J.Partial Diff. Eqs. 11(1998), 46–58
©International Academic Publishers Vol.11, No.1

AN INTEGRABILITY CONDITION FOR MONGE-AMPÈRE EQUATIONS ON A KÄHLER MANIFOLD

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Dedicated to Professor Ding Xiaxi on the occasion of his 70th birthday

(Received May 21, 1997)

Abstract The symmetry group of the Monge-Ampère equation on a Kähler manifold is determined and an integrability condition on the solution is derived as a conservation law.

Key Words Complex Monge-Ampère equation, Futaki's obstruction, Lie symmetry group, Noether's theorem and conservation laws.
Classification 35C, 35J.

0. Introduction

In the study of many geometric differential equations integrability conditions play an important role. For some situations they yield obstructions for solvability, and for the other they provide balancing relation of the problem under consideration among different regions. Notable examples of integrability conditions include Pohozaev's type identities for semi-linear problems in the Euclidean space, the Kazdan-Warner condition for the Nirenberg problem, and the Futaki's obstruction for Kähler-Einstein metrics. It has been known for a long time that all these integrability conditions are intimately related to the symmetry of the underlying manifolds as well as to the canonical differential operators which reflect this symmetry. In [1] we discuss this issue from the viewpoint of Lie's theory of symmetry groups for differential equations and Noether's theorem on conservation laws. The general setting can be briefly described as follows.

Consider a differential operator $\mathcal{F}[u]$ which is the Euler-Lagrange operator for the Lagrangian $\mathcal{L}[u]$ on a manifold with a certain structure. Usually $\mathcal{F}[u]$ is canonical in the sense that it inherits partially or entirely the symmetry of this structure. One would like to look for an integrability condition, or a variational identity as called in [1], for the nonhomogeneous problem

$$\mathcal{F}[u] = g(x, u)$$

The procedure of generating a variational identity corresponding to each (divergence) symmetry of $\mathcal{F}[u] = 0$ consists in the following three steps.

(1) Find the Lie symmetry group for the homogeneous equation

$$\mathcal{F}[u] = 0$$

This can be done by solving a system of linear PDE's satisfied by the infinitesimal generators of the symmetry group.

(2) Determine which infinitesimal symmetry is an infinitesimal variational or divergence symmetry for the Lagrangian $\mathcal{L}$. This step can be easily carried out by direct verification.

(3) Put the infinitesimal divergence symmetry into an expression appearing in a crucial step of the proof of Noether's theorem on conservation laws. After some integration by parts we obtain a variational identity for solutions of the nonhomogeneous problem. In general, each infinitesimal divergence symmetry produces a variational identity.

We have applied this procedure in [1] to derive some variational identities for the p-Laplacian and the conformal Laplacian on a Riemannian manifold. Some of these identities are old, but some are new. In a companion paper [2] we treat the same problem for a class of conformally invariant fourth-order semi-linear equations.

In this note we would like to further illustrate the procedure to the complex Monge-Ampère equation on a compact Kähler manifold. Consider the equation

$$\det\left(g_{\alpha\overline{\beta}} + \frac{\partial^2 u}{\partial z^\alpha \partial \overline{z}^\beta}\right) = \exp F(z, \overline{z}, u) \det\left(g_{\alpha\overline{\beta}}\right) \tag{0.1}$$

which is defined invariantly on a Kähler manifold (M, g). Here $g_{\alpha\overline{\beta}} dz^\alpha \otimes d\overline{z}^\beta$ is the Kähler metric in local coordinates and F is a given function on $M \times \mathrm{R}$. Equation (0.1) was studied by Aubin [3] and Yau [4] independently. Both authors proved the existence of a solution when $\partial F/\partial u > 0$. The case $\partial F/\partial u \geq 0$ is much harder and is solved in [4]. Needless to say, the most important case of (0.1) is

$$\det\left(g_{\alpha\overline{\beta}} + \frac{\partial^2 u}{\partial z^\alpha \partial \overline{z}^\beta}\right) = \exp(\lambda u + \phi(z, \overline{z})) \det\left(g_{\alpha\overline{\beta}}\right) \tag{0.2}$$

whose solvability is equivalent to the existence of a Kähler-Einstein metric on M. The signature of λ is determined by the signature of the first Chern class of M; it is 1, 0, or -1 (under a normalization) depending on whether $c_1(M)$ is negative, zero, or positive. The above mentioned work of Aubin and Yau solved (0.2) for $\lambda = 1$ and $\lambda = 0$, 1 respectively. The case $\lambda = -1$ is unsolved (except when the dimension is two, see Tian [5]). In [6] Futaki discovered an obstruction to this case. It turns out that if (0.2) is solvable, then for all holomorphic vector field ξ on M,

$$\int_M \xi^j \frac{\partial \phi}{\partial x^j} e^{-u+\phi} = 0 \tag{0.3}$$

But then one can produce some ϕ on a Kähler manifold with nontrivial holomorphic vector fields such that (0.3) cannot hold. Recent progress on the solvability of (0.2) in the case of positive $c_1(M)$ can be found in Tian [7].

Using the procedure described above, we shall derive an integrability condition on the solution of the general equation (0.1), which, in the special case (0.2), reduces to (0.3). See Proposition 3.3 for a precise statement.

This note consists of three sections. In Section 1 we consider the Hodge decomposition for a holomorphic vector field. The decomposition formula provides a term needed in the group analysis of the complex Monge-Ampère operator in Section 2. In Section 3 we shall derive the integrability condition for (0.1).

Acknowledgement The support of an Earmarked Grant for Research, H.K., is gratefully acknowledged. The second author is also partially supported by National Nature Science Foundation of China.

1. A Hodge Decomposition

Let M be a compact Kähler manifold of dimension m. In complex local coordinates, its Kähler metric is of the form

$$ds^2 = g_{\alpha\overline{\beta}} dz^\alpha \otimes dz^{\overline{\beta}}, \quad \alpha, \beta = 1, \cdots, m$$

and the corresponding Kähler form is

$$\omega = \frac{\sqrt{-1}}{2\pi} g_{\alpha\overline{\beta}} dz^\alpha \wedge dz^{\overline{\beta}}$$

let $\mathfrak{h}(M)$ be the complex Lie algebra of all holomorphic vector fields on M. For any $(1,0)$-type vector field

$$\xi = \xi^\alpha \frac{\partial}{\partial z^\alpha}$$

in $\mathfrak{h}(M)$ we associate to it a 1-form of $(0,1)$-type by

$$\alpha = g_{\alpha\overline{\beta}}\xi^\alpha dz^{\overline{\beta}} \equiv \alpha_{\overline{\beta}} dz^{\overline{\beta}}$$

According to the Hodge decomposition theorem (See, e.g. Lichnerowicz [8]) for each 1-form of (0.1)-type α there exist two 1-form μ and $H\alpha$ of the same type, $H\alpha$ being harmonic, such that

$$\alpha = \Delta\mu + H\alpha \tag{1.1}$$

The Kähler condition $d\omega = 0$ is equivalent to

$$\frac{\partial g_{\alpha\overline{\beta}}}{\partial z^{\overline{\gamma}}} = \frac{\partial g_{\alpha\overline{\gamma}}}{\partial z^{\overline{\beta}}}$$

Therefore, if α is also holomorphic,

$$d''\alpha = \frac{\partial \alpha_{\overline{\beta}}}{\partial z^{\overline{\gamma}}} dz^{\overline{\gamma}} \wedge dz^{\overline{\beta}} = \left(\frac{\partial g_{\alpha\overline{\beta}}}{\partial z^{\overline{\gamma}}}\xi^\alpha + g_{\alpha\overline{\beta}}\frac{\partial \xi^\alpha}{\partial z^{\overline{\gamma}}}\right) dz^{\overline{\gamma}} \wedge dz^{\overline{\beta}} = 0$$

i.e., α is d''-closed. Applying d'' to both sides of (1.1), we have

$$2d''\delta''d''\mu = 2d''(\delta''d'' + d''\delta'')\mu = d''\Delta\mu = d''\alpha - d''H\alpha = 0$$

Therefore

$$\delta''d''\mu = 0$$

Putting this into (1.1) yields

$$\alpha = d''\rho + H\alpha \tag{1.2}$$

where $\rho = 2\delta''\mu$ is a complex-valued function. Thus we have proved, for any holomorphic vector field ξ of $(1,0)$-type, (1.2) holds for its associated form α where ρ is a function and $H\alpha$ is harmonic. A similar statement holds for fields of (0,1)-type. In particular, when ξ is a real holomorphic vector field, i.e.,

$$\xi = \xi^\alpha \frac{\partial}{\partial z^\alpha} + \xi^{\overline{\alpha}} \frac{\partial}{\partial z^{\overline{\alpha}}}, \quad \xi^{\overline{\alpha}} = \overline{\xi^\alpha}, \quad \alpha = 1, \cdots, m$$

we have the following decomposition

$$\alpha = d\sigma + H\alpha \tag{1.3}$$

where α is the associated 1-form $g_{\alpha\overline{\beta}}\xi^\alpha dz^{\overline{\beta}} + g_{\beta\overline{\alpha}}\xi^{\overline{\alpha}} dz^\beta$, $\sigma = 2\mathrm{Re}\,(\rho)$ and $H\alpha$ is harmonic (ρ is associated to $\xi^\alpha \frac{\partial}{\partial z^\alpha}$ via (1.2).)

2. Symmetries for the Monge-Ampère Equation

Let M be an n-dimensional Riemannian manifold with a Riemannian metric $g_{ij}dx^i \otimes dx^j$. A one-parameter group of transformations on $M \times \mathbf{R}$ is of the form

$$\tilde{x}^i = \Sigma^i(x, u, \varepsilon), \quad i = 1, \cdots, n \tag{2.1}$$

$$\tilde{u} = \Phi(x, u, \varepsilon) \tag{2.2}$$

satisfying $x = \Sigma(x, u, 0)$ and $u = \Phi(x, u, 0)$. Its infinitesimal transformation, i.e., the vector field on $M \times \mathbf{R}$ generating the one-paremeter group, can be obtained by differentiating the relations (2.1) and (2.2). Denoting by

$$v = \xi^i(x, u)\frac{\partial}{\partial x^i} + \phi(x, u)\frac{\partial}{\partial u}$$

we have

$$\xi^i(x, u) = \frac{\partial}{\partial \varepsilon}\bigg|_{\varepsilon=0} \Sigma^i(x, u, \varepsilon) \quad \text{and} \quad \phi(x, u) = \frac{\partial}{\partial \varepsilon}\bigg|_{\varepsilon=0} \Phi(x, u, \varepsilon)$$

Given a function $u(x)$ in M, under the transformation group it goes over to a new function $\tilde{u}(\tilde{x})$ as follows. Since, for ε sufficiently close to 0,

$$\tilde{x} = \Sigma(x, u(x), \varepsilon) \tag{2.3}$$

is invertible in x, we may use (2.2) to define $\tilde{u}(\tilde{x})$ where now the x in Φ is replaced by $\tilde{x}$ using (2.3). By covariant differentiating $\tilde{u}$ we obtain $\begin{pmatrix} N \\ O \end{pmatrix}$-tensor fields $\tilde{u}_{,J}$ at $\tilde{x}$. The following formula, obtained in [1], describes the infinitesimal change of $u_{,J}(x)$ to $\tilde{u}_{,J}(\tilde{x})$ along the group action.

$$\frac{\partial}{\partial \varepsilon}\bigg|_{\varepsilon=0} \tilde{u}_{,j_1\cdots j_N}(\tilde{x}) = (\phi - \xi^k u_k)_{,j_1\cdots j_N} + \xi^k u_{,kj_1\cdots j_N}$$

$$+ (u_{,j_1\cdots j_N k} - u_{,kj_1\cdots j_N})\xi^k + \sum_{k=1}^{N} \Gamma^i_{j_k m} u_{,j_1\cdots i \cdots j_N}\xi^m \tag{2.4}$$

where Γ^k_{ij}'s are the Christofel symbols of the metric g_{ij}.

When one deals with complex manifold M with a Kähler metric $g_{\alpha\overline{\beta}}dz^\alpha \otimes d\overline{z}^\beta$ $(\alpha, \beta = 1, \cdots, m)$, the formula (2.4) is still valid if we let $x^\alpha = z^\alpha$ and $x^{\alpha+m} = \overline{z}^\alpha$ for $\alpha = 1, \cdots, m$. Notice that in this case the infinitesimal transformation is given by

$$v = \xi^\alpha(x, u)\frac{\partial}{\partial z^\alpha} + \xi^{\overline{\alpha}}(x, u)\frac{\partial}{\partial \overline{z}^\alpha} + \phi(x, u)\frac{\partial}{\partial u}$$

A one-parameter group of transformations on M is called a symmetry of a differential equation

$$F(x,u(x),u_{,i}(x),\cdots)=0$$

if $\widetilde{u}(\widetilde{x})$ solves the same equation whenever $u(x)$ is a solution. The infinitesimal criterion for symmetry can be obtained by differentiating the equation $F(\widetilde{x},\widetilde{u}(\widetilde{x}),\widetilde{u}_{,i}(\widetilde{x}),\cdots)=0$ and then putting $\varepsilon=0$. Thus, using (2.4) we have

$$\xi^k\frac{\partial F}{\partial x^k}+\phi\frac{\partial F}{\partial u}+\frac{\partial}{\partial\varepsilon}\Big|_{\varepsilon=0}\widetilde{u}_{,i}(\widetilde{x})\frac{\partial F}{\partial u_{,i}}+\cdots=0 \tag{2.5}$$

on $F(x,u(x),u_{,i}(x),\cdots)=0$.

When it comes to determining the symmetry group for the Monge-Ampère equation

$$\mathcal{M}_g[u]\equiv\frac{\det(g_{\alpha\overline{\beta}}+u_{,\alpha\overline{\beta}})}{\det(g_{\alpha\overline{\beta}})}=0 \quad \text{on } M \tag{2.6}$$

one can, in principle, use the equation (2.5) and the prolongation formula (2.4). However, things become simpler because of the Kähler condition which implies that all mixed Christofel symbols vanish, and therefore $u_{,\alpha\overline{\beta}}=u_{\alpha\overline{\beta}}$. So (2.5) and (2.4) simply become

$$\frac{c^{\alpha\overline{\beta}}}{\det(g_{\alpha\overline{\beta}})}\xi^i\frac{\partial g_{\alpha\overline{\beta}}}{\partial x^i}+\xi^i\frac{\partial}{\partial x^i}\Big(\frac{1}{\det(g_{\alpha\overline{\beta}})}\Big)\det(g_{\alpha\overline{\beta}}+u_{\alpha\overline{\beta}})+\frac{c^{\alpha\overline{\beta}}\phi_{\alpha\overline{\beta}}}{\det(g_{\alpha\overline{\beta}})}=\lambda\mathcal{M}_g[u] \tag{2.7}$$

for some $\lambda=\lambda(x,u,u_i,u_{ij})$, where

$$\phi_{\alpha\overline{\beta}}=\frac{\partial^2}{\partial z^{\overline{\beta}}\partial z^\alpha}(\phi(x,u(x))-\xi^k(x,u(x))u_k(x))+\xi^k(x,u(x))u_{k\alpha\overline{\beta}}(x) \tag{2.8}$$

($c^{\alpha\overline{\beta}}$ is the (α,β)-cofactor of $(g_{\alpha\overline{\beta}}+u_{\alpha\overline{\beta}})$.)

Proposition 2.1 *Let M be a compact Kähler manifold. The symmetry group of the homogeneous Monge-Ampère equation (2.6) is generated by vector fields*

$$v=\xi^\alpha\frac{\partial}{\partial z^\alpha}+\xi^{\overline{\alpha}}\frac{\partial}{\partial z^{\overline{\alpha}}}+(c-\sigma(x))\frac{\partial}{\partial u}$$

where $\xi\in\mathfrak{h}(M)$ is real, $c\in\mathbf{R}$ and σ is determined from ξ via (1.3).

Proof Writing (2.8) out we get

$$\begin{aligned}\phi_{\alpha\overline{\beta}}=&\,\phi_{\alpha\overline{\beta}}+\phi_{u\alpha}u_{\overline{\beta}}+\phi_{u\overline{\beta}}u_\alpha+\phi_{uu}u_\alpha u_{\overline{\beta}}+\phi_u u_{\alpha\overline{\beta}}\\ &-(\xi^k_{\alpha\overline{\beta}}+\xi^k_{u\alpha}u_{\overline{\beta}}+\xi^k_{u\overline{\beta}}u_\alpha+\xi^k_{uu}u_\alpha u_{\overline{\beta}}+\xi^k_u u_{\alpha\overline{\beta}})u_k\\ &-(\xi^k_\alpha+\xi^k_u u_\alpha)u_{k\overline{\beta}}-(\xi^k_{\overline{\beta}}+\xi^k_u u_{\overline{\beta}})u_{k\alpha}\end{aligned}$$

Put this expression into (2.7) and compare the coefficients of the derivatives of u. Notice that now (2.7) holds as an algebraic equation in x, u and its derivatives. First of all, it is easy to see that λ must be independent of u_{ij}. Then a comparison with the coefficients of the higher order derivatives of u shows that ξ must be independent of u, i.e., $\xi = \xi(x)$ and it is holomorphic. Also $\phi_{uu} = 0$. Let's write

$$\phi(x, u) = A(x)u + B(x)$$

Now (2.7) is simplified to

$$\frac{c^{\alpha\overline{\beta}}}{\det(g_{\alpha\overline{\beta}})}\left(\phi_{\alpha\overline{\beta}} + A_\alpha u_{\overline{\beta}} + A_{\overline{\beta}} u_\alpha + A u_{\alpha\overline{\beta}} - \xi^k_\alpha u_{k\overline{\beta}} - \xi^k_{\overline{\beta}} u_{k\alpha} + \xi^k \frac{\partial g_{\alpha\overline{\beta}}}{\partial x^k}\right) = \lambda' \mathcal{M}_g[u]$$

So $A_i = 0$, i.e., $A(x)$ is constant. By using the identity

$$c^{\alpha\overline{\beta}}(g_{\gamma\overline{\beta}} + u_{\gamma\overline{\beta}}) = \det(g_{\alpha\overline{\beta}} + u_{\alpha\overline{\beta}})\delta^\alpha_\gamma$$

the above equation further reduces to

$$\frac{c^{\alpha\overline{\beta}}}{\det(g_{\alpha\overline{\beta}})}\left(B_{\alpha\overline{\beta}} - A g_{\alpha\overline{\beta}} + \xi^\gamma_\alpha g_{\alpha\overline{\beta}} + \xi^{\overline{\gamma}}_{\overline{\beta}} g_{\overline{\gamma}\alpha} + \xi^\gamma \frac{\partial g_{\alpha\overline{\beta}}}{\partial z^\gamma} + \xi^{\overline{\gamma}} \frac{\partial g_{\alpha\overline{\beta}}}{\partial z^{\overline{\gamma}}}\right) = \lambda'' \mathcal{M}_g[u]$$

Therefore

$$B_{\alpha\overline{\beta}} - A g_{\gamma\overline{\beta}} + \xi^\gamma_\alpha g_{\gamma\overline{\beta}} + \xi^{\overline{\gamma}}_{\overline{\beta}} g_{\overline{\gamma}\alpha} + \xi^\gamma \frac{\partial g_{\alpha\overline{\beta}}}{\partial z^\gamma} + \xi^{\overline{\gamma}} \frac{\partial g_{\alpha\overline{\beta}}}{\partial z^{\overline{\gamma}}} = 0$$

for $\alpha, \beta = 1, \cdots, m$. Now,

$$\begin{aligned}\xi^\gamma_\alpha g_{\gamma\overline{\beta}} + \xi^\gamma \frac{\partial g_{\alpha\overline{\beta}}}{\partial z^\gamma} &= \frac{\partial \xi^\gamma}{\partial z^\alpha} g_{\gamma\overline{\beta}} + \xi^\gamma g_{\nu\overline{\beta}} \Gamma^\nu_{\alpha\gamma} = g_{\gamma\overline{\beta}}\left(\frac{\partial \xi^\gamma}{\partial z^\alpha} + \xi^\nu \Gamma^\gamma_{\alpha\nu}\right) \\ &= g_{\gamma\overline{\beta}} \nabla_\alpha \xi^\gamma = \nabla_\alpha\left(\frac{\partial \rho}{\partial z^{\overline{\beta}}} + (H\alpha)_{\overline{\beta}}\right) \qquad \text{(by (1.2))} \\ &= \frac{\partial^2 \rho}{\partial z^\alpha \partial z^{\overline{\beta}}} + d'(H\alpha)_{\alpha\overline{\beta}} = \rho_{\alpha\overline{\beta}}\end{aligned}$$

where $H\alpha = (H\alpha)_{\overline{\beta}} dz^{\overline{\beta}}$ and $d'(H\alpha) = d'(H\alpha)_{\alpha\overline{\beta}} dz^\alpha \wedge dz^{\overline{\beta}} = 0$. Similarly,

$$\xi^{\overline{\gamma}}_{\overline{\beta}} g_{\overline{\gamma}\alpha} + \xi^{\overline{\gamma}} \frac{\partial g_{\alpha\overline{\beta}}}{\partial z^{\overline{\gamma}}} = \overline{\rho}_{\alpha\overline{\beta}}$$

Therefore

$$(B + \sigma)_{\alpha\overline{\beta}} - A g_{\alpha\overline{\beta}} = 0$$

which implies $A = 0$ and, by using the compactness of M,

$$B = -\sigma + \text{constant}$$

The proof of the proposition is completed.

Remark General references on Lie symmetry groups include Olver [9] and Ovsiannikov [10].

3. An Integrability Condition

Consider the following Monge-Ampère equation on a compact Kähler manifold $(M, g_{\alpha\overline{\beta}}dz^\alpha \otimes dz^{\overline{\beta}})$, $\alpha, \beta = 1, \cdots, m$,

$$\mathcal{M}_g[u] = f(x, u) \tag{3.1}$$

where f is a given smooth function in $M \times \mathbf{R}$. The Monge-Ampère operator $\mathcal{M}_g[u]$ can be written as

$$\mathcal{M}_g[u] = \det\left(\delta^\alpha_\beta + \nabla^\alpha_\beta u\right)$$

where $\nabla^\alpha_\beta u = \nabla^\alpha \nabla_\beta u = g^{\alpha\overline{\gamma}}\partial^2 u/\partial z^{\overline{\gamma}}\partial z^\beta$. By expanding the determinant we have

$$\det\left(\delta^\alpha_\beta u + \nabla^\alpha_\beta u\right)$$

$$= 1 + \nabla^\alpha_\alpha u + \frac{1}{2!}\begin{bmatrix} \nabla^{\alpha_1}_{\alpha_1} u & \nabla^{\alpha_1}_{\alpha_2} u \\ \nabla^{\alpha_2}_{\alpha_1} u & \nabla^{\alpha_2}_{\alpha_2} u \end{bmatrix} + \cdots + \frac{1}{m!}\begin{bmatrix} \nabla^{\alpha_1}_{\alpha_1} u & \cdot & \cdot & \cdot & \nabla^{\alpha_1}_{\alpha_m} u \\ \cdot & & & & \cdot \\ \cdot & & & & \cdot \\ \cdot & & & & \cdot \\ \nabla^{\alpha_m}_{\alpha_1} u & \cdot & \cdot & \cdot & \nabla^{\alpha_m}_{\alpha_m} u \end{bmatrix}$$

Setting

$$\varepsilon^{\alpha_1\cdots\alpha_k}_{\beta_1\cdots\beta_k} = \frac{1}{k!}\delta^{\alpha_1\cdots\alpha_k}_{\beta_1\cdots\beta_k}$$

where $\delta^{\alpha_1\cdots\alpha_k}_{\beta_1\cdots\beta_k}$ is the Kronecker tensor, we have

$$\mathcal{M}_g[u] = 1 + \sum_{k=1}^{m} \varepsilon^{\alpha_1\cdots\alpha_k}_{\beta_1\cdots\beta_k} \nabla^{\beta_1}_{\alpha_1} u \cdots \nabla^{\beta_k}_{\alpha_k} u$$

It is well-known (See, e.g. Aubin [11]) that $\mathcal{M}_g[u]$ has a variational structure; namely, it is the Euler-Lagrange operator of the functional

$$J(u) = \int_M L(x, u, u_{,\alpha\overline{\beta}})dv_g$$

where the Lagrange L is given by

$$L[u] = L(x, u, u_{,\alpha,\overline{\beta}}) = \int_0^1 u\mathcal{M}_g[su]ds$$

(See Lemma 3.1 below for a precise description of the first variation of J.)

We are going to follow the steps sketched in the introduction to derive an integrability condition for solution of (3.1). The first step, which is the determination of the symmetries group for the homogeneous Monge-Ampère equation, has been carried in the last section. The next step is to verify which infinitesimal symmetry is in fact variational or divergence symmetry for J. Following a crucial step in the proof of Noether's theorem on conservation laws we examine the expression

$$pr^{(2)}v(L[u])+L[u]\mathrm{div}_g\xi\equiv\frac{\partial}{\partial\varepsilon}\Big|_{\varepsilon=0}L[\tilde{u}]+L[u]\mathrm{div}_g\xi$$

$$=\nabla_{\overline{\beta}}\nabla_\alpha Q\frac{\partial L}{\partial u_{\alpha\overline{\beta}}}+Q\frac{\partial L}{\partial u}+\nabla_\alpha(L\xi^\alpha)+\nabla_{\overline{\alpha}}(L\xi^{\overline{\alpha}}) \tag{3.2}$$

where $Q=\phi-\xi^\alpha u_\alpha-\xi^{\overline{\alpha}}u_{\overline{\alpha}}$, and v is any infinitesimal symmetry of $\mathcal{M}_g[u]=0$. According to Proposition 2.1 we can choose v to be

$$\xi^\alpha(x)\frac{\partial}{\partial z^\alpha}+\xi^{\overline{\alpha}}(x)\frac{\partial}{\partial z^{\overline{\alpha}}}+(-\sigma)\frac{\partial}{\partial u}$$

where ξ^i, $i=1,\cdots,2m$, is a real holomorphic vector field on M and σ is determined by ξ from (1.3).

Lemma 3.1 *For any function η on M,*

$$\nabla_{\overline{\beta}}\nabla_\alpha\eta\frac{\partial L}{\partial u_{\alpha\overline{\beta}}}+\eta\frac{\partial L}{\partial u}=\eta\mathcal{M}_g[u]+\nabla_{\beta_1}\Big(u\sum_k\frac{k}{k+1}\varepsilon^{\beta_1\cdots\beta_k}_{\alpha_1\cdots\alpha_k}\nabla^{\alpha_1}\eta\nabla^{\alpha_2}_{\beta_2}u\cdots\nabla^{\alpha_k}_{\beta_k}u\Big)$$

$$-\nabla^{\alpha_1}\Big(\eta\sum_k\frac{k}{k+1}\varepsilon^{\beta_1\cdots\beta_k}_{\alpha_1\cdots\alpha_k}\nabla_{\beta_1}u\nabla^{\alpha_2}_{\beta_2}u\cdots\nabla^{\alpha_k}_{\beta_k}u\Big)$$

Proof

$$\nabla_{\overline{\beta}}\nabla_\alpha\eta\frac{\partial L}{\partial u_{\alpha\overline{\beta}}}+\eta\frac{\partial L}{\partial u}$$

$$=\frac{d}{dt}\Big|_{t=0}\int_0^1(u+t\eta)\mathcal{M}_g[s(u+t\eta)]ds$$

$$=\frac{d}{dt}\Big|_{t=0}\int_0^1(u+t\eta)\Big(1+\sum_k\varepsilon^{\beta_1\cdots\beta_k}_{\alpha_1\cdots\alpha_k}\nabla^{\alpha_1}_{\beta_1}(u+t\eta)\cdots\nabla^{\alpha_k}_{\beta_k}(u+t\eta)s^k\Big)ds$$

$$=\frac{d}{dt}\Big|_{t=0}(u+t\eta)\Big(1+\sum_k\frac{1}{k+1}\varepsilon^{\beta_1\cdots\beta_k}_{\alpha_1\cdots\alpha_k}\nabla^{\alpha_1}_{\beta_1}(u+t\eta)\cdots\nabla^{\alpha_k}_{\beta_k}(u+t\eta)\Big)$$

$$=\eta\Big(1+\sum_k\frac{1}{k+1}\varepsilon^{\beta_1\cdots\beta_k}_{\alpha_1\cdots\alpha_k}\nabla^{\alpha_1}_{\beta_1}u\nabla^{\alpha_2}_{\beta_2}u\cdots\nabla^{\alpha_k}_{\beta_k}u\Big)$$

$$+u\sum_k \frac{k}{k+1}\varepsilon^{\beta_1\cdots\beta_k}_{\alpha_1\cdots\alpha_k}\nabla^{\alpha_1}_{\beta_1}\eta\nabla^{\alpha_2}_{\beta_2}u\cdots\nabla^{\alpha_k}_{\beta_k}u$$

$$=\eta\mathcal{M}_g[u]+\nabla_{\beta_1}\Big(u\sum_k \frac{k}{k+1}\varepsilon^{\beta_1\cdots\beta_k}_{\alpha_1\cdots\alpha_k}\nabla^{\alpha_1}_{\beta_1}\eta\nabla^{\alpha_2}_{\beta_2}u\cdots\nabla^{\alpha_k}_{\beta_k}u\Big)$$

$$-\nabla^{\alpha_1}\Big(\eta\sum_k \frac{k}{k+1}\varepsilon^{\beta_1\cdots\beta_k}_{\alpha_1\cdots\alpha_k}\nabla_{\beta_1}u\nabla^{\alpha_2}_{\beta_2}u\cdots\nabla^{\alpha_k}_{\beta_k}u\Big)$$

as claimed.

Lemma 3.2 *We have, for* $Q=-\sigma-\xi^\alpha u_\alpha-\xi^{\overline{\alpha}}u_{\overline{\alpha}}$,

$$\nabla_{\overline{\beta}}\nabla_\alpha Q\frac{\partial L}{\partial u_{\alpha\overline{\beta}}}+Q\frac{\partial L}{\partial u}+\mathrm{div}_g(L\xi)$$

$$=-\sigma+\nabla_{\beta_1}\Big(u\sum_k \frac{1}{k+1}\varepsilon^{\beta_1\cdots\beta_k}_{\alpha_1\cdots\alpha_k}\nabla^{\alpha_1}\sigma\nabla^{\alpha_2}_{\beta_2}u\cdots\nabla^{\alpha_k}_{\beta_k}u\Big)$$

$$-\nabla^{\alpha_1}\Big(\sigma\sum_k \frac{1}{k+1}\varepsilon^{\beta_1\cdots\beta_k}_{\alpha_1\cdots\alpha_k}\nabla_{\beta_1}u\nabla^{\alpha_2}_{\beta_2}u\cdots\nabla^{\alpha_k}_{\beta_k}u\Big) \qquad (3.3)$$

Proof For simplicity we shall first consider the (1.0)-type part of ξ and establish (3.3) for $\xi=(\xi^1,\cdots,\xi^m,0\cdots0)$ and $Q=-\rho-\xi^\alpha u_\alpha$.

$$\nabla_{\overline{\beta}}\nabla_\alpha Q\frac{\partial L}{\partial u_{\alpha\overline{\beta}}}+Q\frac{\partial L}{\partial u}+\mathrm{div}_g(L\xi)$$

$$=(-\rho_{\alpha\overline{\beta}}-(\nabla_\alpha\xi^\gamma)u_{\gamma\overline{\beta}}-\xi^\gamma\nabla_\alpha u_{\gamma\overline{\beta}})\int_0^1\frac{uc^{\alpha\overline{\beta}}(s)}{\det(g_{\alpha\overline{\beta}})}ds$$

$$+(-\rho-\xi^\gamma u_\gamma)\int_0^1\mathcal{M}_g[su]ds+\nabla_\alpha\xi^\alpha\int_0^1 u\mathcal{M}_g[su]ds$$

$$+\xi^\alpha u_\alpha\int_0^1\mathcal{M}_g[su]ds+\xi^\gamma\nabla_\gamma u_{\alpha\overline{\beta}}\int_0^1\frac{uc^{\alpha\overline{\beta}}(s)}{\det(g_{\alpha\overline{\beta}})}ds$$

$$=\int_0^1(-\rho_{\alpha\overline{\beta}}s+(\nabla_\alpha\xi^\gamma)g_{\gamma\overline{\beta}}-\nabla_\alpha\xi^\gamma(g_{\gamma\overline{\beta}}+su_{\gamma\overline{\beta}}))\frac{uc^{\alpha\overline{\beta}}(s)}{\det(g_{\alpha\overline{\beta}})}ds$$

$$-\rho\int_0^1\mathcal{M}_g[su]ds+\nabla_\alpha\xi^\alpha\int_0^1 u\mathcal{M}_g[su]ds$$

where we have used the fact $\nabla_\alpha u_{\gamma\overline{\beta}}=\nabla_\gamma u_{\alpha\overline{\beta}}$ on a Kähler manifold.

Now, by the Hodge decomposition (1.2)

$$g_{\gamma\overline{\beta}}\nabla_\alpha\xi^\gamma=\nabla_\alpha\Big(\frac{\partial\rho}{\partial z^{\overline{\beta}}}+(H\alpha)_{\overline{\beta}}\Big)=\frac{\partial^2\rho}{\partial z^\alpha\partial z^{\overline{\beta}}}+d'(H\alpha)_{\alpha\overline{\beta}}=\rho_{\alpha\overline{\beta}}$$

since

$$d'(H\alpha) = d'(H\alpha)_{\alpha\overline{\beta}} dz^{\alpha} \wedge dz^{\overline{\beta}} = 0$$

Consequently the above expression is equal to

$$\rho_{\alpha\overline{\beta}} \int_0^1 \frac{uc^{\alpha\overline{\beta}}(s)}{\det(g_{\alpha\overline{\beta}})} ds + (-\rho)_{\alpha\overline{\beta}} \int_0^1 \frac{uc^{\alpha\overline{\beta}}(s)}{\det(g_{\alpha\overline{\beta}})} ds + (-\rho) \int_0^1 \mathcal{M}_g[su] ds$$

Denote the first term of this expression by I and the rest by II. We have, by Lemma 3.1,

$$\begin{aligned} II &= -\left(\nabla_{\overline{\beta}} \nabla_{\alpha} \rho \frac{\partial L}{\partial u_{\alpha\overline{\beta}}} + \rho \frac{\partial L}{\partial u}\right) \\ &= -\rho \mathcal{M}_g[u] + \nabla^{\alpha_1}\left(\rho \sum_k \frac{k}{k+1} \varepsilon^{\beta_1\cdots\beta_k}_{\alpha_1\cdots\alpha_k} \nabla_{\beta_1} u \nabla^{\alpha_2}_{\beta_2} u \cdots \nabla^{\alpha_k}_{\beta_k} u\right) \\ &\quad - \nabla_{\beta_1}\left(u \sum_k \frac{k}{k+1} \varepsilon^{\beta_1\cdots\beta_k}_{\alpha_1\cdots\alpha_k} \nabla^{\alpha_1} \rho \nabla^{\alpha_2}_{\beta_2} u \cdots \nabla^{\alpha_k}_{\beta_k} u\right) \end{aligned}$$

On the other hand,

$$\begin{aligned} I &= \int_0^1 u \frac{d}{dt}\Big|_{t=0} \frac{\det(g_{\alpha\overline{\beta}} + s(u_{\alpha\overline{\beta}} + ts^{-1}\rho_{\alpha\overline{\beta}}))}{\det(g_{\alpha\overline{\beta}})} ds \\ &= \int_0^1 u \frac{d}{dt}\Big|_{t=0} \left(1 + \sum \varepsilon^{\beta_1\cdots\beta_k}_{\alpha_1\cdots\alpha_k} \nabla^{\alpha_1}_{\beta_1}\left(u + \frac{t\rho}{s}\right) \cdots \nabla^{\alpha_k}_{\beta_k}\left(u + \frac{t\rho}{s}\right) s^k\right) ds \\ &= \sum_k u \varepsilon^{\beta_1\cdots\beta_k}_{\alpha_1\cdots\alpha_k} \nabla^{\alpha_1}_{\beta_1} \rho \nabla^{\alpha_2}_{\beta_2} u \cdots \nabla^{\alpha_k}_{\beta_k} u \\ &= \rho \sum_k \varepsilon^{\beta_1\cdots\beta_k}_{\alpha_1\cdots\alpha_k} \nabla^{\alpha_1}_{\beta_1} u \nabla^{\alpha_2}_{\beta_2} u \cdots \nabla^{\alpha_k}_{\beta_2} u + \nabla_{\beta_1}\left(u \sum_k \varepsilon^{\beta_1\cdots\beta_k}_{\alpha_1\cdots\alpha_k} \nabla^{\alpha_1} \rho \nabla^{\alpha_2}_{\beta_2} u \cdots \nabla^{\alpha_k}_{\beta_k} u\right) \\ &\quad - \nabla^{\alpha_1}\left(\rho \sum_k \varepsilon^{\beta_1\cdots\beta_k}_{\alpha_1\cdots\alpha_k} \nabla_{\beta_1} u \nabla^{\alpha_2}_{\beta_2} u \cdots \nabla^{\alpha_k}_{\beta_k} u\right) \end{aligned}$$

Adding Terms I and II together yields (3.3). So we have proved Lemma 3.2 when ξ is of the type $(1,0)$. A similar identity can be obtained for the $(0,1)$-type part of ξ. By putting these two identities together we obtain (3.3).

It follows from Lemma 3.1 and Lemma 3.2 the following basic identity holds

$$0 = Q\mathcal{M}_g[u] + \sigma + \operatorname{div}_g(L\xi + \vec{A}) \tag{3.4}$$

where

$$\vec{A} = u \sum_k \varepsilon^{\beta_1\cdots\beta_k}_{\alpha_1\cdots\alpha_k} \nabla^{\alpha_2}_{\beta_2} u \cdots \nabla^{\alpha_k}_{\beta_k} u \left(\frac{k}{k+1} \nabla^{\alpha_1} Q - \frac{1}{k+1} \nabla^{\alpha_1} \sigma\right)$$

$$-g^{\overline{\alpha}\alpha_1}\sum_k \varepsilon^{\beta_1\cdots\beta_k}_{\alpha_1\cdots\alpha_k}\nabla_{\beta_1}u\nabla^{\alpha_2}_{\beta_2}u\cdots\nabla^{\alpha_k}_{\beta_k}u\Big(\frac{k}{k+1}Q-\frac{1}{k+1}\sigma\Big)$$

for any infinitesimal symmetry $v=\xi^i\partial/\partial x^i+(-\sigma)\partial/\partial u$ and any smooth function u. When u solves $\mathcal{M}_g[u]=0$ and σ can be expressed in the form $\mathrm{div}_g\vec{B}$ for some vector $\vec{B}$, (3.4) shows that $L\xi+\vec{A}-\vec{B}$ is a conservation law for the Lagrangian L. On the other hand, when u is a solution of the nonhomogeneous equation (3.1), (3.4) yields

$$0=-\sigma f(x,u)+\mathrm{div}_g\xi F(x,u)+\xi^jF_{x_j}(x,u)+\sigma+\mathrm{div}_g((L-F)\xi+\vec{A}) \tag{3.5}$$

where $F(x,u)$ is a primitive function of $f(x,u)$. Recall that

$$\int_M\sigma=0$$

an integration of (3.5) over M gives

$$\int_M\xi^jF_{x_j}(x,u)=\int_M(\sigma f(x,u)-\mathrm{div}_g\xi F(x,u)) \tag{3.6}$$

Proposition 3.3 *Let $\xi^j\partial/\partial x^j$, $j=1,\cdots,2m$, be a real holomorphic vector field on M and σ is determined by* (1.3)*. Then* (3.6) *holds for any smooth solution u of* (3.1)*. In particular, if M is a compact Kähler-Einstein manifold whose Kähler form $\omega=(\sqrt{-1}/2\pi)g_{\alpha\overline{\beta}}dz^\alpha\wedge dz^{\overline{\beta}}$ represents the first Chern class of M, then*

$$\int_M\xi^jF_{x_j}(x,u)=-\int_M\mathrm{div}_g\xi(f(x,u)+F(x,u)) \tag{3.7}$$

Proof Let M be a Kähler-Einstein manifold. Let Λ be the complex vector space that consists of all complex-valued smooth functions satisfying $\Delta_g\varphi=2\varphi$. Then according to [8], Λ is isomorphic to the subalgebra of the all real vector fields of $\mathfrak{h}(M)$ through the correspondence

$$\varphi\mapsto g^{\alpha\overline{\beta}}\varphi_{\overline{\beta}}\frac{\partial}{\partial z^\alpha}+g^{\overline{\beta}\alpha}\overline{\varphi}_{\overline{\beta}}\frac{\partial}{\partial z^{\overline{\alpha}}}$$

Henceforth, as can be seen from (1.2),

$$\alpha=d''\rho\quad\text{with }\Delta\rho=2\rho,\quad 2\delta''\alpha=2\delta''d''\rho=\Delta\rho=2\rho$$

So we have

$$\rho=-\nabla_\alpha\xi^\alpha$$

and

$$\sigma=-\nabla_\alpha\xi^\alpha-\Delta_{\overline{\alpha}}\xi^{\overline{\alpha}}$$

Putting this into (3.6) we get (3.7).

Corollary 3.4 ([6]) *Suppose that M is a compact Kähler-Einstein manifold whose Kähler form represents the first Chern class of M. Then for any solution of the equation*

$$\mathcal{M}_g[u] = e^{-u+\phi(x)}$$

we have

$$\int_M \xi^j \frac{\partial \phi}{\partial x^j} e^{-u+\phi} = 0$$

for any real holomorphic vector field ξ.

References

[1] Chou K.S. and Zhu X.P., Variational identities and conservation laws, preprint 1997.

[2] Chou K.S. and Zhu X.P., A characterization of the Panetz operator, preprint 1997.

[3] Aubin T., Equations du type de Monge-Ampère sur les variétés Kähleriennes compacts, *C. R. Acad. Sci.,* Paris **283** (1976), 119–121.

[4] Yau S.T., On the Ricci curvature of a compact Kähler manifold and the complex Monge-Ampère equation I, *Comm. Pure Applied Math.,* **31** (1987), 339–441.

[5] Tian G., On Calabi's conjecture for complex surfaces with positive first Chern Class, *Invent. Math.,* **101** (1990), 101–172.

[6] Futaki A., An obstruction to the existence of Einstein-Kähler matrices, *Invent Math.,* **73** (1983), 437–443.

[7] Tian G., Kähler-Einstein metrices with positive scalar curvature, preprint 1997.

[8] Lichnerowicz A., Géometrie des Groupes de Transformations, Paris, Dunod 1958.

[9] Olver P., Applications of Lie Groups to Differential Equations, Springer-Verlag, New York 1986.

[10] Ovsiannikov L.V., Group Analysis of Differential Equations, Nauka, Moscow 1978. (English translation published by Academic Press, New York 1982).

[11] Aubin T., Réduction du cas positif de l'équation de Monge-Ampère sur les variétés Kähleriennes compacts à la démonstration d'une inéqalité, *J. Functional Anal.,* **57** (1984), 143–153.