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THE GLOBAL SOLUTION OF THE SCALAR NONCONVEX CONSERVATION LAW WITH BOUNDARY CONDITION (CONTINUATION)

Pan Tao
(Department of Mathematics, Guangxi University, Nanning 530004, China)
Lin Longwei
(Faculty of Science and Technology, Macau University, Macau)

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Abstract Using the Kruskov's method[1], we show the uniqueness for the global weak solution of the initial-boundary value problem (1.1)–(1.3) (in the class of bounded and measurable functions).

Key Words Scalar conservation law; boundary condition; nonconvex; uniqueness.

Classification 35L65.

1. Introduction

The initial-boundary value problem for the scalar conservation law was first discussed by Bardos, Leroux and Nedelec[2] (in several space variables). They showed the existence and uniqueness of the global weak solution by vanishing viscosity method and the Kruskov's method. But they had obtained neither the estimation of the solution in the boundary nor the stability in respect to the boundary data. In 1988, Le Floch[3] considered the initial-boundary value problem for the convex conservation law (in the quarter plane). They derived the explicit formula for the exact solution, and proved the uniqueness of the weak solution (in the class of piecewise regular functions). It is well-known that the nonconvex case is more complicated than the convex case, and that the uniqueness in the class of bounded and measurable functions is more perfect than the uniqueness in the class of piecewise regular functions.

We consider the initial-boundary value problem for the nonconvex conservation law (in the quarter plane):

$$\begin{cases} u_t + f(u)_x = 0 & (0 < x < +\infty,\ t > 0) & (1.1) \\ u(0,t) = u_b(t) & (t \geq 0) & (1.2) \\ u(x,0) = u_0(x) & (0 \leq x < +\infty) & (1.3) \end{cases}$$

In [4], we first have given the definition of the global weak solution for the problem (1.1)–(1.3). And we have proved the existence of the weak solution for the problem (1.1)–(1.3) by the polygonal approximations method (for $u_0(x), u_b(t)$ are bounded variation functions, $f(u)$ is a locally Lipschitz continuous function). In this paper we first give the estimation of the solution in the boundary. Then we prove the uniqueness in the class of bounded and measurable functions by Kruskov's method[1]. And we obtain the stability in respect to the boundary data and the initial data.

The global weak solution of the initial-boundary value problem for the scalar non-convex conservation law plays an important role in the mathematical modeling and computations of the one-dimensional sedimentation processes[5,6].

2. Definition of the Weak Solution

Assume that $f(u)$ is a Lipschitz continuous function on $[-M, M], u_0(x), u_b(t)$ are bounded and measurable functions and

$$|f(u) - f(u')| \le L|u - u'|, \quad \forall u, u' \in [-M, M] \tag{2.1}$$

$$-M \le u_0(x), u_b(t) \le M, \quad 0 \le x < +\infty,\ t \ge 0 \tag{2.2}$$

where M, L are arbitrary positive constants.

Definition 2.1[4] *A locally bounded and measurable function $u(x,t)$ on $[0,+\infty) \times [0,+\infty)$ is called a weak solution of the initial-boundary problem (1.1)–(1.3), if for every $k \in R^1$ and for any nonnegative function $\varphi(x,t) \in C_0^\infty([0,+\infty) \times [0,+\infty))$, it satisfies the following inequality:*

$$\begin{aligned}&\int_0^{+\infty}\int_0^{+\infty} \{|u-k|\varphi_t + \operatorname{sgn}(u-k)(f(u)-f(k))\varphi_x\}dxdt \\ &\quad + \int_0^{+\infty} \operatorname{sgn}(u_b(t)-k)(f(u(0,t))-f(k))\varphi(0,t)dt \\ &\quad + \int_0^{+\infty} |u_0(x)-k|\varphi(x,0)dx \ge 0\end{aligned} \tag{2.3}$$

where $\operatorname{sgn} x = \begin{cases} 1, & x \ge 0 \\ -1, & x < 0 \end{cases}$.

Lemma 2.1[4] *If $u(x,t)$ is a weak solution of the problem (1.1)–(1.3), then*

$$\operatorname{sgn}(u(0,t)-k)(f(u(0,t))-f(k)) \le 0, \quad \forall k \in I(u(0,t), u_b(t)),\ a.e.\ t \ge 0 \tag{2.4}$$

where $I(a,b) = [\min\{a,b\}, \operatorname{Max}\{a,b\}]$.

We have:

Lemma 2.2 *Assume that $u^{(i)}(x,t)$ is the weak solution of the problem (1.1)-(1.3) associated with the data $(u_0^{(i)}(x), u_b^{(i)}(t))$, $i=1,2$. Then*

$$\operatorname{sgn}(u^{(1)}(0,t)-u^{(2)}(0,t))(f(u^{(1)}(0,t))-f(u^{(2)}(0,t)))\le L|u_b^{(1)}(t)-u_b^{(2)}(t)| \quad (2.5)$$

Proof By Lemma 2.1, we have

$$\operatorname{sgn}(u^{(i)}(0,t)-k)(f(u^{(i)}(0,t))-f(k))\le 0,\quad \forall k\in I(u^{(i)}(0,t),u_b^{(i)}(t)),\ a.e.\ t\ge 0 \quad (2.6)$$

$i=1,2$. We distinguish between four cases (for convenience's sake, we let $u_b^{(i)}\equiv u_b^{(i)}(t)$, $u^{(i)}\equiv u^{(i)}(0,t)$, $i=1,2$):

(i) $u^{(i)}=u_b^{(i)}$ $(i=1,2)$

By (2.1), we can prove (2.5), obviously.

(ii) $u^{(1)}=u_b^{(1)}$, $u^{(2)}\ne u_b^{(2)}$

If $u^{(1)}=u_b^{(1)}\in I(u^{(2)},u_b^{(2)})$, from (2.1) and (2.6) we can prove (2.5) obviously.

If $u^{(1)}=u_b^{(1)}\notin I(u^{(2)},u_b^{(2)})$, we distinguish between four cases:

$$\begin{array}{ll}(\mathrm{A}_1) & u^{(1)}=u_b^{(1)}<u_b^{(2)}<u^{(2)}\\ (\mathrm{A}_2) & u^{(2)}<u_b^{(2)}<u^{(1)}=u_b^{(1)}\\ (\mathrm{A}_3) & u^{(1)}=u_b^{(1)}<u^{(2)}<u_b^{(2)}\\ (\mathrm{A}_4) & u_b^{(2)}<u^{(2)}<u^{(1)}=u_b^{(1)}\end{array}$$

For the cases (A_1), (A_2):

$$\begin{aligned}&\operatorname{sgn}(u^{(1)}-u^{(2)})(f(u^{(1)})-f(u^{(2)}))\\ &=\operatorname{sgn}(u^{(1)}-u^{(2)})(f(u_b^{(1)})-f(u_b^{(2)}))+\operatorname{sgn}(u_b^{(2)}-u^{(2)})(f(u_b^{(2)})-f(u^{(2)}))\\ &\le \operatorname{sgn}(u^{(1)}-u^{(2)})(f(u_b^{(1)})-f(u_b^{(2)}))\le |f(u_b^{(1)})-f(u_b^{(2)})|\le L|u_b^{(1)}-u_b^{(2)}|\end{aligned}$$

For the cases (A_3), (A_4):

$$\begin{aligned}&\operatorname{sgn}(u^{(1)}-u^{(2)})(f(u^{(1)})-f(u^{(2)}))\\ &\le |f(u^{(1)})-f(u^{(2)})|\le L|u^{(1)}-u^{(2)}|\le L|u_b^{(1)}-u_b^{(2)}|\end{aligned}$$

(iii) $u^{(1)}\ne u_b^{(1)}$, $u^{(2)}=u_b^{(2)}$

We can prove (2.5) as in the previous case (ii).

(iv) $u^{(i)}\ne u_b^{(i)}$ $(i=1,2)$

If $u^{(1)}\in I(u^{(2)},u_b^{(2)})$ or $u^{(2)}\in I(u^{(1)},u_b^{(1)})$, by (2.6) we can prove (2.5), obviously.

If $u^{(1)}\notin I(u^{(2)},u_b^{(2)})$, $u^{(2)}\notin I(u^{(1)},u_b^{(1)})$ and either $u_b^{(1)}\in I(u^{(2)},u_b^{(2)})$ or $u_b^{(2)}\in I(u^{(1)},u_b^{(1)})$, we distinguish between two cases:

$$(\mathrm{B}_1)\qquad u^{(1)}<u_b^{(2)}\le u_b^{(1)}\le u^{(2)}$$

$$(\mathrm{B}_2) \qquad u^{(2)} < u_b^{(1)} \le u_b^{(2)} \le u^{(1)}$$

In these two cases, we have

$$\operatorname{sgn}(u^{(1)} - u^{(2)}) = \operatorname{sgn}(u^{(1)} - u_b^{(1)}) = \operatorname{sgn}(u_b^{(2)} - u^{(2)})$$

Hence

$$\begin{aligned}
&\operatorname{sgn}(u^{(1)} - u^{(2)})(f(u^{(1)}) - f(u^{(2)})) \\
&\quad = \operatorname{sgn}(u^{(1)} - u_b^{(1)})(f(u^{(1)}) - f(u_b^{(1)})) + \operatorname{sgn}(u^{(1)} - u^{(2)})(f(u_b^{(1)}) - f(u_b^{(2)})) \\
&\qquad + \operatorname{sgn}(u_b^{(2)} - u^{(2)})(f(u_b^{(2)}) - f(u^{(2)})) \\
&\quad \le \operatorname{sgn}(u^{(1)} - u^{(2)})(f(u_b^{(1)}) - f(u_b^{(2)})) \le |f(u_b^{(1)}) - f(u_b^{(2)})| \le L|u_b^{(1)} - u_b^{(2)}|
\end{aligned}$$

If $u^{(1)}, u_b^{(1)} \notin I(u^{(2)}, u_b^{(2)})$, $u^{(2)}, u_b^{(2)} \notin I(u^{(1)}, u_b^{(1)})$, we distinguish between eight cases:

$$\begin{aligned}
&(\mathrm{C}_1) \qquad u_b^{(1)} < u^{(1)} < u^{(2)} < u_b^{(2)} \\
&(\mathrm{C}_2) \qquad u_b^{(2)} < u^{(2)} < u^{(1)} < u_b^{(1)}
\end{aligned}$$

In these two cases, we can prove (2.5) as in the previous cases (A_3), (A_4).

$$\begin{aligned}
&(\mathrm{C}_3) \qquad u^{(1)} < u_b^{(1)} < u_b^{(2)} < u^{(2)} \\
&(\mathrm{C}_4) \qquad u^{(2)} < u_b^{(2)} < u_b^{(1)} < u^{(1)}
\end{aligned}$$

In these two cases, we can prove (2.5) as in the previous cases (B_1), (B_2).

$$\begin{aligned}
&(\mathrm{C}_5) \qquad u_b^{(1)} < u^{(1)} < u_b^{(2)} < u^{(2)} \\
&(\mathrm{C}_6) \qquad u_b^{(2)} < u^{(2)} < u_b^{(1)} < u^{(1)} \\
&(\mathrm{C}_7) \qquad u^{(1)} < u_b^{(1)} < u^{(2)} < u_b^{(2)} \\
&(\mathrm{C}_8) \qquad u^{(2)} < u_b^{(2)} < u^{(1)} < u_b^{(1)}
\end{aligned}$$

These four cases are similar, we only discuss the case (C_5):

$$\begin{aligned}
&\operatorname{sgn}(u^{(1)} - u^{(2)})(f(u^{(1)}) - f(u^{(2)})) \\
&\quad = \operatorname{sgn}(u^{(1)} - u_b^{(2)})(f(u^{(1)}) - f(u_b^{(2)})) + \operatorname{sgn}(u_b^{(2)} - u^{(2)})(f(u_b^{(2)}) - f(u^{(2)})) \\
&\quad \le \operatorname{sgn}(u^{(1)} - u_b^{(2)})(f(u^{(1)}) - f(u_b^{(2)})) \le |f(u^{(1)}) - f(u_b^{(2)})| \\
&\quad \le L|u^{(1)} - u_b^{(2)}| \le L|u_b^{(1)} - u_b^{(2)}|
\end{aligned}$$

3. The Uniqueness of the Weak Solution

The main result of this paper is contained in the following

Theorem 3.1 *Let $u^{(i)}(x,t)$ be the weak solution of the problem (1.1)–(1.3) associated with the data $(u_0^{(i)}(x), u_b^{(i)}(t))$, where $u_0^{(i)}(x), u_b^{(i)}(t)$ are bounded and measurable functions satisfying (2.2), $i=1,2$. And $f(u)$ is a Lipschitz continuous function of $[-M,M]$ satisfying (2.1). If*

$$|u^{(i)}(x,t)| \le M, \ a.e. \ (x,t) \in (0,X)\times(0,T), \quad i=1,2 \tag{3.1}$$

then for almost all $t \in (0,T_0)$, we have

$$\int_0^{X-Lt} |u^{(1)}(x,t)-u^{(2)}(x,t)|dx \le \int_0^X |u_0^{(1)}(x)-u_0^{(2)}(x)|dx + L\int_0^t |u_b^{(1)}(t)-u_b^{(2)}(t)|dt \tag{3.2}$$

where X,T are arbitrary positive constants, $T_0=\min\left(T,\dfrac{X}{L}\right)$.

Proof Let $Q_T=\{(x,t)\mid 0\le x\le X-Lt, 0<t<T_0\}$. By Definition 2.1, for every $k\in R^1$ and any nonnegative function, $\varphi(x,t)\in C_0^\infty(Q_T)$, $\varphi(0,t)=0$. Then

$$\int_0^{+\infty}\int_0^{+\infty} \{|u^{(i)}(x,t)-k|\varphi_t + \operatorname{sgn}(u^{(i)}(x,t)-k) \cdot (f(u^{(i)}(x,t))-f(k))\varphi_x\}dxdt \ge 0, \quad i=1,2 \tag{3.3}$$

Let $w_h(x)=\dfrac{1}{h}w\left(\dfrac{x}{h}\right)$, $j_h(x-\xi,t-\tau)=w_h(x-\xi)-w_h(t-\tau)$, where $h>0$, $w\in C_0^\infty(-\infty,+\infty)$, $w(x)\ge 0$, $\operatorname{supp} w\subset[-1,1]$, $\displaystyle\int_{-\infty}^{+\infty} w(x)dx=1$, ξ and τ are positive parameters. Using the method of the papers [1] and [7], we can obtain

$$\int_0^{+\infty}\int_0^{+\infty} \{|u^{(1)}(x,t)-u^{(2)}(x,t)|\varphi_t + \operatorname{sgn}(u^{(1)}(x,t)-u^{(2)}(x,t)) \cdot (f(u^{(1)}(x,t))-f(u^{(2)}(x,t)))\varphi_x\}dxdt \ge 0 \tag{3.4}$$

We select $t_1,t_2\in(0,T_0)$, $t_1<t_2$, and define

$$\alpha_h(x)=\int_{-\infty}^x w_h(s)ds=\int_{-\infty}^{\frac{x}{h}} w(s)ds$$

$$G_h(t)=\alpha_h(t-t_1)-\alpha_h(t-t_2)=\int_{\frac{t-t_2}{h}}^{\frac{t-t_1}{h}} w(s)ds$$

$$X_\varepsilon(x,t) = 1 - \alpha_\varepsilon(x + Lt - X + \varepsilon) = \int_{1+\frac{x+Lt-X}{\varepsilon}}^{+\infty} w(s)ds$$

$$H_\delta(x) = 1 - \alpha_\delta(-x + \delta) = \int_{1-\frac{x}{\delta}}^{+\infty} w(s)ds$$

where $\varepsilon > 0$, $h > 0$, $\delta > 0$. Obviously, for sufficiently small $h, \varepsilon, \delta, G_t(t) \in C_0^\infty(0, T_0)$, $X_\varepsilon(x,t) \in C_0^\infty(Q_T)$, $H_\delta(x) \in C_0^\infty[0, X)$, $H_\delta(0) = 0$. If in (3.4) we set

$$\varphi(x,t) = G_h(t)X_\varepsilon(x,t)H_\delta(x)$$

then for sufficiently small h, ε, δ,

$$\varphi(x,t) \in C_0^\infty(Q_T), \quad \varphi(x,t) \geq 0, \quad \varphi(0,t) = 0$$

Obtaining

$$\begin{aligned}
&\iint_{Q_T} |u^{(1)}(x,t) - u^{(2)}(x,t)|G_h'(t)X_\varepsilon(x,t)H_\delta(x)dxdt \\
&\quad + \iint_{Q_T} \Big\{ |u^{(1)}(x,t) - u^{(2)}(x,t)| \frac{\partial X_\varepsilon}{\partial t} + \operatorname{sgn}(u^{(1)}(x,t) - u^{(2)}(x,t)) \\
&\quad\quad \cdot (f(u^{(1)}(x,t)) - f(u^{(2)}(x,t))) \frac{\partial X_\varepsilon}{\partial x} \Big\} G_h(t)H_\delta(x)dxdt \\
&\quad + \iint_{Q_T} \operatorname{sgn}(u^{(1)}(x,t) - u^{(2)}(x,t))(f(u^{(1)}(x,t)) - f(u^{(2)}(x,t))) \\
&\quad\quad \cdot H_\delta'(x)X_\varepsilon(x,t)G_h(t)dxdt \equiv I_1 + I_2 + I_3 \geq 0
\end{aligned} \tag{3.5}$$

we note that when $u^{(1)}(x,t) \neq u^{(2)}(x,t)$:

$$\begin{aligned}
&|u^{(1)}(x,t) - u^{(2)}(x,t)| \frac{\partial X_\varepsilon}{\partial t} + \operatorname{sgn}(u^{(1)}(x,t) - u^{(2)}(x,t)) \\
&\quad \cdot (f(u^{(1)}(x,t)) - f(u^{(2)}(x,t))) \frac{\partial X_\varepsilon}{\partial x} \\
&= |u^{(1)}(x,t) - u^{(2)}(x,t)| \Big\{ \frac{\partial X_\varepsilon}{\partial t} + \frac{f(u^{(1)}(x,t)) - f(u^{(2)}(x,t))}{u^{(1)}(x,t) - u^{(2)}(x,t)} \frac{\partial X_\varepsilon}{\partial x} \Big\} \\
&\leq |u^{(1)}(x,t) - u^{(2)}(x,t)| \Big\{ \frac{\partial X_\varepsilon}{\partial t} + L\Big|\frac{\partial X_\varepsilon}{\partial x}\Big| \Big\} = 0
\end{aligned}$$

then the second integration in (3.5)

$$I_2 \leq 0 \tag{3.6}$$

Because, for every $(x,t)\in\Omega=\{(x,t)\mid 0\le x<X-Lt, 0<t<T_0\}$, $X_\varepsilon(x,t)\equiv 1$, and $H'_\delta(x)=\frac{1}{\delta}w\Big(1-\frac{x}{\delta}\Big)\ge 0$, $H'_\delta(x)\in C_0^\infty[0,2\delta)$, hence the third integration in (3.5):

$$I_3=\int_0^{T_0}\int_0^{2\delta}\operatorname{sgn}(u^{(1)}(x,t)-u^{(2)}(x,t))\cdot(f(u^{(1)}(x,t)) - f(u^{(2)}(x,t)))H'_\delta(x)G_h(t)dxdt \tag{3.7}$$

Note that

$$\begin{aligned} m(t)\int_0^{2\delta}H'_\delta(x)dx &\le \int_0^{2\delta}\operatorname{sgn}(u^{(1)}(x,t)-u^{(2)}(x,t))\cdot(f(u^{(1)}(x,t)) \\ &\quad - f(u^{(2)}(x,t)))H'_\delta(x)dx \\ &\le M(t)\int_0^{2\delta}H'_\delta(x)dx \end{aligned}$$

where:

$$m(t)=\inf_{0\le x\le 2\delta}\{\operatorname{sgn}(u^{(1)}(x,t)-u^{(2)}(x,t))(f(u^{(1)}(x,t))-f(u^{(2)}(x,t)))\}$$

$$M(t)=\sup_{0\le x\le 2\delta}\{\operatorname{sgn}(u^{(1)}(x,t)-u^{(2)}(x,t))(f(u^{(1)}(x,t))-f(u^{(2)}(x,t)))\}$$

and that

$$\int_0^{2\delta}H'_\delta(x)dx=\int_0^{2\delta}\frac{1}{\delta}w\Big(1-\frac{x}{\delta}\Big)dx=1$$

Thus

$$\begin{aligned} &\lim_{\delta\to 0}\int_0^{2\delta}\operatorname{sgn}(u^{(1)}(x,t)-u^{(2)}(x,t))(f(u^{(1)}(x,t))-f(u^{(2)}(x,t)))\cdot H'_\delta(x)dx \\ &= \operatorname{sgn}(u^{(1)}(0,t)-u^{(2)}(0,t))(f(u^{(1)}(0,t))-f(u^{(2)}(0,t))) \end{aligned}$$

Letting $\delta\to 0$ in (3.7) and using Lemma 2.2, we obtain:

$$I_3\le\int_0^{T_0}L|(u_b^{(1)}(t)-u_b^{(2)}(t)|G_h(t)dt=L\int_{t_1}^{t_2}|u_b^{(1)}(t)-u_b^{(2)}(t)|G_h(t)dt \tag{3.8}$$

Note that for sufficiently small h:

$$G_h(t)\equiv 1,\quad t\in(t_1,t_2)$$

and

$$G'_h(t)=w_h(t-t_1)-w_h(t-t_2)$$

By (3.5), (3.7) and (3.8), we have

$$\int_0^{T_0}\int_0^{X-Lt}|u^{(1)}(x,t)-u^{(2)}(x,t)|(w_h(t-t_1)-w_h(t-t_2))dxdt$$

$$+L\int_{t_2}^{t_1}|u_b^{(1)}(t)-u_b^{(2)}(t)|dt\geq 0 \tag{3.9}$$

Let

$$\mu(t)=\int_0^{X-Lt}|u^{(1)}(x,t)-u^{(2)}(x,t)|dx$$

then

$$\begin{aligned}\lim_{h\to 0}\int_0^{T_0}\mu(t)w_h(t-t_i)dt&=\lim_{h\to 0}\int_{|t-t_i|<h}\mu(t)w_h(t-t_i)dt\\&=\mu(t_i),\quad i=1,2\end{aligned} \tag{3.10}$$

Letting $h\to 0$ in (3.9) and using (3.10), we obtain

$$\begin{aligned}\int_0^{X-Lt_2}|u^{(1)}(x,t_2)-u^{(2)}(x,t_2)|dx&\leq\int_0^{X-Lt_1}|u^{(1)}(x,t_1)-u^{(2)}(x,t_1)|dx\\&+L\int_{t_1}^{t_2}|u_b^{(1)}(t)-u_b^{(2)}(t)|dt\end{aligned} \tag{3.11}$$

Following [1], let $t_1\to 0$ and exchange t_2 for t in (3.11), then for almost all $t\in(0,T_0)$

$$\begin{aligned}\int_0^{X-Lt}|u^{(1)}(x,t)-u^{(2)}(x,t)|dx&\leq\int_0^{X}|u_0^{(1)}(x)-u_0^{(2)}(x)|dx\\&+L\int_0^{t}|u_b^{(1)}(t)-u_b^{(2)}(t)|dt\end{aligned}$$

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