.4}$$

$$|(\varphi, \chi) - (\varphi, \chi)_h| \le Ch\|\varphi\|_{L^2(\Omega)}\|\chi\|_{H^1(\Omega)} \quad \forall \varphi, \chi \in V_h \tag{2.5}$$

$$|\langle \varphi, \chi \rangle - \langle \varphi, \chi \rangle_h| \le Ch\|\varphi\|_{H^1(\Omega)}\|\chi\|_{H^1(\Omega)} \quad \forall \varphi, \chi \in V_h \tag{2.6}$$

where C is a positive constant independent of h.

Now we denote by $a(\cdot,\cdot)$ the following inner product in V

$$a(w,v) = (\nabla w, \nabla v) + \langle pw, v\rangle \quad \forall w, v \in V \tag{2.7}$$

and introduce the Green operator $G : V^* \to V$ defined by

$$a(G\psi, v) = (\psi, v) \quad \forall z \in V,\ \psi \in V^* \tag{2.8}$$

where now $(\cdot,\cdot)$ stands for the duality pairing between V^* and V. It is easy to show that for any $\psi \in L^2(\Omega)$ we have $G\psi \in H^2(\Omega)$ and $\|G\psi\|_{H^2(\Omega)} \le C\|\psi\|_{L^2(\Omega)}$. We also introduce the discrete Green operator $G_h : V^* \to \mathring{V}_h$ defined by

$$a(G_h\psi, v) = a(G\psi, v) \quad \forall v \in \mathring{V}_h,\ \psi \in V^* \tag{2.9}$$

It is known that

$$\|[G - G_h]\psi\|_{H^m(\Omega)} \le Ch^{2-m}\|G\psi\|_{H^2(\Omega)} \le Ch^{2-m}\|\psi\|_{L^2(\Omega)}, \quad m = 0, 1 \tag{2.10}$$

We will also use the L^2-projection operator $P_h : L^2(\Omega) \to V_h$ defined by

$$(P_h w, v)_h = (w, v) \quad \forall v \in V_h,\ w \in L^2(\Omega) \tag{2.11}$$

It is easy to show that (cf. e.g. [9])

$$\|w - P_h w\|_{V^*} \le Ch\|w\|_{L^2(\Omega)} \quad \forall w \in L^2(\Omega) \tag{2.12}$$

for some constant $C > 0$ independent of h.

As indicated in Section 1 we will use a difference scheme in time to discretize the problem (1.1)–(1.5) which combines an implicit discretization of the diffusion $-\Delta\beta(u)$ and an explicit approximation of the convection $b(t)\partial u/\partial z$. Let $\tau = T/N$ be the time step (N integer) and set $t^n = n\tau$, $I^n = (t^{n-1}, t^n]$ and $\partial w^n = (w^n - w^{n-1})/\tau$ for any given family $\{w^n\}_{n=0}^N$. We also set $w^n = w(\cdot, t^n)$ and $[[w]]^n = (1/\tau)\int_{I^n} w(\cdot, t)dt$ for any continuous (respectively, integrable) function in time defined in Q_T.

Now we are able to introduce the fully discrete scheme.

Problem ($\mathbf{P}_{h,\tau}$) For any $1 \le n \le N$, find $(U^n, \Theta^n) \in V_h \times V_h$ such that $U^0 = P_h u_0$, $\Theta^n = \Pi_h\beta(U^n)$, $\Theta^n - \Pi_h g_D^n \in \mathring{V}_h$ and

$$(\partial U^n, \varphi_h)_h - \left(b^{n-1}U^{n-1}, \frac{\partial \varphi_h}{\partial z}\right) + (\nabla\Theta^n, \nabla\varphi_h) + \langle p\Theta^n + g_N^n, \varphi_h\rangle_h = 0 \quad \forall \varphi_h \in \mathring{V}_h \tag{2.13}$$

Note that since $\varphi_h \in \mathring{V}_h$, we have $(b^{n-1}U^{n-1}, \partial\varphi_h/\partial z) = (b^{n-1}U^{n-1}, \partial\varphi_h/\partial z)_h$. It is clear that (2.13) is a system of nonlinear algebraic equations associated with a continuous and uniformly monotone operator, thus for each $1 \le n \le N$, (2.13) has a unique solution. For computing the discrete solutions we can use the nonlinear Gauss-Seidel method as in the case of solving Stefan problem without convection (See [9], [8]).

3. The Error Estimates

In this section we derive the error estimates for discrete Problem $(P_{h,\tau})$. Throughout we will always denote by C the generic constants independent of h, τ. The first step is the following stability estimate for the discrete problem.

Lemma 3.1 *Under the assumptions* (H1)-(H6) *we have*

$$\max_{1\le n\le N}\|U^n\|^2_{L^2(\Omega)}+\sum_{n=1}^{N}\tau\|\Theta^n\|^2_{H^1(\Omega)}\le C \tag{3.1}$$

Proof We take $\varphi_h=\tau(\Theta^n-\Pi_h g_D^n)\in \mathring{V}_h$ in (2.13) and sum it over n from 1 to n_0, for a generic $n_0\le N$ to obtain that

$$\sum_{n=1}^{n_0}\tau(\partial U^n,\Theta^n-\Pi_h g_D^n)_h-\sum_{n=1}^{n_0}\tau\left(b^{n-1}U^{n-1},\frac{\partial}{\partial z}(\Theta^n-\Pi_h g_D^n)\right)$$
$$+\sum_{n=1}^{n_0}\tau(\nabla\Theta^n,\nabla(\Theta^n-\Pi_h g_D^n))+\sum_{n=1}^{n_0}\tau\langle p\Theta^n+g_N^n,\Theta^n-\Pi_h g_D^n\rangle_h=0 \tag{3.2}$$

Let J be the dimension of $\mathring{V}_h$, $\{x_j\}_{j=1}^J$ the nodes of the triangulation $\mathcal{T}_h$ on $\overline{\Omega}\backslash\Gamma_D$, and $\{\psi_j\}_{j=1}^J$ the corresponding canonical basis of $\mathring{V}_h$. Denote by $m_{ij}=(\psi_i,\psi_j)_h$, $1\le i,j\le J$, then it is clear that $m_{ij}=0, \forall i\ne j, 1\le i, j\le J$. From (H1) we know that $\beta(s)\ge\alpha_1(s-\lambda)$ for $s\ge 0$ and $\beta(s)\le\alpha_1 s$ for $s\le 0$. Thus using (H1), (H3) (2.2) and (2.3) we have

$$\sum_{n=1}^{n_0}\tau(\partial U^n,\Theta^n)_h=\sum_{n=1}^{n_0}\left[\sum_{j=1}^{J}m_{jj}(U_j^n-U_j^{n-1})\Theta_j^n\right]$$
$$=\sum_{n=1}^{n_0}\left[\sum_{j=1}^{J}m_{jj}\int_{U_j^{n-1}}^{U_j^n}\beta(U_j^n)d\xi\right]$$
$$\ge\sum_{n=1}^{n_0}\left[\sum_{j=1}^{J}m_{jj}\int_{U_j^{n-1}}^{U_j^n}\beta(\xi)d\xi\right]$$
$$=\sum_{j=1}^{J}m_{jj}\left[\int_0^{U_j^{n_0}}\beta(\xi)d\xi-\int_0^{U_j^0}\beta(\xi)d\xi\right]$$
$$\ge\sum_{j=1}^{J}m_{jj}\min\left(\int_0^{U_j^{n_0}}\alpha_1(\xi-\lambda)d\xi,\int_0^{U_j^{n_0}}\alpha_1\xi d\xi\right)-\sum_{j=1}^{J}m_{jj}\Theta_j^0U_j^0$$

$$\geq \frac{1}{4}\alpha_1 \sum_{j=1}^{J} m_{jj}|U_j^{n_0}|^2 - C$$
$$\geq C\|U^{n_0}\|^2_{L^2(\Omega)} - C$$

where $U_j^n = U^n(x_j)$ and $\Theta_j^n = \Theta^n(x_j)$, $1 \leq j \leq J$, $1 \leq n \leq N$. Now using the summation by parts formula

$$\sum_{n=1}^{n_0} a_n[b_n - b_{n-1}] = a_{n_0}b_{n_0} - a_0 b_0 - \sum_{n=1}^{n_0} b_{n-1}[a_n - a_{n-1}] \tag{3.3}$$

and (H6) we can easily obtain that

$$\sum_{n=1}^{n_0} \tau(\partial U^n, \Theta^n - \Pi_h g_D^n)_h \geq C\|U^{n_0}\|^2_{L^2(\Omega)} - C\sum_{n=1}^{n_0} \tau\|U^{n-1}\|^2_{L^2(\Omega)} - C \tag{3.4}$$

By using (H6) and Young's inequality we get

$$\left|\sum_{n=1}^{n_0} \tau\Big(b^{n-1}U^{n-1}, \frac{\partial}{\partial z}(\Theta^n - \Pi_h g_D^n)\Big)\right|$$
$$\leq \eta \sum_{n=1}^{n_0} \tau\|\Theta^n\|^2_{H^1(\Omega)} + \frac{C}{\eta}\sum_{n=1}^{n_0} \tau\|U^{n-1}\|^2_{L^2(\Omega)} + C \tag{3.5}$$

where $\eta > 0$ will be specified later.

By applying Poincaré inequality and (H6) we obtain that

$$\sum_{n=1}^{n_0} \tau(\nabla\Theta^n, \nabla(\Theta^n - \Pi_h g_D^n)) \geq C\sum_{n=1}^{n_0} \tau\|\Theta^n\|^2_{H^1(\Omega)} - C \tag{3.6}$$

The last term can be treated by using (H4)–(H6), (2.4) and the trace theorem to get

$$\sum_{n=1}^{n_0} \tau\langle p\Theta^n - g_N^n, \Theta^n - \Pi_h g_D^n\rangle_h \geq -\eta\sum_{n=1}^{n_0} \tau\|\Theta^n\|^2_{H^1(\Omega)} - C \tag{3.7}$$

Now the desired estimate (3.1) follows by substituting (3.4)–(3.7) into (3.2), choosing η appropriately small and using Gronwall inequality. □

Let $e_u^n = [[u]]^n - U^n$ and $e_\theta^n = [[\theta]]^n - \Theta^n$, $1 \leq n \leq N$. Recall that $[[u]]^n = (1/\tau)\int_{I^n} u(\cdot,t)dt$ and $[[\theta]]^n = (1/\tau)\int_{I^n} \theta(\cdot,t)dt$. The following lemma is the key ingredient to derive the error estimates for Problem $(P_{h,\tau})$.

Lemma 3.2 *Under the assumptions* (H1)–(H7) *we have*

$$\left\|\frac{\partial}{\partial z}Ge_u^n\right\|^2_{L^2(\Gamma_0)} \leq \eta\Big[\frac{1}{\tau}\int_{I^n} \|\beta(u) - \beta(U^n)\|^2_{L^2(\Omega)}dt\Big] + \frac{C}{\eta}\|e_u^n\|^2_{V^*}, \quad 1 \leq n \leq N \tag{3.8}$$

where $\eta > 0$ is an arbitrary constant.

Proof By (H7) we know that $\theta = \beta(u) > 0$ on $\overline{\Gamma}_0 \times [0,T]$. But $\theta \in C(\overline{Q}_T)$, thus there exist two positive constants ρ and δ such that $\beta(u) \geq \rho > 0$ in the strip $\overline{\Omega}_\delta \times [0,T]$, where $\Omega_\delta = \{(x', z) \in \Omega : x' \in \Gamma, 0 < z < \delta\}$. From (H1) it is easy to check that

$$|r - s| \leq \left(\frac{2}{\alpha_1} + \frac{\lambda}{\rho}\right)|\beta(r) - \beta(s)| =: \gamma|\beta(r) - \beta(s)| \tag{3.9}$$

for any $r, s \in \mathbf{R}$ such that $\beta(r) \geq \rho > 0$. For example, for any $s \in [0, \lambda]$,

$$\begin{aligned} |r - s| &\leq |r - \lambda| + \lambda \leq \frac{1}{\alpha_1}|\beta(r) - \beta(\lambda)| + \frac{\lambda}{\rho}\rho \\ &\leq \frac{1}{\alpha_1}|\beta(r)| + \frac{\lambda}{\rho}|\beta(r)| = \left(\frac{1}{\alpha_1} + \frac{\lambda}{\rho}\right)|\beta(r) - \beta(s)| \end{aligned} \tag{3.10}$$

where we have used the fact that $r > \lambda$ since $\beta(r) \geq \rho > 0$. From (3.9) we now get

$$|e_u^n| \leq \frac{1}{\tau}\int_{I^n} |u - U^n| dt \leq \frac{\gamma}{\tau}\int_{I^n} |\beta(u) - \beta(U^n)| dt \quad \text{a.e. in } \Omega_\delta \tag{3.11}$$

Let $w = Ge_u^n$. Then from the definition (2.8) we know that

$$-\Delta w = e_u^n \quad \text{in } \Omega \tag{3.12}$$

$$w = 0 \quad \text{on } \Gamma_D, \quad \frac{\partial w}{\partial \mathbf{n}} + pw = 0 \quad \text{on } \Gamma_N \tag{3.13}$$

Let $\zeta \in C_0^\infty(\mathbf{R})$ be the cut-off function such that $\zeta(z) = 1$ in $(-\delta/2, \delta/2)$, $\zeta(z) = 0$ outside $(-\delta, \delta)$, and $0 \leq \zeta \leq 1$ in $\mathbf{R}$. Define $\widehat{w} = \zeta w$, then it is easy to check that $\widehat{w}$ satisfies the relations:

$$-\Delta\widehat{w} = -\Delta\zeta \cdot w - \nabla\zeta \cdot \nabla w + \zeta e_u^n \quad \text{in } \Omega \tag{3.14}$$

$$\widehat{w} = 0 \quad \text{on } \Gamma_D, \quad \frac{\partial\widehat{w}}{\partial \mathbf{n}} + p\widehat{w} = 0 \quad \text{on } \Gamma_N \tag{3.15}$$

where we have used (3.12)–(3.13) and the fact that $\partial\zeta/\partial\mathbf{n} = 0$ on Γ_N, since ζ depends only on z. From (3.14)–(3.15) we have

$$\begin{aligned} \|\widehat{w}\|_{H^2(\Omega)} &\leq C\|\Delta\zeta \cdot w\|_{L^2(\Omega)} + C\|\nabla\zeta \cdot \nabla w\|_{L^2(\Omega)} + C\|\zeta e_u^n\|_{L^2(\Omega)} \\ &\leq C\|w\|_{H^1(\Omega)} + C\|e_u^n\|_{L^2(\Omega_\delta)} \\ &\leq C\|e_u^n\|_{V^*} + C\frac{1}{\sqrt{\tau}}\left[\int_{I^n} \|\beta(u) - \beta(U^n)\|_{L^2(\Omega)}^2 dt\right]^{\frac{1}{2}} \end{aligned}$$

where we have used (3.11) and the fact that $\|Ge_u^n\|_{H^1(\Omega)} \leq C\|e_u^n\|_{V^*}$, which can be easily proved from (3.12)–(3.13). Hence, by trace theorem, we obtain

$$\left\|\frac{\partial w}{\partial z}\right\|_{L^2(\Gamma_0)}^2 = \left\|\frac{\partial(\zeta w)}{\partial z}\right\|_{L^2(\Gamma_0)}^2$$

$$\leq \eta\|\zeta w\|^2_{H^2(\Omega)} + \frac{C}{\eta}\|\zeta w\|^2_{H^1(\Omega)}$$

$$\leq \eta\Big[\frac{1}{\tau}\int_{I^n}\|\beta(u)-\beta(U^n)\|^2_{L^2(\Omega)}dt\Big] + \frac{C}{\eta}\|e^n_u\|^2_{V^*}.$$

This completes the proof. □

Now we are in the position to prove the main result of the paper. We set

$$u_{h,\tau}(\cdot,t) := U^n(\cdot), \quad \theta_{h,\tau}(\cdot,t) := \Theta^n(\cdot) \quad \text{for } t\in(t^{n-1},t^n], \quad 1\leq n\leq N$$

Theorem 1 *Under the assumptions* (H1)–(H7) *we have*

$$\|u-u_{h,\tau}\|^2_{L^\infty(0,T;V^*)} + \|\theta-\theta_{h,\tau}\|^2_{L^2(Q_T)} \leq C\Big(h+\tau+\frac{h^2}{\tau}\Big) \tag{3.16}$$

Proof Recall that $e^n_u = [[u]]^n - U^n$ and $e^n_\theta = [[\theta]]^n - \Theta^n$. We multiply (1.1) by Ge^n_u and integrate over $\Omega\times I^n$, then take $\varphi_h = \tau G_h e^n_u \in \overset{\circ}{V}_h$ in (2.13), take the difference and finally sum over n from 1 to n_0, for a generic $n_0\leq N$. We easily obtain

$$\sum_{n=1}^{n_0}\tau(\partial(u^n-U^n), Ge^n_u) + \sum_{n=1}^{n_0}\tau([[\theta]]^n-\Theta^n, e^n_u) =: (\mathrm{I}) + (\mathrm{II})$$

$$= \sum_{n=1}^{n_0}\tau(\partial U^n, [G_h-G]e^n_u) + \sum_{n=1}^{n_0}\tau[(\partial U^n, G_h e^n_u)_h - (\partial U^n, G_h e^n_u)]$$

$$+ \sum_{n=1}^{n_0}\tau\Big([[bu]]^n - b^{n-1}U^{n-1}, \frac{\partial}{\partial z}Ge^n_u\Big) + \sum_{n=1}^{n_0}\tau\Big(b^{n-1}U^{n-1}, \frac{\partial}{\partial z}[G-G_h]e^n_u\Big)$$

$$+ \sum_{n=1}^{n_0}\tau[\langle g^n_N, G_h e^n_u\rangle_h - \langle [[g_N]]^n, Ge^n_u\rangle]$$

$$+ \sum_{n=1}^{n_0}\tau[\langle p\Theta^n, G_h e^n_h\rangle_h - \langle p\Theta^n, G_h e^n_u\rangle]$$

$$+ \sum_{n=1}^{n_0}\tau[(\nabla\Pi_h g^n_D, \nabla G_h e^n_u) - (\nabla[[g_D]]^n, \nabla Ge^n_u)]$$

$$+ \sum_{n=1}^{n_0}\tau[\langle p\Pi_h g^n_D, G_h e^n_u\rangle - \langle p[[g_D]]^n, Ge^n_u\rangle] =: (\mathrm{III}) + \cdots + (\mathrm{X}) \tag{3.17}$$

Now we estimate separately the terms (I), $\cdots$, (X). We will concentrate on the new terms (V) and (VI). The other terms are more or less standard and can be treated by the same methods as that for Stefan problem (cf. e.g. [9]). Here we will not give the details. The first two terms can be estimated as in [9] as follows:

$$(\mathrm{I}) + (\mathrm{II}) \geq C\|e_u^{n_0}\|_{V^*}^2 + C\sum_{n=1}^{n_0}\|e_u^n - e_u^{n-1}\|_{V^*}^2$$

$$+ C\sum_{n=1}^{n_0}\int_{I^n}\|\beta(u) - \beta(U^n)\|_{L^2(\Omega)}^2 dt - C(h+\tau) \tag{3.18}$$

By using Lemma 3.1 and (2.1) we can obtain that

$$|(\mathrm{III})| + |(\mathrm{IV})| \leq \eta\|e_u^{n_0}\|_{V^*}^2 + \eta\sum_{n=1}^{n_0}\|e_u^n - e_u^{n-1}\|_{V^*}^2 + \frac{C}{\eta}\cdot\frac{h^2}{\tau} \tag{3.19}$$

To proceed further, we split (V) as follows:

$$(\mathrm{V}) = \sum_{n=1}^{n_0}\tau\Big([[bu]]^{n-1} - b^{n-1}[[u]]^{n-1}, \frac{\partial}{\partial z}Ge_u^n\Big) + \sum_{n=1}^{n_0}\tau\Big(b^{n-1}[[u]]^{n-1}, \frac{\partial}{\partial z}G[e_u^n - e_u^{n-1}]\Big)$$

$$+ \sum_{n=1}^{n_0}\tau\Big(b^{n-1}e_u^{n-1}, \frac{\partial}{\partial z}Ge_u^{n-1}\Big) =: (\mathrm{V})_1 + \cdots + (\mathrm{V})_3 \tag{3.20}$$

It follows easily from (H2) and (2.1) that

$$|(\mathrm{V})_1| \leq C\sum_{n=1}^{n_0}\tau\|[[bu]]^{n-1} - b^{n-1}[[u]]^{n-1}\|_{L^2(\Omega)}\|Ge_u^n\|_{H^1(\Omega)}$$

$$\leq \eta\sum_{n=1}^{n_0}\tau\|e_u^n\|_{V^*}^2 + \frac{C}{\eta}\cdot\tau^2 \tag{3.21}$$

$$|(\mathrm{V})_2| \leq C\sum_{n=1}^{n_0}\tau\|[[u]]^{n-1}\|_{L^2(\Omega)}\|e_u^n - e_u^{n-1}\|_{V^*}$$

$$\leq \eta\sum_{n=1}^{n_0}\|e_u^n - e_u^{n-1}\|_{V^*}^2 + \frac{C}{\eta}\tau \tag{3.22}$$

To estimate $(\mathrm{V})_3$, note first that $-\Delta Ge_u^{n-1} = e_u^{n-1}$ a.e. in Ω from the definition (2.8) of the operator G. Thus we have

$$\Big(e_u^{n-1}, \frac{\partial}{\partial z}Ge_u^{n-1}\Big) = \Big(-\Delta Ge_u^{n-1}, \frac{\partial}{\partial z}Ge_u^{n-1}\Big)$$

$$= \Big(\nabla Ge_u^{n-1}, \frac{\partial}{\partial z}\nabla Ge_u^{n-1}\Big) - \int_\Gamma\Big(\frac{\partial}{\partial \mathbf{n}}Ge_u^{n-1}\Big)\Big(\frac{\partial}{\partial z}Ge_u^{n-1}\Big)d\sigma$$

$$= \frac{1}{2}\int_\Omega\frac{\partial}{\partial z}|\nabla Ge_u^{n-1}|^2 dx + \int_{\Gamma_0}\Big|\frac{\partial}{\partial z}Ge_u^{n-1}\Big|^2 d\sigma$$

$$
\begin{aligned}
&-\int_{\Gamma_L}\left|\frac{\partial}{\partial z}Ge_u^{n-1}\right|^2 d\sigma+\frac{1}{2}\int_{\Gamma_N}p\frac{\partial}{\partial z}|Ge_u^{n-1}|^2 d\sigma\\
&=\frac{1}{2}\int_{\Gamma_0}\left|\frac{\partial}{\partial z}Ge_u^{n-1}\right|^2 d\sigma-\frac{1}{2}\int_{\Gamma_L}\left|\frac{\partial}{\partial z}Ge_u^{n-1}\right|^2 d\sigma\\
&\quad-\frac{1}{2}\int_{\Gamma_N}|Ge_u^{n-1}|^2\frac{\partial p}{\partial z}d\sigma\\
&\le\frac{1}{2}\int_{\Gamma_0}\left|\frac{\partial}{\partial z}Ge_u^{n-1}\right|^2 d\sigma-\frac{1}{2}\int_{\Gamma_N}|Ge_u^{n-1}|^2\frac{\partial p}{\partial z}d\sigma
\end{aligned}\tag{3.23}
$$

where in the third inequality we have used the fact that $\mathbf{n}=\mathbf{e}_d$ or $\mathbf{n}=-\mathbf{e}_d$ on Γ_D and the boundary condition $\partial Ge_u^{n-1}/\partial\mathbf{n}+pGe_u^{n-1}=0$ on Γ_N; and in the forth equality we have used the boundary condition $Ge_u^{n-1}=0$ on Γ_D and the fact that $Ge_u^{n-1}\in H^2(\Omega)$ thus $Ge_u^{n-1}\in C(\overline{\Omega})$ by the embedding theorem. Now by using Lemma 3.2 and the trace theorem we obtain that

$$
(\mathrm{V})_3\le\eta\sum_{n=1}^{n_0}\int_{I^n}\|\beta(u)-\beta(U^n)\|^2_{L^2(\Omega)}dt+\frac{C}{\eta}\sum_{n=1}^{n_0}\tau\|e_u^{n-1}\|^2_{V^*}\tag{3.24}
$$

By using (2.10), (2.1) and Lemma 3.2 we also have

$$
\begin{aligned}
|(\mathrm{VI})|&\le C\sum_{n=1}^{n_0}\tau\|U^{n-1}\|_{L^2(\Omega)}\|[G-G_h]e_u^n\|_{H^1(\Omega)}\\
&\le Ch\sum_{n=1}^{n_0}\tau\|U^{n-1}\|_{L^2(\Omega)}\|e_u^n\|_{L^2(\Omega)}\le Ch
\end{aligned}\tag{3.25}
$$

The other four terms can be estimated by using (H5)–(H6), (2.4), (2.6), (2.1) and Lemma 3.1 to get

$$
|(\mathrm{VII})|+|(\mathrm{VIII})|+|(\mathrm{IX})|+|(\mathrm{X})|\le\eta\sum_{n=1}^{n_0}\tau\|e_u^n\|^2_{V^*}+\frac{C}{\eta}(\tau+h)\tag{3.26}
$$

Now substituting (3.18)–(3.26) into (3.17) and choosing $\eta>0$ appropriately small, we can easily obtain

$$
\begin{aligned}
&\|e_u^{n_0}\|^2_{V^*}+\sum_{n=1}^{n_0}\|e_u^n-e_u^{n-1}\|^2_{V^*}+\sum_{n=1}^{n_0}\int_{I^n}\|\beta(u)-\beta(U^n)\|^2_{L^2(\Omega)}dt\\
&\le C\sum_{n=1}^{n_0}\tau\|e_u^{n-1}\|^2_{V^*}+C\Big(h+\tau+\frac{h^2}{\tau}\Big)
\end{aligned}\tag{3.27}
$$

Now the theorem follows from the Gronwall inequality and the following observations:

$$
\sum_{n=1}^{n_0}\int_{I^n}\|\beta(u)-\beta(U^n)\|^2_{L^2(\Omega)}dt
$$

$$\geq \frac{1}{2}\int_{I^n}\|\theta-\Theta^n\|^2_{L^2(\Omega)}dt-\sum_{n=1}^{n_0}\int_{I^n}\|\beta(U^n)-\Pi_h\beta(U^n)\|^2_{L^2(\Omega)}dt$$

$$\geq \frac{1}{2}\sum_{n=1}^{n_0}\int_{I^n}\|\theta-\theta_{h,\tau}\|^2_{L^2(\Omega)}dt-Ch^2\sum_{n=1}^{n_0}\tau\|\Theta^n\|^2_{H^1(\Omega)}$$

$$\geq \frac{1}{2}\sum_{n=1}^{n_0}\int_{I^n}\|\theta-\theta_{h,\tau}\|^2_{L^2(\Omega)}dt-Ch^2 \tag{3.28}$$

and

$$\max_{1\leq n\leq n_0}\|e_u^n\|^2_{V^*}\geq \sup_{t\in[0,t^{n_0}]}\|u-u_{h,\tau}\|^2_{V^*}-C\tau \tag{3.29}$$

by using Lémma 3.1 and (2.1) and the following estimate [8]

$$\|\beta(\chi)-\Pi_h\beta(\chi)\|_{L^2(\Omega)}\leq Ch\|\nabla\Pi_h\beta(\chi)\|_{L^2(\Omega)}\quad \forall\chi\in V_h \tag{3.30}$$

This completes the proof. □

4. Some Extensions

One popular method in solving Stefan problem is to smooth the enthalpy first and then discretize it by using finite element methods (cf. e.g. [7] and [9]). Thus, for $\varepsilon>0$, let $H_\varepsilon:\mathbf{R}\to\mathbf{R}$ be the regularization of the maximal monotone graph $H=\beta^{-1}$:

$$H_\varepsilon(s)=\begin{cases}\min(s/\varepsilon,H(s)) & \text{if } s>0\\ 0 & \text{if } s=0\\ \max(s/\varepsilon,H(s)) & \text{if } s<0\end{cases}$$

We can solve the continuous casting problem (1.1)–(1.5) by the following method.

Problem $(\mathrm{P}_{\varepsilon,h,\tau})$ For any $1\leq n\leq N$, find $(U^n,\Theta^n)\in V_h\times V_h$ such that $U^0=P_h[H_\varepsilon(\theta_0)]$, $U^n=\Pi_h[H_\varepsilon(\Theta^n)]$, $\Theta^n-\Pi_h g_D^n\in \mathring{V}_h$ and

$$(\partial U^n,\varphi_h)_h-\left(b^{n-1}U^{n-1},\frac{\partial\varphi_h}{\partial z}\right)+(\nabla\Theta^n,\nabla\varphi_h)+\langle p\Theta^n+g_N^n,\varphi_h\rangle_h=0\quad\forall\varphi_h\in\mathring{V}_h \tag{4.1}$$

Note that (4.1) can also be easily solved by the nonlinear Gauss-Seidel method. By modifying the method in Section 3 we can show the following error estimates for Problem $(\mathrm{P}_{\varepsilon,h,\tau})$:

$$\|u-u_{\varepsilon,h,\tau}\|^2_{L^\infty(0,T;V^*)}+\|\theta-\theta_{\varepsilon,h,\tau}\|^2_{L^2(Q_T)}\leq\left(\varepsilon+h+\tau+\frac{h^2}{\tau}\right)$$

which is again of the same convergence order as that proved for Stefan problem without convection in [9]. Here we have set

$$u_{\varepsilon,h,\tau}(\cdot,t) := U^n(\cdot), \quad \theta_{\varepsilon,h,\tau}(\cdot,t) := \Theta^n(\cdot) \quad \text{for } t \in (t^{n-1}, t^n], \quad 1 \le n \le N$$

The finite element method with ε-regularization of the enthalpy is important for solving the optimal control problem governed by the continuous casting problem (See [20]) because now the discrete solution operator is Fréchet differentiable. Another important fact is that it often happens in practical situations that the latent heat does not release at the melting temperature instantly but rather release in a narrow temperature interval around the melting temperature (See e.g. [12]). This corresponds to a natural regularization of the enthalpy. The numerical results of the method in this paper will be reported in [21] where the continuous casting problem with nonlinear flux including the practically important Stefan-Boltzman radiation law will be also considered.

We can also extend the method in this paper to study the numerical solutions of the following general Stefan problem with prescribed convection in [18] and [19]

$$\frac{\partial u}{\partial t} + \operatorname{div}(u\mathbf{v} - \nabla\theta) = f \quad \text{in} \quad \Omega \times (0,T) \tag{4.2}$$

$$\theta = \beta(u) \quad \text{in} \quad \Omega \times (0,T) \tag{4.3}$$

$$\theta = g_D(x,t) \quad \text{on} \quad \Gamma_D \times (0,T) \tag{4.4}$$

$$(\mathbf{v}\cdot\mathbf{n})u - \frac{\partial\theta}{\partial\mathbf{n}} = p(x)\theta + g_N(x,t) \quad \text{on} \quad \Gamma_N \times (0,T) \tag{4.5}$$

$$u(x,0) = u_0(x) \quad \text{in} \quad \Omega \tag{4.6}$$

where $\Omega \subset \mathbf{R}^d$ is a bounded polygonal domain if $d=2$ or a bounded polyhedral domain if $d=3$, $\mathbf{v}: \Omega\times(0,T) \to \mathbf{R}^d$ is the prescribed velocity. In order to obtain the existence and uniqueness of the weak solutions of (4.2)–(4.6), it is assumed in [18] and [19] that

$$\mathbf{v}\cdot\mathbf{n} \ge 0 \quad \text{on } \Gamma_D \times (0,T), \quad \mathbf{v}\cdot\mathbf{n} \le 0 \quad \text{on } \Gamma_N \times (0,T)$$

and $\mathbf{v} = 0$ on $(\Gamma_D \cap \Gamma_N)\times(0,T)$. This problem will be addressed in a separate paper.

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References

[1] Friedman A., The Stefan problem in several space variables, *Trans. Amer. Math. Soc.*, **133** (1968), 51–87.

[2] Jerome J. W., Approximation of Nonlinear Evolutionary Systems, Academic Press, New York, 1983.

[3] Kamenomostskaja S.L., On the Stefan problems, *Mat. Sb.*, **53** (1961), 489–514.

[4] Meyer G.H., Multidimensional Stefan problem, *SIAM J. Numer. Anal.*, **10** (1973), 522-538.

[5] Ciavaldini J.F., Analyse numérique d'un probléme de Stefan á deux phases par une méthode l'elements finis, *SIAM J. Numer. Anal.*, **12** (1975), 464–487.

[6] White R.E., A numerical solution of the enthalpy formulation of the Stefan problem, *SIAM J. Numer. Anal.*, **19** (1982), 1158–1172.

[7] Jerome J. W. and Rose M.E., Error estimates for the multidimensional two phase Stefan problem, *Math. Comp.*, **39** (1982), 377-414.

[8] Elliott C.M., Error analysis of the enthalpy method for the Stefan problem, *IMA J. Numer. Anal.*, **7** (1987), 61–71.

[9] Nochetto R.H. and Verdi C., Approximation of degenerate parabolic problems using numerical integration, *SIAM J. Numer. Anal.*, **25** (1988), 784–814.

[10] Verdi C. and Visintin A., Error estimates for a semi-explicit numerical scheme for Stefan-type problems, *Numer. Math.*, **52** (1988), 165–185.

[11] Jiang X. and Nochetto R.H., A P^1-P^1 finite element method for a phase relaxation model. Part I: quasi-uniform mesh, Preprint (1996).

[12] Louhenkilpi S., Laitinen E. and Nieminen R., Real-time simulation of heat transfer in continuous casting, *Etallurgical Trans. B,* **24B** (1993), 685–693.

[13] Rodrigues J.F., Obstacle Problems in Mathematical Physics, North-Holland, Amsterdam, 1987.

[14] Yi F., An evolutionary continuous casting problem of two-phase and its periodic behaviour, *J. Part. Diff. Eq.*, **2** (1989), 7–22.

[15] Rodrigues J.F. and Yi F., On a two-phase continuous casting Stefan problem with nonlinear flux, *Euro. J. Appl. Math.*, **1** (1990), 259–278.

[16] Chen Z., Numerical solutions of a two-phase continuous casting problem, in Numerical Methods for Free Boundary Problem, P. Neittaanmaki, eds., International Series of Numerical Mathematics 99, Birkhäuser, Basel, 1991, 103–121.

[17] Douglas J., Jr. and Russell T.F., Numerical methods for convection-dominated diffusion problem based on combining the method of characteristic with finite element or finite difference procedures, *SIAM J. Numer. Anal.*, **19** (1982), 871–885.

[18] Rulla J., Weak solutions to Stefan problems with prescribed convection, *SIAM J. Math. Anal.*, **18** (1987), 1784–1800.

[19] Yi F. and Qiu Y., On Stefan problem with prescribed convection, *Acta Mathematica Scientia,* **14** (1994), 153–166.

[20] Mäkelä M., Männikkö T. and Schramm H., Application of nonsmooth optimization methods to continuous casting of steel, DFG Report No. 421, Universität Bayreuth, 1993.

[21] Chen Z., Shih T.M. and Yue Y.X., Numerical methods for Stefan problems with prescribed convection and nonlinear flux, to appear.

[22] Ciarlet P.G., The finite Element Method for Elliptic Problems, North-Holland, Amsterdam, 1978.