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Some Results on the Stability of Non-classical Shock Waves

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Dedicated to Professor Ding Xiaxi on the occasion of his 70th birthday

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Abstract Part 1 of this paper establishes the infinite-time stability of a class of over-compressive viscous shock waves; the equations studied here are a mathematical analogue of those of magnetohydrodynamics. Part 2 communicates a rather general short-time stability result for undercompressive shock waves in several space dimensions; technically, this is an easy extension of Majda's corresponding result for Laxian shock waves.

Key Words Conservation laws, shock waves, stability, overcompressive, under-compressive.

Classification 35L, 35K, 76W.

1. Infinite-time Stability of Non-classical Viscous Shock Waves

A traveling viscous shock wave solution

$$u^*(x,t) = \phi^*(x - st), \quad \phi^*(\pm\infty) = u^\pm \tag{1}$$

of a "viscous" system of n conservation laws

$$u_t + (f(u))_x = (B(u)u_x)_x \tag{2}$$

is called stable for infinite time if with some appropriate norm $\|\cdot\|$ and some $\delta > 0$, the following holds for any perturbation $\overline{u}_0 : \mathbf{R} \to \mathbf{R}^n$: If $\|\overline{u}_0\| < \delta$, then the solution u of (2) with data

$$u(x,0) = \phi^*(x) + \overline{u}_0(x), \quad x \in \mathbf{R} \tag{3}$$

exists for all times $t > 0$ and converges in the sense

$$\lim_{t\to\infty} \sup_{x\in\mathbf{R}} |u(x,t) - \phi(x - st)| = 0 \tag{4}$$

to another viscous shock wave of profile ϕ with the same end states $\phi(\pm\infty) = u^\pm$. For classical shock waves, stability in this sense has been proved by Goodman, Matsumura and Nishihara, Liu, Szepessy and Xin in [1–4] under various assumptions; certain non-classical shock waves were shown to be stable by Liu and co-authors in [5–7]. The purpose of this paper consists in establishing an infinite-time stability result of certain non-classical viscous shock waves in the "cylindrical model" introduced by the author in [8]. This model (see also [9]) is given by the equations

$$\begin{aligned} y_t + (zy)_x &= \mu y_{xx} \\ z_t + \frac{1}{2}(|y|^2 + z^2)_x &= \zeta z_{xx} \end{aligned} \tag{5}$$

where $x \in \mathbf{R}$, $t \in [0, \infty)$, $y(x,t) \in \mathbf{R}^{n-1}$ $(n \geq 3)$, $z(x,t) \in \mathbf{R}$, and $\mu, \zeta > 0$. We abbreviate (5) as (2), with $u \equiv (y, z)$ and $B \equiv \operatorname{diag}(\mu, \cdots, \mu, \zeta) \in \mathbf{R}^{n\times n}$. The inviscid part of (5) is hyperbolic, with characteristic speeds

$$\lambda_{3/1}(u) = z \pm |y|, \quad \lambda_2(u) = z \tag{6}$$

We consider the maximally overcompressive case, i.e., shock waves which satisfy

$$\lambda_1(u^-) > s > \lambda_3(u^+) \tag{7}$$

A solution $\phi : \mathbf{R} \to \mathbf{R}^n$ of the boundary value problem

$$B\phi' = f \circ \phi - s\phi - q, \quad \phi(\pm\infty) = u^\pm \tag{8}$$

with

$$q = f(u^-) - su^- = f(u^+) - su^+ \tag{9}$$

will be called a profile for the pair (u^-, u^+). We write $q = (q_1, q_2)$, $q_1 \in \mathbf{R}^{n-1}$, and, w. l. o. g., fix from now on

$$s = 0 \quad \text{and} \quad q_2 > 0 \tag{10}$$

Lemma 1 *If $q_1 \in \mathbf{R}^{n-1}$ is sufficiently small, then there exists* (i) *a unique pair (u^-, u^+) with* (7), (9), *and* (ii) *a unique profile ϕ_0^* for (u^-, u^+) with and $\phi_0^*(0) = 0$.*

Proof (i) The equation $f(u) - su = q$ reads

$$\begin{aligned} zy &= q_1 \\ \frac{1}{2}(|y|^2 + z^2) &= q_2 \end{aligned} \tag{11}$$

If $q_1 = 0$, then its solution set consists of the two points

$$u^\pm = (0, \mp(2q_2)^{1/2})$$

and the circle

$$\chi = \{((2q_2)^{1/2}\theta, 0), \theta \in S^{n-2}\}$$

Points $u \in \chi$ have $\lambda_2(u) = s$ and can thus satisfy neither $\lambda_1(u) > s$ nor $\lambda_3(u) < s$. For u^- and u^+, (7) holds. Now perturb q_1 away from 0. (7) and the Implicit Function Theorem imply that the points u^-, u^+ perturb regularly. While χ does not, nevertheless no new solutions of (11) are generated near χ which would satisfy $\lambda_1(u) > s$ or $\lambda_3(u) < s$. (ii) In case $q_1 = 0$, the existence of ϕ_0^* is obvious, since u^-, u^+ are (the only) fixed points of (8) contained in the invariant line $y = 0$. In other words, the point 0 is an element of the unstable manifold of u^- as well of the stable manifold of u^+. As these are both open sets, this property persists under perturbation of q_1 away from 0.

Theorem 1 *consider u^-, u^+ as in Lemma 1 and a corresponding profile ϕ^* and let*

$$m^* = \int_{-\infty}^{\infty} (\phi^*(x) - \phi_0^*(x))dx \tag{12}$$

If now $|q_1|$ and $|m^*|$ are sufficiently small, then there exists a $\delta > 0$ such that the following holds: If

$$\overline{m} = \int_{-\infty}^{\infty} \overline{u}_0(x)dx \tag{13}$$

satisfies

$$|\overline{m}| < \delta$$

then a unique (other) profile ϕ for (u^-, u^+) is determined through the relation

$$\int_{-\infty}^{\infty} (\phi(x) - \phi^*(x))dx = \int_{-\infty}^{\infty} \overline{u}_0(x)dx \tag{14}$$

If $\overline{u}_0$ is furthermore also small in the sense that the function U_0 given by

$$U_0(x) = \int_{-\infty}^{x} (\phi^*(\tilde{x}) + \overline{u}_0(\tilde{x}) - \phi(\tilde{x}))d\tilde{x} \tag{15}$$

satisfies

$$\|U_0\|_2 < \delta$$

then the solution u of (5) with data (3) converges in the sense (4) to ϕ. Briefly speaking, (1) is stable for infinite time.

A principal motivation to study this problem is the fact that in the analogy between the cylindrical model and the equations of MHD, the shocks whose stability we prove here correspond to almost gasdynamic MHD shocks. The rest of this paper is devoted to proving Theorem 1.

Lemma 2 *Consider u^-, u^+ and ϕ_0^* as in Lemma 4.2. If $m \in \mathbf{R}^n$ is sufficiently small, then there exists a unique profile ϕ with*

$$\int_{-\infty}^{\infty} (\phi(x) - \phi_0^*(x))dx = m$$

For a proof of this, we invoke the following abstract observation, which is elementary from the point of view of invariant manifold theory [10].

Lemma 3 *Let Φ be the flow of a smooth vector field on a smooth submanifold $\mathcal{S}$ of $\mathbf{R}^n$, I^- and I^+ normally hyperbolic invariant manifolds for Φ, and $\mathcal{M} \subset \mathcal{S}$ a submanifold which is contained in the intersection of the unstable manifold of I^- and the stable manifold of I^+. Then for any $u \in \mathcal{M}$,*

$$\gamma(u) := \int_{-\infty}^{\infty} \left(\frac{\partial \Phi}{\partial u}(u,t) \Big| T_u\mathcal{M} \right) dt \ \in \ Lin(T_u\mathcal{M}, \mathbf{R}^n)$$

exists, and the mapping

$$\gamma : \mathcal{M} \to Lin(T\mathcal{M}, \mathbf{R}^n)$$

is smooth.

Consider first the flow Φ of (8) with (10) and $q_1 = 0$. We know already that a neighborhood of the point $0 \in \mathbf{R}^n$ is contained in the unstable manifold of u^- and in the stable manifold of u^+. We compute

$$\gamma(0) = \int_{-\infty}^{\infty} \frac{\partial \Phi}{\partial u}(0,t)dt$$

Along the profile $\phi_0^* = \Phi(0,.)$, the derivative $\dfrac{\partial \Phi}{\partial u}(0,.)$ satisfies the variation equation

$$\frac{\partial}{\partial t}\frac{\partial \Phi}{\partial u}(0,t) = B^{-1}Df(\Phi(0,t))\frac{\partial \Phi}{\partial u}(0,t)$$

with

$$\frac{\partial \Phi}{\partial u}(0,0) = I_n$$

As

$$\phi(\mathbf{R}) \subset \{0\} \times \mathbf{R}$$

and

$$Df(0,z)) = zI_n$$

is diagonal, we see that $\gamma(0)$ is a diagonal, positive definite matrix. In particular, we have

$$\det \gamma(0) \neq 0$$

According to the Implicit Function Theorem, this proves Lemma 2 for the case $q_1 = 0$. To prove it for small $q_1 \neq 0$, we consider the augmented system

$$\begin{aligned} B\phi' &= f \circ \phi - s\phi - q \\ q' &= 0 \end{aligned} \tag{16}$$

defined on $\mathbf{R}^n \times Q$, Q a sufficiently small neighborhood of $q_* = (0, q_2)$. For $q \in Q$, u^- and u^+ are smooth functions of q. Obviously,

$$I^\pm = \{(u^\pm(q), q) : q \in Q\}$$

are normally hyperbolic invariant manifolds for (16). Applying Lemma 3 to this new situation, we see that the map

$$(u, q) \mapsto \widehat{m}(u, q) = \int_{-\infty}^{\infty} (\phi(u, q, t) - \phi(0, q, t))dt$$

is differentiable. As

$$\frac{\partial \widehat{m}}{\partial u}(0, q_*) = \gamma(0)$$

is non-singular, there exists a smooth function $u = \widehat{u}(m, q)$ on a neighborhood of $(0, q_*)$ such that

$$\int_{-\infty}^{\infty} (\phi(u, q, t) - \phi(0, q, t))dt = m \Leftrightarrow u = \widehat{u}(m, q)$$

The proof of Lemma 2 is complete.

We keep investigating the same situation. Consider a profile ϕ^* and a perturbation $\overline{u}_0$ such that the quantities m^* and $\overline{m}$ given by (12), (13) are small. Using Lemma 2, we find a unique profile ϕ with (14). It remains to show that if the function U_0 given by (15) is sufficiently small in $H^2(\mathbf{R})$, then the solution of (5) with initial data $\phi^* + \overline{u}_0$ satisfies (4). Instead of u, one considers the integrated perturbation

$$U(x, t) = \int_{-\infty}^{x} (u(\widetilde{x}, t) - \phi(\widetilde{x}))d\widetilde{x}$$

U satisfies the equation

$$U_t(x, t) + A(\phi(x))U_x(x, t) + Q(\phi(x), U_x(x, t)) = BU_{xx}(x, t) \tag{17}$$

where $A \equiv Df$ and Q is a quadratic remainder satisfying

$$Q(\phi, z) \leq |z|^2 \tag{18}$$

System (17) is uniformly parabolic. Thus [11, 12], it has a unique solution $U \in C^0([0, T], H^2(\mathbf{R}))$ for some time $T > 0$ with (arbitrary) data $U_0 \in H^2(\mathbf{R})$, as soon as $\|U_0\|_2$ is sufficienty small. For given U_0 and any $\beta > 0$, let $T_\beta > 0$ be the supremum of all such T which moreover satisfy

$$\sup_{\mathbf{R}\times[0,T]} \{|U|, |U_x|\} \le \beta \tag{19}$$

A standard short time estimate shows that for every $\beta > 0$, there are constants $\gamma_\beta, \tau_\beta > 0$ such that

$$\|U(.,0)\|_2 \le \gamma_\beta \Rightarrow T_\beta \ge \tau_\beta \tag{20}$$

For such U_0, well-known considerations (e.g. [1]) on (17) yield the energy estimate

$$\|U_x(.,T)\|_1^2 \le c\Big(\|U_x(.,0)\|_1^2 + \int_0^T \|U_x(.,t)\|_1^2 dt\Big) \tag{21}$$

for all $T \in [0, T_\beta)$ with some constant $c > 0$ which does not depend on T. Suppose now one has also

Lemma 4 *There is a $\beta_1 > 0$ such that for all $\beta \in (0, \beta_1)$ and U_0 with $\|U_0\|_2 \le \gamma_\beta$, the solution U of (4.20) with data U_0 satisfies*

$$\|U(.,T)\|_0^2 + \int_0^T \|U_x(.,t)\|_0^2 dt \le c\|U(.,0)\|_0^2 \tag{22}$$

for all $T \in [0, T_\beta)$ with some $c > 0$ which does not depend on T.

Then one can combine (21) and (22) to find β, δ such that

$$\|U_0\|_2 \le \delta \Rightarrow \|U(.,T)\|_2 \le \gamma_\beta \quad \text{for all } T \in [0, T_\beta)$$

This, however, means by (20) that for such data $T_\beta = \infty$. Then, (22) implies

$$\int_0^\infty \|U_x(.,t)\|_0^2 dt < \infty \tag{23}$$

Through an integration step of the form

$$|U_x(x,T)|^2 \le c(T_1, T_2) \int_{T-T_2}^{T-T_1} \|U_x(.,t)\|_0^2 dt, \quad T > T_2 > T_1 > 0$$

(23) implies the desired decay result

$$\lim_{t\to\infty} \sup_{x\in\mathbf{R}} |U_x(x,t)| = 0$$

It remains to show Lemma 4. To do so, we let $w : \mathbf{R} \to (0,\infty)$ be a smooth weight function (following [13]) and define $V = V(x,t)$ through

$$U(x,t) = w(x)V(x,t) \tag{24}$$

Substituting (24) into (17) and dividing by w yields

$$V_t + \frac{1}{w}(A(\phi))(wV)_x + \frac{1}{w}Q(\phi,(wV)_x) = B\Big(V_{xx} + 2\frac{w'}{w}V_x + \frac{w''}{w}V\Big)$$

Multiplying by V^t and integrate with respect to x and t, we obtain

$$\frac{1}{2}\int_{-\infty}^{\infty}|V(x,T)|^2dx + \int_0^T\int_{-\infty}^{\infty}\frac{1}{w}V^t(A(\phi))(wV)_x dxdt$$

$$+\int_0^T\int_{-\infty}^{\infty}\frac{1}{w}V^tQ(\phi,(wV)_x)dxdt$$

$$= \frac{1}{2}\int_{-\infty}^{\infty}|V(x,0)|^2dx - \int_0^T\int_{-\infty}^{\infty}V^tBV_{xx}dxdt$$

$$+\int_0^T\int_{-\infty}^{\infty}V^tB\Big(2\frac{w'}{w}V_x + \frac{w''}{w}V\Big)dxdt$$

Integrating by parts, the second term on the left hand side of this equation becomes

$$\int_0^T\int_{-\infty}^{\infty}V^t\Big(-\frac{1}{2}(A\circ\phi)' + \frac{w'}{w}(A(\phi))\Big)Vdxdt$$

Using (18) and (19), we estimate the third term on the left hand side as

$$\geq -\int_0^T\int_{-\infty}^{\infty}\frac{1}{w}|V||(wV)_x|^2dxdt$$

$$\geq -2\int_0^T\int_{-\infty}^{\infty}|wV|\Big(|V_x|^2 + \Big(\frac{w'}{w}\Big)^2|V|^2\Big)dxdt$$

$$\geq -2\beta\Big[\int_0^T\int_{-\infty}^{\infty}|V_x|^2dxdt + \int_0^T\int_{-\infty}^{\infty}\Big(\frac{w'}{w}\Big)^2|V|^2dxdt\Big]$$

The second term on the right hand side is

$$= -\int_0^T\int_{-\infty}^{\infty}V_x^tBV_xdxdt$$

$$\leq -\min\{\mu,\zeta\}\int_0^T\int_{-\infty}^{\infty}|V_x|^2dxdt$$

Finally, the last term on the right hand side can be written as

$$\int_0^T\int_{-\infty}^{\infty}\Big(\frac{w'}{w}\Big)^2V^tBVdxdt$$

Combining these observations, we obtain

$$\frac{1}{2}\int_{-\infty}^{\infty}|V(x,T)|^2dx$$

$$+\int_0^T\int_{-\infty}^{\infty} V^t\Big[-\frac{1}{2}(A\circ\phi')+\frac{w'}{w}A(\phi)-(\max\{\mu,\zeta\}+2\beta)\Big(\frac{w'}{w}\Big)^2 I\Big]V\,dxdt$$
$$+(\min\{\mu,\zeta\}-2\beta)\int_0^T\int_{-\infty}^{\infty}|V_x|^2dxdt$$
$$\le\frac{1}{2}\int_{-\infty}^{\infty}|V(x,0)|^2dx \tag{25}$$

Lemma 5 *Let $\kappa>0$. If q_1, m^*, and $\overline{m}$ are sufficiently small, and the weight w is chosen as solving*

$$\frac{w'(x)}{w(x)}=\omega(\phi(x))|\phi'(x)|,\quad x\in\mathbf{R}$$

and

$$w(0)=1$$

with $\omega:\mathbf{R}^n\to\mathbf{R}$ a certain appropriate function, then there are constants $c_1, c_2, c_3, k>0$ such that

$$c_1^{-1}\le w(x)\le c_1,\quad |w'(x)|\le c_2 \tag{26}$$

and

$$|w'(x)|\le c_3|\phi'(x)| \tag{27}$$

as well as

$$\overline{A}(x)\equiv-\frac{1}{2}\frac{d}{dx}(A(\phi(x)))+\frac{w'(x)}{w(x)}A(\phi(x))-\kappa\Big(\frac{w'(x)}{w(x)}\Big)^2 I\ge k|\phi'(x)|I \tag{28}$$

hold for all $x\in\mathbf{R}$.

Proof Consider first the special case $q_1=0$, $m^*=\overline{m}=0$. In this case,

$$\phi'(x)=|\phi'(x)|(0,-1)\in\mathbf{R}^{n-1}\times\mathbf{R}$$

and thus

$$-\frac{1}{2}\frac{d}{dx}(A(\phi(x)))=\frac{1}{2}|\phi'(x)|\frac{\partial A}{\partial z}(\phi(x))=\frac{1}{2}|\phi'(x)|I$$

Consequently, the conclusion of the lemma holds in this case with $\omega=0$, $w\equiv1$, $c_1=1$, $c_2=c_3=0$, $k=\frac{1}{2}$. By regular perturbation, there are numbers $k_1>0$ and $\overline{z}>0$ such that

$$A(\phi(x))\ge k_1 I\quad\text{if}\quad \check{\pi}(\phi(x))<-\overline{z}$$
$$A(\phi(x))\le -k_1 I\quad\text{if}\quad \check{\pi}(\phi(x))>\overline{z}$$

while

$$-\frac{1}{2}\frac{d}{dx}(A(\phi(x)))\ge\frac{1}{4}|\phi'(x)|I\quad\text{if}\quad -\overline{z}<\check{\pi}(\phi(x))<\overline{z}$$

if $|q_1|, |m^*|, |\overline{m}|$ are small enough. By possibly making these three quantities still smaller, the value $\overline{z}$ can, for every $k_2 > 0$, moreover be adjusted so that also

$$|\phi'(x)| < k_2 \quad \text{if} \quad |\bar{\pi}(\phi(x))| > \overline{z}$$

holds. Since ϕ stays uniformly in a bounded region and f is smooth, there trivially exists a constant $\widetilde{k} > 0$ such that

$$-\frac{1}{2}\frac{d}{dx}(A(\phi(x))) \geq -\widetilde{k}|\phi'(x)|I$$

We choose

$$\omega(y,z) = \begin{cases} 3\widetilde{k}/k_1, & z < -\overline{z} \\ 0, & -\overline{z} < z < \overline{z} \\ -3\widetilde{k}/k_1, & z > \overline{z} \end{cases}$$

and

$$k_2 = k_1^2/(9\kappa\widetilde{k})$$

For x with $|\bar{\pi}(\phi(x))| < \overline{z}$, these choices yield

$$\overline{A}(x) \geq (-\widetilde{k} + 3\widetilde{k} - 9\kappa\widetilde{k}^2 k_1^{-2} k_2)|\phi'(x)|I \geq \widetilde{k}|\phi(x)|I$$

Thus, (28) holds with

$$k = \min\left\{\widetilde{k}, \frac{1}{4}\right\}$$

Properties (26) and (27) are obvious, since $|\phi'|$ decays exponentially at both infinities. Lemma 5 is proved.

Using Lemma 5 in (25) and choosing β appropriately small, we obtain

$$\int_{-\infty}^{\infty} |V(x,T)|^2 dx + \int_0^T \int_{-\infty}^{\infty} (|\phi'(x)|\,|V(x,t)|)^2 dx dt + |V_x|^2 dx dt \leq \widetilde{c} \int_{-\infty}^{\infty} |V(x,0)|^2 dx$$

with some $\widetilde{c} > 0$. By means of (26), (27), this implies

$$\int_{-\infty}^{\infty} |w(x)V(x,T)|^2 dx + \int_0^T \int_{-\infty}^{\infty} |(w(x)V(x,t))_x|^2 dx dt \leq c \int_{-\infty}^{\infty} |w(x)V(x,0)|^2 dx$$

with some $c > 0$. Since $U = wV$, Estimate (22), and thus Theorem 1, are proved.

We conclude by a simple

Observation *Let $\Phi_B(u^-, u^+)$ be the set of all profiles for (u^-, u^+) with respect to B and set*

$$M_B(u^-, u^+) \equiv \left\{ m \in \mathbf{R}^n : \int_{-\infty}^{\infty} (\phi(x) - \phi^*(x)) = m \quad \textit{for some } \phi \in \Phi_B(u^-, u^+) \right\}$$

Then $M_B = \mathbf{R}^n$ for $q_1 = 0$ while M_B has finite width if $q_1 \neq 0$.

Proof For $q_1 = \widehat{q}_1\theta$ with $q_1 > 0$, all rest points lie in $U_\theta = \mathbf{R}\theta \times \mathbf{R}$ and the width of M_B in any direction transverse to U_θ is bounded. For $q_1 = 0$, the circle χ of further rest points mentioned in the proof of Lemma 1 makes M_B unbounded.

2. Persistence of Multidimensional Non-classical Shock Waves

Consider a hyperbolic system

$$\frac{\partial}{\partial t}u(x,t) + \sum_{j=1}^{d} \frac{\partial}{\partial x_j} F_j(u(t,x)) = 0 \tag{1}$$

of n conservation laws in d space variables — induced by a mapping $F = (F_1, \cdots, F_d) : U \to \mathbf{R}^{n\times d}$ of a convex open set $U \subset \mathbf{R}^n$ into the space of $n \times d$ matrices — and a "transition condition"

$$G(s, N, u^-, u^+) = 0 \tag{2}$$

— defined by some function $G : \mathbf{R} \times S^{d-1} \times U \times U \to \mathbf{R}^m$. For $(s_0, N_0, u_*) \in \mathbf{R} \times S^{d-1} \times U$ and $(\tau, \xi) \in \mathbf{C} \times N_0^\perp$, let $E^-(\tau, \xi; s_0, N_0, u_*)$ $[E^+(\tau, \xi; s_0, N_0, u_*)]$ denote the linear space of all $\mu \in \mathbf{C}^n$ for which the linear constant-coefficients problem

$$\frac{\partial}{\partial t}\tilde{u}(x,t) + \sum_{j=1}^{d} \frac{\partial F_j}{\partial u}(u_*)\frac{\partial}{\partial x_j}\tilde{u}(t,x) - s_0\frac{\partial \tilde{u}}{\partial x}N_0 = 0 \tag{3}$$

has a solution of the form

$$\tilde{u}(t,x) = \mu e^{i\xi x + \tau t}\widehat{u}(N_0^\top x), \quad \widehat{u} : \mathbf{R} \to \mathbf{C} \quad \text{with } \widehat{u}(-\infty) = 0[\widehat{u}(\infty) = 0] \tag{4}$$

Definition *A quadruple $q_0 = (s_0, N_0, u_0^-, u_0^+) \in \mathbf{R} \times S^{d-1} \times U \times U$ satisfies the uniform Lopatinski stability condition with respect to F and G if a $\gamma > 0$ exists such that for all $(\tau, \xi) \in \mathbf{C} \times N_0^\perp$ with $|\tau|^2 + |\xi|^2 = 1$ and $\Re\tau > 0$, the inequality*

$$\begin{aligned}&\left|\left(\tau\frac{\partial G}{\partial s}(q_0) + \left(i\xi \cdot \frac{\partial}{\partial N}\right)G(q_0)\right)\kappa + \frac{\partial G}{\partial u^-}(q_0)\mu^- + \frac{\partial G}{\partial u^+}(q_0)\mu^+\right|^2 \\ &\geq \gamma^2(|\kappa|^2 + |\mu^-|^2 + |\mu^+|^2)\end{aligned} \tag{5}$$

holds for arbitrary $(\kappa, \mu^-, \mu^+) \in \mathbf{C} \times E^-(\tau, \xi; s_0, N_0, u_0^-) \times E^+(\tau, \xi; s_0, N_0, u_0^+)$.

The first purpose of this note is to communicate this notion, together with the following related fact.

Theorem 2 *Consider a smooth closed hypersurface $\mathcal{M}_0 \subset \mathbf{R}^d$ which lies inside a compact ball $B(0, R_0) \subset \mathbf{R}^d$ and let Ω_0^-, Ω_0^+ denote the interior of $\mathcal{M}_0$ and the intersection of its exterior with $B(0, 4R_0)$. Assume*

(i) *System* (1) *satisfies certain structural conditions* (*cf. below*).

(ii) *There is a function* $u_0 : \mathbf{R}^d \to U$ *such that* $u_0|\Omega_0^\pm \in H^{k+1}(\Omega_0^\pm)$ *for a fixed integer* $k \geq 2\left[\frac{d}{2}\right] + 7$.

(iii) *There is a function* $\sigma_0 \in H^{k+1}(\mathcal{M}_0)$ *such that for each fixed* $\alpha \in \mathcal{M}_0$, *condition* (2) *holds with* $s = \sigma_0(\alpha)$ *and* $u^\pm = u_0^\pm(\alpha)$, *the limiting value of* u_0 *at the point* α *from within* $\Omega_0^\pm$, *respectively, and* $N = N_0(\alpha) \in S^{d-1}$ *the exterior normal of* $\mathcal{M}_0$ *at* α.

(iv) *For every* $\alpha \in \mathcal{M}_0$, *the quadruple* $(\sigma_0(\alpha), N_0(\alpha), u_0^-(\alpha), u_0^+(\alpha))$ *satisfies the uniform Lopatinski stability condition with respect to* F *and* G.

(v) σ_0 *and* u_0 *satisfy certain higher order compatibility conditions* (*cf. below*).

Then there exist a $T > 0$ *and a* H^{k+1} *manifold* $\mathcal{M} = \cup_{0\leq t \leq T}(\{t\} \times \mathcal{M}_t)$, *composed of closed hypersurfaces* $\mathcal{M}_t \subset B(0, 2R_0)$,*, and a function* $u : \Omega \equiv [0,T) \times B(0,3R_0) \to \mathbf{R}^n$ *such that the restrictions* $u \mid \Omega^\pm$ *to the exterior* Ω^+ *resp. interior* Ω^- *of* $\mathcal{M}$ *of* Ω *belong to* $H^k(\Omega^\pm)$ *and satisfy* (1) *in the classical sense*, u *assumees initial data* u_0 (*i.e.*, $u(0,.) = u_0$ *on* $B(0,3R_0)$) *and every quadruple* $(\sigma(\alpha), N(\alpha), u^-(\alpha), u^+(\alpha))$ — *with* $(\sigma(\alpha), N(\sigma))$ *the exterior normal* ($|N(\alpha)| = 1$) *to* $\mathcal{M}$ *at* α— *and* $u^\pm(\alpha)$ *the limiting values, within* $\Omega^\pm$, *of* u *at* α—, $\alpha \in \mathcal{M}$, *satisfies the transition condition* (2).

This theorem has been proved by Majda [14, 15] for the (most) important special case

$$G(s, N, u^-, u^+) \equiv (F(u^+) - F(u^-))N - s(u^+ - u^-); \tag{6}$$

in this case, (2) are the Rankine Hugoniot jump conditions, which express conservation across the discontinuity. At the technical level, our more general statement is an almost trivial side-remark: It is proved in completely the same way as Majda's original theorem*, and instead of addressing any of the many details of Majda's argumentation, we simply invite the reader to go back to [14, 15] and verify that this transfer readily works at every single one of its steps.

The result, however, is interesting in a number of further contexts, out of which cases of the form

$$G(s, N, u^-, u^+) = \begin{pmatrix} (F(u^+) - F(u^-))N - s(u^+ - u^-) \\ H(s, N, u^-, u^+) \end{pmatrix} \tag{7}$$

with $m > n$ seem particularly useful. This form corresponds to non-classical shock waves of a kind often referred to as "undercompressive"; the vanishing of H that transition condition (2) imposes in addition to the Rankine-Hugoniot conditions corresponds

* In particular, the structural and compatibility conditions in (i), (v) are the lengthy, but natural conditions mentioned in [14, 15].

to further extraneous requirements on the jump discontinuity: In these cases, conservation alone is not sufficient to determine the dynamics of the shock front, but these additional requirements restore wellposedness. The forthcoming parper [16] will contain examples of this type. One of these is provided by undercompressive shock waves for the Complex Burgers Equation, a system of two conservation laws in two space dimensions; the uniform Lopatinski stability condition is satisfied with the constraint $H = 0$ corresponding to the requirement that each planar shock wave $(\sigma(\alpha), N(\alpha), u^-(\alpha), u^+(\alpha))$, $\alpha \in \mathcal{M}$, has a viscous profile.

The second purpose of this note consists in relating the above observations back to the C^k type results obtained in [17] for the case of one space dimension. More specifically, we have:

Lemma 6 *In the case $d = 1$ (one space dimension), the uniform Lopatinski stability condition with respect to, say, f and g reduces to*

$$\mathbf{R}\frac{\partial g}{\partial s}(q_0) \oplus \frac{\partial g}{\partial u^-}(q_0)R^-(u_0^-, s_0) \oplus \frac{\partial g}{\partial u^+}(q_0)R^+(u_0^+, s_0) = \mathbf{R}^m \tag{8}$$

with

$$R^\pm(u, s) \equiv \sum_{\pm(\lambda - s) > 0} \ker{}_{\mathbf{R}}(f'(u) - \lambda I)$$

Note that independently of this justification as a uniform Lopatinski criterion, condition (8) arises naturally in the direct study — see [17] — of the one-dimensional case via the method of characteristics.

To establish the Lemma, we first fix $\tau \in \mathbf{C}$ with $|\tau| = 1$ and $\Re\tau > 0$, and determine $E^+ \equiv E^+(\tau; s_0, 1, u_0^+)$. $\mu \in \mathbf{C}^n$ belongs to E^+ if and only if there is a smooth function $\check{v} : \mathbf{R} \to \mathbf{C}$ with $\check{v}(+\infty) = 0$ such that

$$v(t, x) = \mu e^{\tau t}\check{v}(x) \tag{9}$$

satisfies

$$\frac{\partial v}{\partial t} + (f'(u_0^+) - s_0 I)\frac{\partial v}{\partial x} = 0 \tag{10}$$

Now, (10) with (9) is equivalent to

$$(f'(u_0^+) - \lambda I)\mu = 0, \quad \lambda = s_0 - \tau\frac{\check{v}(x)}{\check{v}'(x)}$$

i.e., (λ, μ) is an (eigenvalue, eigenvector)-pair of $f'(u_0^+)$ and

$$\check{v}(x) = \check{v}(0)e^{-\frac{\tau}{\lambda - s_0}x}$$

This is decaying for $x \to \infty$ if and only if λ is one of those eigenvalues of $f'(u_0^+)$ which are bigger than s_0. In other words,

$$E^+ = \sum_{\lambda > s_0} \ker_{\mathbf{C}} (f'(u_0^+) - \lambda I)$$

Analogously,

$$E^- = \sum_{\lambda > s_0} \ker_{\mathbf{C}} (f'(u_0^-) - \lambda I)$$

The uniform Lopatinski stability condition is thus equivalent to

$$\left| \kappa \frac{\partial g}{\partial s}(q_0) + \frac{\partial g}{\partial u^-}(q_0)\mu^- + \frac{\partial g}{\partial u^+}(q_0)\mu^+ \right|^2 \geq \gamma^2(|\kappa|^2 + |\mu^-|^2 + |\mu^+|^2) \tag{11}$$

holding for all

$$(\kappa, \mu^-, \mu^+) \in \mathbf{C} \times \sum_{\lambda < s_0} \ker_{\mathbf{C}}(f'(u_0^-) - \lambda I) \times \sum_{\lambda > s_0} \ker_{\mathbf{C}} (f'(u_0^+) - \lambda I)$$

But this is equivalent to (11) holding for all

$$(\kappa, \mu^-, \mu^+) \in \mathbf{R} \times \sum_{\lambda < s_0} \ker_{\mathbf{R}}(f'(u_0^-) - \lambda I) \times \sum_{\lambda > s_0} \ker_{\mathbf{R}} (f'(u_0^+) - \lambda I)$$

which in turn is the same as (8).

Remark While it is here that they first appear printed in a journal, the results of Sections 1 and 2 were first obtained in [18] and [19], respectively. The results of Section 2 were announced in [17, 18] and communicated at the Sixth International Conference on Nonlinear Hyperbolic Problems in Hong Kong, June 1996.

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