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EXISTENCE OF C^1-SOLUTIONS TO CERTAIN NON-UNIFORMLY DEGENERATE ELLIPTIC EQUATIONS*

Lee Junjie
(Center for Mathematical Sciences, Zhejiang University, Hangzhou 310027, China)
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Abstract We are concerned with the Dirichlet problem of

$$\begin{cases} \operatorname{div} A(x, Du) + B(x) = 0 & \text{in } \Omega \\ u = u_0 & \text{on } \partial\Omega \end{cases} \tag{0.1}$$

Here $\Omega \subset \mathbf{R}^N$ is a bounded domain, $A(x,p) = (A^1(x,p), \cdots, A^N(x,p))$ satisfies

$$\min\{|p|^{1+\alpha}, |p|^{1+\beta}\} \le A(x,p) \cdot p \le \alpha_0(|p|^{1+\alpha} + |p|^{1+\beta})$$

with $0 < \alpha \le \beta$.

We show that if A is Lipschitz, B and u_0 are bounded and $\beta < \max\left\{\frac{N+2}{N}\alpha + \frac{2}{N}, \alpha + 2\right\}$, then there exists a C^1-weak solution of (0.1).

Key Words Elliptic equation; non-uniformly degenerate.
Classification 35D05, 35J70.

1. Introduction and Statement of Main Results

Recently many authors have studied the existence and regularity of weak solutions for uniformly degenerate elliptic equations

$$\operatorname{div} A(x, u, Du) + B(x, u, Du) = 0 \quad \text{in } \Omega \subset \mathbf{R}^N \tag{1.1}$$

with structure conditions on the principal part

$$\lambda |p|^{\beta-1}|\xi|^2 \le \frac{\partial A^i}{\partial p_j}(x,z,p)\xi_i\xi_j \le \Lambda |p|^{\beta-1}|\xi|^2 \quad (\beta > 0) \tag{1.2}$$

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see for instance [1-7]. Under the additional hypotheses on A and B, these authors established the $C^{1,\alpha}$ regularity of weak solutions. Lieberman[8] has got similar results for more general equations; that is, the eigenvalues of the matrix $\left(\dfrac{\partial A_i}{\partial p_j}\right)$ needn't be subject to the power law behavior in (1.2).

For non-uniformly equations, Marcellini[9] considered the following non-degenerate case:

$$\lambda(1+|p|)^{\alpha-1}|\xi|^2 \le \frac{\partial A^i}{\partial p_j}(x,p)\xi_i\xi_j \le \Lambda(1+|p|)^{\beta-1}|\xi|^2 \tag{1.3}$$

with $1 \le \alpha \le \beta$.

For α, β and A_x satisfying

$$1 \le \alpha \le \beta < \frac{N+2}{N}\alpha + \frac{2}{N} \tag{1.4}$$

and

$$|A_x(x,p)| \le C(1+|p|)^{\frac{\alpha+\beta}{2}} \tag{1.5}$$

Marcellini established a local $\|Du\|_{L^\infty}$-estimate in terms of the quantities $\|Du\|_{L^{\alpha+1}}, \Omega$, α, β, λ and Λ, and the existence of Lipschitz continuous weak solution for the Dirichlet problem.

In this work, we consider the Dirichlet problems for the non-uniformly degenerate elliptic equations of the form

$$\operatorname{div} A(x,Du) + B(x) = 0 \quad \text{in } \Omega \tag{1.6}$$

$$u = u_0 \quad \text{on } \partial\Omega \tag{1.7}$$

where $\Omega \subset \mathbf{R}^N$ is a bounded domain, A is Lipschitz with $A(x,0)=0$ and satisfies

$$\min\{|p|^{\alpha-1}, |p|^{\beta-1}\}|\xi|^2 \le \frac{\partial A^i(x,p)}{\partial p_j}\xi_i\xi_j \le a_0(|p|^{\alpha-1}+|p|^{\beta-1})|\xi|^2 \tag{1.8}$$

for all $p \in \mathbf{R}^N\backslash\{0\}$, $\xi \in \mathbf{R}^N$, $x \in \Omega$, $(0 < \alpha \le \beta)$, and

$$\left|\frac{\partial A(x,p)}{\partial x_k}\cdot\lambda\right| \le a_0\Big(\frac{\partial A^i(x,p)}{\partial p_j}\lambda_i\lambda_j\Big)^{\frac{1}{2}}(1+|p|)^{\frac{1+\beta}{2}} \tag{1.9}$$

for all $p \in \mathbf{R}^N\backslash\{0\}$, $\lambda \in \mathbf{R}^N$, $1 \le k \le N$.

An example exhibiting the above structure conditions is:

$$\sum_{i=1}^N \frac{\partial}{\partial x_i}(b(x)|Du|^{\alpha-1}u_{x_i} + (1-b(x))|Du|^{\beta-1}u_{x_i} + c_i(x)|u_{x_i}|^{\alpha_i-1}u_{x_i}) = 0$$

where $b(x)$ and $c_i(x)$ are nonnegative C^2-functions, $b(x) \le 1$, and $0 < \alpha \le \alpha_1 \le \cdots \le \alpha_N \le \beta$, $\alpha_1 \ge 1$.

It is known that the gradient estimates and the smoothness of the derivatives for solutions to elliptic equations under standard growth conditions can be reduced to some a priori C^α-estimate of the solutions. There are some results on the gradient bounds of the solutions to certain non-uniformly elliptic equations without using the C^α-estimate of the solutions (See, e.g. [10, 11]). To our knowledge, there is no result on the smoothness of the derivatives of the solutions to elliptic equations under non standard growth conditions as those in (1.8). In this paper we apply Moser's iteration and make some modifications of the argument presented in [9] to infer the local gradient bounds only using the a priori bounds of the solutions, and then we exploit the techniques developed in [1, 5, 12] to prove the continuity of the derivatives.

Definition 1.1 *By a weak solution to* (1.6), (1.7) *we mean a function* $u \in W^{1,1+\alpha}(\Omega) \cap W^{1,1+\beta}_{\rm loc}(\Omega)$ *with* $u - u_0 \in W_0^{1,1+\alpha}(\Omega)$ *such that for every* $\Omega' \subset\subset \Omega$,

$$\int_\Omega \{A(x, Du)\cdot D\varphi - B(x)\varphi\} = 0, \quad \forall \varphi \in W_0^{1,1+\beta}(\Omega')$$

In this paper, we assume that

$$B(x) \in L^\infty(\Omega); \quad u_0 \in L^\infty(\Omega) \cap W^{1,(1+\alpha)\beta/\alpha}(\Omega) \tag{1.10}$$

and

$$0 < \alpha \le \beta < \max\Big\{\frac{N+2}{N}\alpha + \frac{2}{N}, \alpha + 2\Big\} \tag{1.11}$$

Our main result is the following:

Theorem 1.1 *Let* (1.8)–(1.11) *hold. Then there exists a weak solution u to* (1.6), (1.7). *Moreover u_{x_i} are continuous in Ω, $i = 1, 2, \cdots, N$.*

2. Preliminaries and Approximating Problems

We state two lemmas which will be used as we proceed.

Lemma 2.1 ([3, Theorem 2.2.1]) *Let $h \in W_0^{1,2}(\Omega)$ then for $\kappa = \dfrac{N+2}{N}$*

$$\Big(\int_\Omega |h|^{2\kappa}\Big)^N \le C(N)\Big(\int_\Omega |Dh|^2\Big)^N \Big(\int_\Omega h^2\Big)^2$$

Lemma 2.2 ([13, Lemma 3.1]) *Let $f(t)$ be a nonnegative bounded function defined in $[r_0, r_1]$, $r_0 \ge 0$. Suppose that for $r_0 \le t < s \le r$, we have*

$$f(t) \le \theta f(s) + [C_0(s-t)^{-\tau} + C_1]$$

where C_0, C_1, θ *and* τ *are nonnegative constants with* $0 \le \theta < 1$. *Then for all* $r_0 \le \rho < R \le r_1$, *we have*

$$f(\rho) \le C(\tau, \theta)[C_0(R-\rho)^{-\tau} + C_1]$$

We construct an approximation of A as follows. For $\varepsilon \in (0,1)$, we define

$$\begin{aligned}\overline{A}_\varepsilon(x,p) =& (1-\eta(|p|))A(x,p) + \eta(|p|)(\varepsilon + |p|)^{\alpha-1}p \\ &+ \frac{C_1}{\ln \varepsilon^{-1/2}}[(|p|^2+\varepsilon^2)^{\frac{\alpha-1}{2}}p + (|p|^2+\varepsilon^2)^{\frac{\beta-1}{2}}p]\end{aligned}$$

where $\eta(t)$ is Lipschitz with $\eta(t) = 1$ for $t < \varepsilon$ and $t > \frac{1}{2\varepsilon}$, $0 \le \eta \le 1$; C_1 a constant at our disposal. From (1.8), a simple calculation gives

$$\begin{aligned}\frac{\partial \overline{A}^i_\varepsilon}{\partial p_j}(x,p)\xi_i\xi_j \ge& \frac{1}{C_0(\alpha,\beta,a_0,N)}\min\{(|p|+\varepsilon)^{\alpha-1}, (|p|+\varepsilon)^{\beta-1}\}|\xi|^2 \\ &+ \frac{C_1\alpha}{\ln \varepsilon^{-1/2}}\{(\varepsilon^2+|p|^2)^{\frac{\alpha-1}{2}} + (\varepsilon^2+|p|^2)^{\frac{\beta-1}{2}}\}|\xi|^2 \\ &- C_0(\alpha,\beta,a_0,N)|\eta'||p|(|p|^{\alpha-1}+|p|^{\beta-1})\end{aligned}$$

Now set $k = \frac{1}{\ln \varepsilon^{-1/2}}$ and fix $C_1 = C_1(\alpha,\beta,C_0)$ so large that

$$\frac{1}{4}C_1\alpha\{(\varepsilon^2+|p|^2)^{\frac{\alpha-1}{2}} + (\varepsilon^2+|p|^2)^{\frac{\beta-1}{2}}\} > C_0(|p|^{\beta-1}+|p|^{\alpha-1}) \quad \text{for } |p| \ge \varepsilon$$

Then the choice

$$\eta(t) = \begin{cases} 1 & 0 \le t \le \varepsilon \\ 1 - k\ln\frac{t}{\varepsilon} & \varepsilon \le t < \varepsilon e^{\frac{1}{k}} \\ 0 & \varepsilon e^{\frac{1}{k}} \le t < (2\varepsilon e^{\frac{1}{k}})^{-1} \\ 1 + k\ln 2\varepsilon t & (2\varepsilon e^{\frac{1}{k}})^{-1} \le t < \frac{1}{2\varepsilon} \\ 1 & \frac{1}{2\varepsilon} \le t \end{cases}$$

yields

$$\begin{aligned}\frac{\partial \overline{A}^i_\varepsilon}{\partial p_j}(x,p)\xi_i\xi_j \ge& \frac{1}{C_0}\min\{(|p|+\varepsilon)^{\alpha-1}, (|p|+\varepsilon)^{\beta-1}\}|\xi|^2 \\ &+ \frac{C_1\alpha}{2\ln \varepsilon^{-1/2}}\Big\{(\varepsilon^2+|p|^2)^{\frac{\alpha-1}{2}} + (\varepsilon^2+|p|^2)^{\frac{\beta-1}{2}}\Big)|\xi|^2\end{aligned} \tag{2.1}$$

It is easy to check that

$$\left|\frac{\partial \overline{A}^i_\varepsilon}{\partial p_j}(x,p)\right| \le C_0\Big(1 + \frac{C_1\alpha}{\ln \varepsilon^{-1/2}}\Big)\{(|p|+\varepsilon)^{\alpha-1} + (|p|+\varepsilon)^{\beta-1}\} \tag{2.2}$$

We take a usual smoothing approximation of $\overline{A}_\varepsilon(x,p)$, denoted by $A_\varepsilon(x,p)$, such that

$$\begin{cases} \frac{1}{2}\Big(\frac{\partial \overline{A}_\varepsilon^i}{\partial p_j}(x,p)\Big) \le \Big(\frac{\partial A_\varepsilon^i}{\partial p_j}(x,p)\Big) \le 2\Big(\frac{\partial \overline{A}_\varepsilon^i}{\partial p_j}(x,p)\Big) \\ \Big|\frac{\partial A_\varepsilon(x,p)}{\partial x_k}\cdot\lambda\Big| \le C(\alpha,\beta,a_0,N)\Big(\frac{\partial A_\varepsilon^i}{\partial p_j}(x,p)\lambda_i\lambda_j\Big)^{\frac{1}{2}}(1+|p|)^{\frac{1+\beta}{2}} \end{cases} \tag{2.3}$$

and $A_\varepsilon(x,p)$ converges uniformly to $A(x,p)$ on compact subsets of $\Omega\times\mathbf{R}^N$.

Consider

$$\operatorname{div} A_\varepsilon(x,Du) + B_\varepsilon(x) = 0 \quad \text{in } \Omega \tag{2.4}$$

$$u = u_{0\varepsilon} \qquad\qquad \text{on } \partial\Omega \tag{2.5}$$

where B_ε and $u_{0\varepsilon}$ are respectively the smoothing approximations of B and u_0.

From a well-known existence theory (See [3]) there exists a unique classical solution u^ε of problem (2.4), (2.5).

Lemma 2.3 *There exists a constant $M_0(\alpha,\beta,a_0,N,\|B\|_{L^\infty})$ such that*

$$\|u^\varepsilon\|_{L^\infty(\Omega)} \le M_0(1+\|u_0\|_{L^\infty})$$

and

$$\int_\Omega |Du^\varepsilon|^{1+\alpha} \le M_0\Big\{\|u_0\|_{L^\infty} + \int_\Omega (1+|Du_0|^{\frac{\beta}{\alpha}(1+\alpha)})\Big\}$$

Proof The first estimate follows from the maximum principle.

The differential equation for u^ε yields

$$\int_\Omega \{A_\varepsilon(x,Du^\varepsilon)D(u^\varepsilon-u_{0\varepsilon}) - B_\varepsilon(x)(u^\varepsilon-u_{0\varepsilon})\} = 0$$

By (2.1)–(2.3) we estimate $\int_\Omega A_\varepsilon D(u^\varepsilon-u_{0\varepsilon})$ as follows:

$$\begin{aligned} &\int_\Omega A_\varepsilon(x,Du^\varepsilon)(Du^\varepsilon - Du_{0\varepsilon}) \\ &= \int_\Omega\int_0^1 a_\varepsilon^{ij}(x,Du_{0\varepsilon}+tD(u^\varepsilon-u_{0\varepsilon}))dtD_i(u^\varepsilon-u_{0\varepsilon})\cdot D_j(u^\varepsilon-u_{0\varepsilon}) \\ &\quad + \int_\Omega A_\varepsilon(x,Du_{0\varepsilon})(Du^\varepsilon-Du_{0\varepsilon}) \\ &\ge \frac{1}{C}\int_E |Du^\varepsilon - Du_{0\varepsilon}|^{1+\alpha} - C\int_\Omega[(|Du_0|+\varepsilon)^\beta + (|Du_0|+\varepsilon)^\alpha]|Du^\varepsilon - Du_{0\varepsilon}| \end{aligned}$$

where $a_\varepsilon^{ij}(x,p)=\dfrac{\partial A_\varepsilon^i}{\partial p_j}(x,p)$, $E=\{x\in\Omega; |D(u^\varepsilon-u_{0\varepsilon})|\ge 2|Du_{0\varepsilon}|, |D(u^\varepsilon-u_{0\varepsilon})|\ge 1\}$.
From Hölder inequality,

$$\int_\Omega [(|Du_0|+\varepsilon)^\beta+(|Du_0|+\varepsilon)^\alpha]|D(u^\varepsilon-u_{0\varepsilon})|$$
$$\le \delta\int_E |D(u^\varepsilon-u_{0\varepsilon})|^{1+\alpha}+C(\delta,\alpha,\beta,N)\int_\Omega (1+|Du_0|^{\frac{\beta}{\alpha}(1+\alpha)})\quad (\forall\delta>0)$$

Then by taking δ small enough we deduce

$$\int_E |Du^\varepsilon-Du_{0\varepsilon}|^{1+\alpha}\le C\Big\{\int_\Omega (1+|Du_0|^{\frac{\beta}{\alpha}(1+\alpha)})+\|u_0\|_{L^\infty}\Big\}$$

and hence

$$\int_\Omega |Du^\varepsilon|^{1+\alpha}\le C\Big\{\int_E |Du^\varepsilon-Du_{0\varepsilon}|^{1+\alpha}+\int_\Omega |Du_0|^{1+\alpha}\Big\}$$
$$\le M_0\Big\{\|u_0\|_{L^\infty}+\int_\Omega (1+|Du_0|^{\frac{\beta}{\alpha}(1+\alpha)})\Big\}$$

3. Gradient Estimate

Let us denote by B_ρ, B_R balls compactly contained in Ω, of radii respectively ρ, R and with the same center.

Theorem 3.1 *Let (1.11) hold, then there exist constants $C=C(\alpha,\beta,a_0,N,\|u^\varepsilon\|_{L^\infty},\|B\|_{L^\infty})$ and $\tau_2=\tau_2(\alpha,\beta,N)\in(0,1)$ such that for any $B_R\subset\subset\Omega$*

$$\sup_{B_{\frac{R}{4}}}|Du^\varepsilon|\le C\left\{1+\left(\frac{\int_{B_R}(1+|Du^\varepsilon|^{1+\alpha})}{R}\right)^{\frac{1}{\tau_2}}\right\}$$

Proof We focus our attention on the case $\beta<\alpha+2$ because the case $\beta<\dfrac{N+2}{N}\alpha+\dfrac{2}{N}$ follows by modifying Marcellini's work[9].

The strategy is the following: we first prove that for $\dfrac{R}{4}\le t<s\le R$,

$$\left(\int_{B_{\frac{s+t}{2}}}|Du^\varepsilon|^{h_0}\right)^{1/h_0}\le C(h_0,s,t)\|Du^\varepsilon\|_{L^\infty(B_s)}^{\gamma_0/h_0},\quad \forall h_0\ge 1 \tag{3.1}$$

And then we prove that

$$\|Du^\varepsilon\|_{L^\infty(B_t)}\le C(h_0,s,t)\left(\int_{B_{\frac{s+t}{2}}}|Du^\varepsilon|^{h_0}\right)^{\gamma_1/h_0},\quad \forall h_0\ge 1 \tag{3.2}$$

(γ_0 and γ_1 are independent of h_0). Finally, we fix h_0 large enough such that $\frac{\gamma_0\gamma_1}{h_0} < 1$ and use Lemma 2.2 to obtain the desired gradient bound. (The precise forms of inequalities (3.1) and (3.2) will be given respectively in (3.11) and (3.15) below).

Step 1 From the differential of equation for u^ε it follows that

$$\int_\Omega \left(a_\varepsilon^{ij}(x, Du^\varepsilon)D_{jk}u^\varepsilon + \frac{\partial A_\varepsilon^i}{\partial x_k}\right)D_i\eta = \int_\Omega B_\varepsilon D_k\eta \quad \forall \eta \in C_0^{0,1}(\Omega)$$

Replace η by $\eta D_k u^\varepsilon$ and then add over k to get

$$\int_\Omega b_\varepsilon^{ij} D_jW D_j\eta + \int_\Omega a_\varepsilon^{ij} D_{ik}u^\varepsilon D_{jk}u^\varepsilon\eta + \int_\Omega \frac{\partial A_\varepsilon^i}{\partial x_k} D_i(D_k u^\varepsilon \eta) \le \int_\Omega B_\varepsilon D_k(D_k u^\varepsilon \eta) \tag{3.3}$$

where

$$W = \int_0^{(|Du^\varepsilon|-1)_+} (t+1)(t+1+\varepsilon)^{\alpha-1}dt, \quad b_\varepsilon^{ij} = \frac{a_\varepsilon^{ij}}{(|Du^\varepsilon|+\varepsilon)^{\alpha-1}}$$

and $\eta = W^s\xi$ $(s \ge 1)$, $\xi \in C_0^{0,1}(\Omega)$ is nonnegative.

Taking $\eta = W^{q+\frac{2\alpha}{1+\alpha}-1}\xi^2$ (for any $q \ge \frac{2}{1+\alpha}$, ξ vanishing on $\partial\Omega$) in (3.3), and then by (2.1)–(2.3) and Hölder inequality we estimate the terms of the resulting equation as follows:

$$\begin{aligned}
&\int_\Omega b_\varepsilon^{ij} D_jW D_j(W^{q+\frac{2\alpha}{1+\alpha}-1}\xi^2) + \int_\Omega a_\varepsilon^{ij} D_{ik}u^\varepsilon D_{jk}u^\varepsilon W^{q+\frac{2\alpha}{1+\alpha}-1}\xi^2 \\
&\ge \frac{q}{C}\int_\Omega b_\varepsilon^{ij} D_iW D_jW W^{q+\frac{2\alpha}{1+\alpha}-2}\xi^2 + \frac{1}{C}\int_\Omega a_\varepsilon^{ij} D_{ik}u^\varepsilon D_{jk}u^\varepsilon W^{q+\frac{2\alpha}{1+\alpha}-1}\xi^2 \\
&\quad - \frac{C}{q}\int_\Omega |D\xi|^2(1+W^{q+\frac{\beta+\alpha}{1+\alpha}})
\end{aligned} \tag{3.4}$$

$$\begin{aligned}
&\int_\Omega \frac{\partial A_\varepsilon^i}{\partial x_k} D_i(D_k u^\varepsilon W^{q+\frac{2\alpha}{1+\alpha}-1}\xi^2) \\
&\le C\int_\Omega (a_\varepsilon^{ij} D_{ik}u^\varepsilon D_{jk}u^\varepsilon)^{\frac12} W^{q+\frac{2\alpha}{1+\alpha}-1}|Du^\varepsilon|^{\frac{\beta+1}{2}}\xi^2 \\
&\quad + Cq\int_\Omega (b_\varepsilon^{ij} D_iW D_jW)^{\frac12} W^{q+\frac{2\alpha}{1+\alpha}-2}|Du^\varepsilon|^{\frac{\alpha+\beta+2}{2}}\xi^2 \\
&\quad + C\int_\Omega (a_\varepsilon^{ij} D_i\xi D_j\xi)^{\frac12} W^{q+\frac{2\alpha}{1+\alpha}-1}|Du^\varepsilon|^{\frac{3+\beta}{2}}\xi \\
&\le \delta\int_\Omega a_\varepsilon^{ij} D_{ik}u^\varepsilon D_{jk}u^\varepsilon W^{q+\frac{2\alpha}{1+\alpha}-1}\xi^2 + \delta q\int_\Omega b_\varepsilon^{ij} D_jW D_jW W^{q+\frac{2\alpha}{1+\alpha}-2}\xi^2 \\
&\quad + C(\delta)q\int_\Omega (1+W^{q+\frac{\beta+\alpha}{1+\alpha}})(\xi^2+|D\xi|^2), \quad \forall\delta > 0
\end{aligned} \tag{3.5}$$

and

$$\int_\Omega B_\varepsilon D_k(D_k u^\varepsilon W^{q+\frac{2\alpha}{1+\alpha}-1}\xi^2)$$

$$\leq \delta\int_\Omega a_\varepsilon^{ij} D_{ik}u^\varepsilon D_{jk}u^\varepsilon W^{q+\frac{2\alpha}{1+\alpha}-1}\xi^2 + \delta q\int_\Omega b_\varepsilon^{ij} D_i W D_j W W^{q+\frac{2\alpha}{1+\alpha}-2}\xi^2$$

$$+ C(\delta)q\int_\Omega (1+W^{q+\frac{\beta+\alpha}{1+\alpha}})(\xi^2+|D\xi|^2), \quad \forall\delta>0 \tag{3.6}$$

In (3.2)–(3.4) we have used the inequality: $\frac{1}{C}W^{\frac{1}{1+\alpha}} \leq |Du^\varepsilon| \leq CW^{\frac{1}{1+\alpha}}$ for $W \geq 1$.,

From (3.2)–(3.4) we take δ in (3.5), (3.6) small enough to obtain

$$\int_\Omega |D(W^{\frac{q}{2}+\frac{\alpha}{1+\alpha}}\xi)|^2 + q\int_\Omega |Du^\varepsilon|^{\alpha-1}|D^2u|^2 W^{q+\frac{2\alpha}{1+\alpha}-1}\xi^2$$

$$\leq Cq^2\int_\Omega (\xi^2+|D\xi|)^2)(1+W^{q+\frac{\beta+\alpha}{1+\alpha}}) \tag{3.7}$$

Let us denote $f(\rho) = \sup_{B_\rho} W$, $\widehat{N} = N$ if $N > 2$; $\widehat{N} = 3$ if $N = 2$. For $\frac{R}{4} \leq t < s \leq R$, set

$$s_0 = \frac{s+t}{2}, \quad 2\theta = 2+\alpha+\beta$$

$$q_0 = \frac{2}{1+\alpha}, \quad p_0 = \frac{\widehat{N}}{1+\alpha}, \quad \kappa = \frac{\widehat{N}}{\widehat{N}-2}$$

and for $h = 0, 1, 2, \cdots$, define

$$q_{h+1} + \frac{\beta+\alpha}{1+\alpha} = q_h + \frac{\beta+\alpha+\theta}{1+\alpha}, \quad p_{h+1} + \frac{\beta+\alpha}{1+\alpha} = \left(p_h + \frac{2\alpha}{1+\alpha}\right)\kappa$$

$$R_h = s_0 + \frac{1}{2^{h+1}}(s-s_0), \quad B_h = B_{R_h}$$

$$\widehat{R}_h = t + \frac{1}{2^{h+1}}(s_0-t), \quad \widehat{B}_h + B_{\widehat{R}_h}$$

We take $\xi_h \in C_0^{0,1}(B_h)$ so that $0 \leq \xi_h \leq 1$, $\xi_h = 1$ on B_{h+1} and $|D\xi_h| \leq C\frac{2^h}{s-t}$, and then using the integration by parts we estimate the integral $\int_{B_h} W^{q_h+\frac{\beta+\alpha+\theta}{1+\alpha}}\xi_h^2$ as follows.

$$\int_{B_h} W^{q_h+\frac{\beta+\alpha+\theta}{1+\alpha}}\xi_h^2 \leq C\int_{B_h} W^{q_h+\frac{\beta+\alpha+\theta-2}{1+\alpha}}|Du^\varepsilon|^2\xi_h^2 + C\int_{B_h}\xi_h^2$$

(integration by parts)

$$= -C\int_{B_h} u^\varepsilon[D_{ii}u^\varepsilon W^{q_h+\frac{\beta+\alpha+\theta-2}{1+\alpha}}\xi_h^2 + 2\xi_h W^{q_h+\frac{\beta+\alpha+\theta-2}{1+\alpha}} D_iu^\varepsilon D_i\xi_h$$

$$+ \Big(q_h + \frac{\beta+\alpha+\theta-2}{1+\alpha}\Big)W^{q_h+\frac{\beta+\alpha+\theta-2}{1+\alpha}-1}D_iu^\varepsilon D_iW\xi_h^2] + C\int_{B_h}\xi_h^2$$

$$\le C\int_{B_h}|Du^\varepsilon|^{\alpha-1}|D^2u^\varepsilon|^2W^{q_h+\frac{2\alpha}{1+\alpha}-1}\xi_h^2 + Cq_h\int_{B_h}|DW|^2W^{q_h+\frac{2\alpha}{1+\alpha}-2}\xi_h^2$$

$$+ C\frac{2^hq_h}{(s-t)}\int_{B_h}(1+W^{q_h+\frac{\beta+\alpha}{1+\alpha}}) \tag{3.8}$$

Replacing q and ξ in (3.7) respectively by q_h and ξ_h we have

$$\int_{B_h}|Du^\varepsilon|^{\alpha-1}|D^2u^\varepsilon|^2W^{q_h+\frac{2\alpha}{1+\alpha}-1}\xi_h^2 + q_h\int_{B_h}|DW|^2W^{q_h+\frac{2\alpha}{1+\alpha}-2}\xi_h^2$$

$$\le C\frac{4^hq_h^2}{(s-t)^2}\int_{B_h}(1+W^{q_h+\frac{\beta+\alpha}{1+\alpha}}) \tag{3.9}$$

From (3.8) and (3.9),

$$\int_{B_{h+1}}(1+W^{q_{h+1}+\frac{\beta+\alpha}{1+\alpha}}) \le \frac{C4^hq_h^2}{(s-t)^2}\int_{B_h}(1+W^{q_h+\frac{\beta+\alpha}{1+\alpha}})$$

Let us define

$$A_h = \Big(\int_{B_h}(1+W^{q_h+\frac{\beta+\alpha}{1+\alpha}})\Big)^{\frac{1}{q_h+\frac{\beta+\alpha}{1+\alpha}}}$$

then the above inequality can be rewritten as follows

$$A_{h+1} \le \Big(C\frac{4^hq_h^2}{(s-t)^2}\Big)^{\frac{1}{q_{h+1}+\frac{\beta+\alpha}{1+\alpha}}}A_h^{\frac{q_h+\frac{\beta+\alpha}{1+\alpha}}{q_{h+1}+\frac{\beta+\alpha}{1+\alpha}}} \tag{3.10}$$

By the definition of q_h and by iterating (3.10) we can easily arrive at

$$\Big(\int_{B_{S_0}}(1+W^{q_0+\frac{\beta+\alpha+h_0\theta}{1+\alpha}})\Big)^{\frac{1}{q_0+\frac{\beta+\alpha+h_0\theta}{1+\alpha}}} \le A_{h_0} \le \frac{C(h_0)}{(s-t)^{\frac{2(1+\alpha)}{\theta}}}A_0^{\frac{q_0+\frac{\beta+\alpha}{1+\alpha}}{q_{h_0}+\frac{\beta+\alpha}{1+\alpha}}}$$

$$= \frac{C(h_0)}{(s-t)^{\frac{2(1+\alpha)}{\theta}}}\Big(\int_{B_s}(1+W^{\frac{2+\beta+\alpha}{1+\alpha}})\Big)^{\frac{1+\alpha}{2+\beta+\alpha+h_0\theta}}$$

$$\le \frac{C(h_0)(f(s))^{\frac{1+\beta}{2+\beta+\alpha+h_0\theta}}}{(s-t)^{\frac{2(1+\alpha)}{\theta}}}\Big(\int_{B_R}(1+|Du^\varepsilon|^{\alpha+1})\Big)^{\frac{1+\alpha}{2+\beta+\alpha+h_0\theta}} \tag{3.11}$$

provided that $f(s) \ge 1$, where $C(h_0) \to \infty$ as $h_0 \to \infty$.

Step 2 By the definition of p_h we find that

$$p_h + \frac{2\alpha}{1+\alpha} = \Big(p_0 + \frac{2\alpha}{1+\alpha}\Big)\kappa^h - \frac{\beta-\alpha}{1+\alpha}\cdot\frac{\kappa^h - 1}{\kappa - 1}$$

$$\tau_0 := \sum_{h=0}^{\infty} \frac{1}{p_h + \frac{2\alpha}{1+\alpha}} \le \frac{(1+\alpha)\kappa}{2+\alpha-\beta} \tag{3.12}$$

and

$$\tau_1 := \prod_{i=0}^{\infty} \frac{p_i + \frac{\beta+\alpha}{1+\alpha}}{p_i + \frac{2\alpha}{1+\alpha}} = \lim_{h\to\infty} \prod_{i=0}^{h} \frac{p_i + \frac{\beta+\alpha}{1+\alpha}}{p_i + \frac{2\alpha}{1+\alpha}}$$

$$= \lim_{h\to\infty} \kappa^h \frac{p_0 + \frac{\beta+\alpha}{1+\alpha}}{p_h + \frac{2\alpha}{1+\alpha}} = \frac{p_0 + \frac{\beta+\alpha}{1+\alpha}}{p_0 + \frac{2\alpha}{1+\alpha} - \frac{\beta-\alpha}{2(1+\alpha)}(\hat{N}-2)} \tag{3.13}$$

Now we choose $\widehat{\xi}_h \in C_0^{0,1}(\widehat{B}_h)$ so that $0 \le \widehat{\xi}_h \le 1$, $\widehat{\xi}_h = 1$ on $\widehat{B}_{h+1}$ and $|D\widehat{\xi}_h| \le C\dfrac{2^h}{s-t}$, and then apply (3.7) with $q = p_h$, $\xi = \widehat{\xi}_h$ and use Sobolev's inequality to obtain

$$\widehat{A}_{h+1} \le \Big(\frac{C4^h p_h^2}{(s-t)^2}\Big)^{\frac{1}{p_h + \frac{2\alpha}{1+\alpha}}} \widehat{A}_h^{\frac{p_h + \frac{\beta+\alpha}{1+\alpha}}{p_h + \frac{2\alpha}{1+\alpha}}} \tag{3.14}$$

where

$$\widehat{A}_h = \Big(\int_{\widehat{B}_h} (1 + W^{p_h + \frac{\beta+\alpha}{1+\alpha}})\Big)^{\frac{1}{p_h + \frac{\beta+\alpha}{1+\alpha}}}$$

We iterate (3.14) and use (3.12), (3.13) and Hölder inequality to obtain

$$f(t) \le \widehat{A}_\infty \le C(s-t)^{-2\tau_0\tau_1} \widehat{A}_0^{\tau_1}$$

$$\le C(s-t)^{-2\tau_0\tau_1}\Big(\int_{B_{S_0}} (1 + W^{q_0 + \frac{\beta+\alpha+h_0\theta}{1+\alpha}})\Big)^{\frac{\tau_1}{q_0 + \frac{\beta+\alpha+h_0\theta}{1+\alpha}}} \tag{3.15}$$

We are now in a position to prove the gradient bound. From (3.11) and (3.15),

$$f(t) \le \frac{C(h_0)(f(s))^{\frac{(1+\beta)\tau_1}{2+\beta+\alpha+h_0\theta}}}{(s-t)^{2\tau_0\tau_1 + \frac{2(1+\alpha)\tau_1}{\theta}}}\Big(\int_{B_R} (1 + |Du^\varepsilon|^{1+\alpha})\Big)^{\frac{(1+\alpha)\tau_1}{2+\beta+\alpha+h_0\theta}}$$

We fix h_0 so large that $\dfrac{(1+\beta)\tau_1}{2+\beta+\alpha+h_0\theta} < 1$ and then from Hölder inequality we have

$$f(t) \le \frac{1}{2} f(s) + C\left\{\frac{\Big[\int_{B_R} (1 + |Du^\varepsilon|^{1+\alpha})\Big]^{\widehat{\tau}_1}}{(s-t)^{\widehat{\tau}_2}}\right\}^{\frac{1}{1-\gamma_0}}$$

where

$$\gamma_0 = \frac{(1+\beta)\tau_1}{2+\beta+\alpha+h_0\theta}, \quad \widehat{\tau}_1 = \frac{(1+\alpha)\tau_1}{2+\beta+\alpha+h_0\theta}, \quad \widehat{\tau}_2 = 2\tau_0\tau_1 + \frac{2\tau_1(1+\alpha)}{\theta}$$

Hence we infer from Lemma 2.2 that

$$\begin{aligned} \sup_{B_{\frac{R}{4}}} |Du^\varepsilon| &\le C\Big(1 + \Big(f\Big(\frac{R}{4}\Big)\Big)^{\frac{1}{1+\alpha}}\Big) \\ &\le C\left\{ 1 + \left(\frac{\left(\int_{B_R} (1 + |Du^\varepsilon|^{1+\alpha})\right)^{\widehat{\tau}_1}}{R^{\widehat{\tau}_2}} \right)^{\frac{1}{(1-\gamma_0)(1+\alpha)}} \right\} \end{aligned} \tag{3.16}$$

which completes the proof of Theorem 3.1.

4. Equicontinuity of Du^ε; Proof of Theorem 1.1

It is well known that the key step to the proof of uniformly degenerate equations having $C^{1,\alpha}$ solutions is to show that there are universal constants $\tau, \delta \in (0,1)$ and C such that one of the inequalities

$$\max_{i\le N} \operatorname{osc}_{B_{\frac{R}{4}}} D_i u \le \max\left\{\delta \max_{i\le N} \operatorname{osc}_{B_R} D_i u, CR^\tau\right\}$$

$$\max_{i\le N} \sup_{B_{\frac{R}{4}}} |D_i u| \le \delta \max_{i\le N} \sup_{B_R} |D_i u|$$

holds (See, e.g. [1]). But in our nonuniformly degenerate case the constants τ, δ and C will depend on $m_R = \inf_{B_R} |Du|$ if $m_R > 0$, and the case $m_R = 0$ will be very difficult to treat. In this section we exploit the techniques developed in [1, 5, 12] to establish the equicontinuity of Du^ε and thereby prove Theorem 1.1 by a standard argument.

Let δ be an arbitrary positive number, and for any ball $B_R \subset\subset \Omega' \subset\subset \Omega$, let $M(R) = \max_{i\le N} \sup_{B_R} |D_i u^\varepsilon|$. Our argument follows lines of [12] with improvement by the introduction of δ (the role will be seen in Lemmas 4.1 and 4.2), and our aim is to show that the oscillation of Du^ε can be made less than δ in a ball of sufficiently small radius.

Lemma 4.1 *There exists a $\mu(\alpha, \beta, a_0, N, \delta, \|Du^\varepsilon\|_{L^\infty(\Omega')}, \|B\|_{L^\infty}) \in (0,1)$ such that if $M(R) \ge \delta$ and*

$$\left|\left\{x \in B_R; D_k u^\varepsilon < \frac{M(R)}{2}\right\}\right| \le \mu|B_R| \quad \text{for some } k \tag{4.1}$$

or

$$\left|\left\{x \in B_R; D_k u^\varepsilon > \frac{-M(R)}{2}\right\}\right| \le \mu|B_R| \quad \text{for some } k \tag{4.2}$$

then there are constants $\gamma_1 \in (0,1)$ *and* C *depending only on* $\delta, \alpha, \beta, a_0, N, \|Du^\varepsilon\|_{L^\infty(\Omega')}$ *and* $\|B\|_{L^\infty}$ *such that for* $\varepsilon \le \delta$,

$$\max_{i\le N} \operatorname{osc}_{B_{\frac{R}{4}}} D_i u^\varepsilon \le \max\left\{\gamma_1 \max_{i\le N} \operatorname{osc}_{B_R} D_i u^\varepsilon, CR\right\}$$

Proof It is no loss of generality to assume that (4.1) is valid. From (2.4)

$$\int_{B_R}\left(a_\varepsilon^{ij} D_{jk}u^\varepsilon + \frac{\partial A_\varepsilon^i}{\partial x_k}\right) D_i\eta = \int_{B_R} B_\varepsilon D_k\eta \quad \text{for } k = 1, 2, \cdots, N \tag{4.3}$$

holds for all Lipschitz function η with $\eta = 0$ on ∂B_R.

For $q \ge \dfrac{N+2}{2}$ we set

$$W = \min\left\{\frac{M(R)}{4}, \max\left\{\frac{M(R)}{2} - D_k u^\varepsilon, 0\right\}\right\}, \quad g(D_k u^\varepsilon) = W^{2q-N-1}$$

and pick $\eta \in C_0^{1,1}(B_R)$ with $0 \le \eta \le 1$, $\eta = 1$ on $B_{\frac{R}{2}}$, $|D\eta| \le \dfrac{C}{R}$, $|D^2\eta| \le \dfrac{C}{R^2}$, and then replace η by $g\eta^{(2q-N)}$ in (4.3) to obtain

$$\begin{aligned}\int_{B_R}&\left(a_\varepsilon^{ij} D_{jk}u^\varepsilon + \frac{\partial A_\varepsilon^i}{\partial x_k}\right) g' D_{jk}u^\varepsilon \eta^{2q-N}\\ &= \int_{B_R} A_\varepsilon^i [g' D_{kk}u^\varepsilon D_i(\eta^{2q-N}) + g D_{ik}(\eta^{2q-N})]\\ &\quad + \int_{B_R} B_\varepsilon [g' D_{kk}u^\varepsilon \eta^{2q-N} + g D_k(\eta^{2q-N})]\end{aligned} \tag{4.4}$$

By (2.1)–(2.3) and noting that $|Du| \ge \dfrac{M(R)}{4} \ge \dfrac{1}{4}\delta \ge \dfrac{1}{4}\varepsilon$ wherever $g' \ne 0$, we compute:

$$\begin{aligned}\int_{B_R}&\left(a_\varepsilon^{ij} D_{jk}u^\varepsilon + \frac{\partial A_\varepsilon^i}{\partial x_k}\right) g' D_{jk}u^\varepsilon \eta^{2q-N}\\ &\ge \frac{qF(M(R))}{C}\int_{B_R} |DW|^2 W^{2q-N-2}\eta^{2q-N}\\ &\quad - \frac{Cq(1+M^{2\beta}(R))}{F(M(R))}\int_{B_R} W^{2q-N-2}\eta^{2q-N}\end{aligned} \tag{4.5}$$

$(F(M(R)) = \min\{M^{\alpha-1}(R), M^{\beta-1}(R)\})$,

$$\begin{aligned}\int_{B_R}&\{A_\varepsilon^i [g' D_{kk}u^\varepsilon D_i(\eta^{2q-N}) + g D_{ik}(\eta^{2q-N})] + B_\varepsilon [g' D_{kk}u^\varepsilon \eta^{2q-N} + g D_k(\eta^{2q-N})]\}\\ &\le q\tilde{\varepsilon} F(M(R))\int_{B_R} |DW|^2 W^{2q-N-2}\eta^{2q-N}\end{aligned}$$

$$+C(\tilde{\varepsilon})q^2\frac{1+M^{2\beta}(R)}{F(M(R))}\int_{B_R}W^{2q-N-2}\eta^{2q-N-2}$$

$$+Cq^2\frac{1+M^{1+\beta}(R)}{R^2}\Big(1+\frac{1}{F(M(R))}\Big)\int_{B_R}W^{2q-N-2}\eta^{2q-N-2}\quad(\forall\tilde{\varepsilon}>0) \tag{4.6}$$

By taking $\tilde{\varepsilon}$ in (4.6) small enough and noting that $M(R)>\delta$, we find that

$$\int_{B_R}|D[(W\eta)^{\frac{q-N}{2}}]|^2\le C(\delta)\frac{q^2(1+M^2(R))}{R^2}M^2(R)\int_{B_R}(W\eta)^q(W\eta)^{-N-2}\quad(q\ge N+2)$$

From Lemma 2.1, ($\kappa=\dfrac{N+2}{N}$)

$$\Big(\int_{B_R}(W\eta)^{q\kappa}(W\eta)^{-N-2}\Big)^{1/\kappa}$$

$$\le C\Big(\frac{q^2M^{\frac{2N+4}{N}}(R)(1+M^2(R))}{R^2\delta^{1+2\alpha}}\Big)^{1/\kappa}\int_{B_R}(W\eta)^q(W\eta)^{-N-2} \tag{4.7}$$

For $q_0=N+4$, $q_{i+1}=q_i\kappa$, $i=0,1,\cdots$, then a standard Moser's iteration yields

$$\sup_{B_{\frac{R}{2}}}W\le C(\delta)\Big\{[R^{-N}M^{N+2}(R)(1+M^N(R))]\int_{B_R}W^2\Big\}^{\frac{1}{N+4}}$$

$$\le C(\delta)(1+M^N(R))^{\frac{1}{N+4}}\mu^{\frac{1}{N+4}}M(R)$$

By taking $\mu=\mu(\alpha,\beta,a_0,N,\delta,\|Du^\varepsilon\|_{L^\infty(\Omega')},\|B\|_{L^\infty})$ sufficiently small, we infer that $W\le\frac{1}{8}M(R)$ on $B_{\frac{R}{2}}$ and hence $D_ku^\varepsilon>\frac{3}{8}M(R)$ on $B_{\frac{R}{2}}$. Now u^ε satisfies a uniformly elliptic equation in B_R and therefore, the lemma follows from the well-known results, (See, e.g. [3]).

Lemma 4.2 *Assume that*

$$\Big|\Big\{x\in B_R;D_ku^\varepsilon<\frac{M(R)}{2}\Big\}\Big|,\Big|\Big\{x\in B_R;D_ku^\varepsilon>-\frac{M(R)}{2}\Big\}\Big|>\mu|B_R|$$

hold for all k. Then there exists an integer $s^*=s^*(\alpha,\beta,a_0,N,\delta,\|Du^\varepsilon\|_{L^\infty(\Omega')},\|B\|_{L^\infty})$ such that $\max\{\delta,2^{s^*}R\}\le M(R)$ *implies*

$$M\Big(\frac{R}{2}\Big)\le(1-2^{-s^*-1})M(R)$$

Proof Form the hypotheses, there is a constant $b=b(\mu,N)\in\big(\frac{3}{4},1\big)$ such that $\Big|\Big\{x\in B_{bR};D_ku^\varepsilon<\dfrac{M(R)}{2}\Big\}\Big|>\dfrac{\mu}{2}|B_{bR}|$.

Let

$$W_h = \left(D_k u^\varepsilon - \left(1 - \frac{1}{2^h}\right) M(R)\right)_+, \quad 1 \le h \le s^*$$

In (4.3) replace η with $W_h^{q-N-1}\eta^{q-N}$ $(q \ge N+2,\ \eta \in C_0^{0,1}(B_{bR}),\ 0 \le \eta \le 1,\ \eta = 1$ on $B_{\frac{R}{2}}, |D\eta| \le \frac{C}{R})$ to obtain

$$\int_{B_{bR}} \left(a_\varepsilon^{ij} D_{jk}u^\varepsilon + \frac{\partial A_\varepsilon^i}{\partial x_k}\right)[(q-N-1)W_h^{q-N-2}D_{ik}u^\varepsilon\eta^{q-N} + (q-N)W_h^{q-N-1}\eta^{q-N-1}D_i\eta]$$
$$= \int_{B_{bR}} B_\varepsilon[(q-N-1)W_h^{q-N-2}D_{kk}u^\varepsilon\eta^{q-N} + (q-N)W_h^{q-N-1}\eta^{q-N-1}D_k\eta] \tag{4.8}$$

Using $M(R) \ge \max\{\delta, 2^{s^*}R\}$, we estimate the integrals in (4.8) similarly as before: the left-hand side of (4.8) is bounded from below by

$$\frac{1}{C}qF(M(R))\int_{B_{bR}} |DW_h|^2 W_h^{q-N-2}\eta^{q-N}$$
$$- \frac{Cq(1+M^{2\beta}(R))(2^{-h}M(R))^2}{\delta^2 F(M(R))R^2}\int_{B_{bR}} W_h^{q-N-2}\eta^{q-N-2} \tag{4.9}$$

and the right-hand side of (4.8) is bounded from above by

$$q\tilde{\varepsilon}F(M(R))\int_{B_{bR}} |DW_h|^2 W_h^{q-N-2}\eta^{q-N}$$
$$+ \frac{C(\tilde{\varepsilon})q(1+M^{2\beta}(R))(2^{-h}M(R))^2}{\delta^2 F(M(R))R^2}\int_{B_{bR}} W_h^{q-N-2}\eta^{q-N-2} \tag{4.10}$$

Hence by taking $\tilde{\varepsilon}$ small enough we obtain

$$\int_{B_{bR}} |D[(W_h\eta)^{\frac{q-N}{2}}]|^2 \le C(\delta)\frac{q^2(1+M^2(R))}{R^2}(2^{-h}M(R))^2\int_{B_{bR}} (W_h\eta)^q (W_h\eta)^{-N-2} \tag{4.11}$$

It follows in the same way as in the proof of Lemma 4.1 that

$$\sup_{B_{\frac{R}{2}}} W_h \le C(\alpha, \beta, a_0, N, \delta, \|B\|_{L^\infty})(1+M^N(R))^{\frac{1}{N+4}}(2^{-h}M(R))\left(\frac{|D(h)|}{|B_R|}\right)^{\frac{1}{N+4}} \tag{4.12}$$

where $D(h) = \{x \in B_{bR}; W_h = (D_k u^\varepsilon - (1-2^{-h}M(R))_+ \ne 0\}$. To estimate $|D(h)|$, we notice that (4.11) holds when $q = N+2$, B_{bR} is replaced by B_R and $\eta \in C_0^{0,1}(B_R)$ satisfies $0 \le \eta \le 1$, $\eta = 1$ on $B_{bR}, |D\eta| \le \frac{C}{(1-b)R}$, then

$$\int_{B_{bR}} |DW_h|^2 \le C(\alpha, \beta, a_0, N, \delta, \|B\|_{L^\infty}, \|Du^\varepsilon\|_{L^\infty(\Omega')})R^{N-2}(2^{-h}M(R))^2 \tag{4.13}$$

for $1 \le h \le s^*$. Since $|D(h)| \le \left(1 - \frac{\mu}{2}\right)|B_{bR}|$, an application of a lemma of De Giorgi [3, Lemma 2.3.5] yields

$$2^{-h-1}M(R)|D(h)|^{\frac{N-1}{N}} \le C(N,\mu)\int_{D(h)\setminus D(h+1)} |DW_h| \tag{4.14}$$

From (4.13) and (4.14), we may conclude that

$$|D(h)| \le C(\alpha,\beta,\alpha_0,N,\delta,\|Du^\varepsilon\|_{L^\infty(\Omega')},\|B\|_{L^\infty})R^{\frac{N}{2}}|D(h)\setminus D(h+1)|^{\frac{1}{2}}$$

We take square both sides of the preceding inequality and add the resulting inequality for $h = 1, 2, \cdots, s^*$ to obtain

$$(s^*-1)|D(s^*)|^2 \le C(\alpha,\beta,a_0,N,\delta,\|Du^\varepsilon\|_{L^\infty(\Omega')},\|B\|_{L^\infty})R^{2N} \tag{4.15}$$

From (4.12) and (4.15), by fixing $s^* = s^*(\alpha,\beta,a_0,N,\delta,\|Du^\varepsilon\|_{L^\infty},\|B\|_{L^\infty})$ sufficiently large, we have

$$\sup_{B_{\frac{R}{2}}} D_k u^\varepsilon \le (1 - 2^{-s^*-1})M(R) \quad \text{for all } k$$

Similarly, we have

$$\sup_{B_{\frac{R}{2}}} (-D_k u^\varepsilon) \le (1 - 2^{-s^*-1})M(R) \quad \text{for all } k$$

Therefore

$$M\left(\frac{R}{2}\right) \le (1 - 2^{-s^*-1})M(R)$$

as claimed.

Combining Lemmas 4.1 with 4.2, by standard arguments, we conclude:

Theorem 4.1 *For any $\Omega' \subset\subset \Omega$ and any number $\delta \in (0,1)$, there exist $\gamma = \gamma(\alpha,\beta,a_0,N,\delta,\|Du^\varepsilon\|_{L^\infty(\Omega')},\|B\|_{L^\infty}) \in (0,1)$ and $C = C(\alpha,\beta,a_0,N,\delta,\|Du^\varepsilon\|_{L^\infty(\Omega')},$ $\|B\|_{L^\infty}, \mathrm{dist}(\Omega',\partial\Omega))$ such that for any ball $B_R \subset \Omega'$, $(\varepsilon \le \delta)$*

$$\max_{i\le N} \mathrm{osc}_{B_R} D_i u^\varepsilon \le \max\{2\delta, CR^\gamma\}$$

Now Theorem 1.1 is a simple consequence of Lemma 2.3, Theorems 3.1 and 4.1.

Remark 4.1 It is easily seen that Theorem 4.1 remains true if the right-hand side of (1.8) is replaced by any function $f(|p|)$ of $|p|$ such that $f(|p|) \ge \min\{|p|^{\alpha-1}, |p|^{\beta-1}\}$ (with the constants depending on f).

Remark 4.2 From our proofs, Theorem 1.1 remains true if the right-hand side of (1.8) is replaced by $a_0(1 + |p|^{\beta-1})|\xi|^2$ (this condition is weaker than the original when $\alpha \ge 1$), and this time, the approximation of $A(x,p)$, $\overline{A}_\varepsilon(x,p)$ is constructed as following

$$\overline{A}_\varepsilon(x,p) = (1-\eta)A(x,p) + \eta(\varepsilon + |p|)^{\alpha-1}p + \frac{C_1}{\ln \varepsilon^{-1/2}}[p + (|p|)^2 + \varepsilon^2)^{\frac{\beta-1}{2}}p]$$

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References

[1] DiBenedetto E., $C^{1+\alpha}$ local regularity of weak solutions of degenerate elliptic equations, *Nonlinear Analysis,* **7** (1983), 827–850.

[2] Evans L.C., A new proof of local $C^{1+\alpha}$-regularity for solutions of Certain degenerate elliptic P.D.E., *J. Diff. Eqns.,* **45** (1982), 356–373.

[3] Ladyzhenskaya O.A. & Ural'tseva N.N., Linear and Quasilinear Elliptic Equations, Academic Press, New York, 1968.

[4] Lewis J.L., Regularity of the derivatives of solutions to certain degenerate elliptic equations, *Indiana Univ. Math. J.,* **32** (1983), 849–858.

[5] Tolksdorf P., Regularity for a more general class of quasilinear elliptic equations. *J. Diff. Eqns.,* **51** (1984), 126–150.

[6] Uhlenbeck K., Regularity for a class of nonlinear elliptic systems, *Acta. Math.,* **138** (1977), 219–240.

[7] Ural'tseva N.N., Degenerate quasilinear systems, *English translation in Sem Math. V. A. Steklov Mat. Inst. Leningrad,* **7** (1968), 83–99.

[8] Lieberman G.M., The natural generalization of the natural conditions of Ladyzhenskaya and Ural'tseva for elliptic equations. *Commum. in P.D.E.,* **16** (2&3) (1991), 311–361.

[9] Marcellini P., Regularity and existence of solutions of elliptic equations with p, q-growth conditions, *J. Diff. Eqns.,* **90** (1) (1991), 1–30.

[10] Oskolkov A.P., A priori estimate for first derivatives of the Dirichlet problem for quasilinear nonuniformly elliptic equations, *Trudy Mat. Inst. Steklov,* **102** (1967), 105–127; *English transl. in Proc. Steklov Math. Inst.,* **102** (1967).

[11] Simon L.M., Interior gradient bounds for non-uniformly elliptic equations. *Indiana Univ. Math. J.,* **25** (1976), 821–855.

[12] Lieberman G.M., Boundary regularity for solutions of degenerate elliptic equations, *Nonlinear Anal.,* **12** (1988), 1203–1219.

[13] Giaquinta M., Multiple Integrals in the Calculus and Nonlinear Elliptic Systems, Annals of Mathematics Studies, Princeton, New Jersey, 1983.

[14] Gilbarg D. & Trudinger N.S., Elliptic Partial Differential Equations of Second Order, Second Edition, Berlin-Heidelberg-New York-Tokyo, Springer-Verlag, 1983.