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GLOBAL $W^{2,p}$ $(2 \leq p < \infty)$ SOLUTIONS OF GBBM EQUATIONS IN ARBITRARY DIMENSIONS

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Abstract This paper studies the initial-boundary value problem of GBBM equations

$$u_t - \Delta u_t = \operatorname{div} f(u) \tag{a}$$

$$u(x,0) = u_0(x) \tag{b}$$

$$u\,|_{\partial\Omega} = 0 \tag{c}$$

in arbitrary dimensions, $\Omega \subset \mathbf{R}^n$. Suppose that $f(s) \in C^1$ and $|f'(s)| \leq C(1+|s|^\gamma)$, $0 \leq \gamma \leq \dfrac{2}{n-2}$ if $n \geq 3$, $0 \leq \gamma < \infty$ if $n = 2$, $u_0(x) \in W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$ $(2 \leq p < \infty)$, then $\forall T > 0$ there exists a unique global $W^{2,p}$ solution $u \in W^{1,\infty}(0,T; W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega))$, so the known results are generalized and improved essentially.

Key Words GBBM equation; initial-boundary value; global $W^{2,p}$ solution.

Classification 35Q.

1. Introduction

There are already many results $[1-7]$ on the existence and uniqueness of global solutions of the initial-boundary value problem for GBBM equations

$$u_t - \Delta u_t = \operatorname{div} f(u) \tag{1}$$

$$u(x,0) = u_0(x) \tag{2}$$

$$u\,|_{\partial\Omega} = 0 \tag{3}$$

where $\Omega \subset \mathbf{R}^n$ is a smooth bounded domain. In [5–7] Chen Yunmei, Goldstein and Guo Boling et al. all studied global $W^{2,p}$ solutions of the problem (1)–(3) respectively, the results obtained by them are as follows: Assume that $\partial\Omega$ is sufficiently smooth, $f(s) \in C^2$, $f'(0)=0$ and satisfies the hypothesis

(H) $$|f'(s)| \le C(1+|s|^\gamma),\ 0 \le \gamma \le \frac{2}{n-2} \text{ if } n \ge 3,\ 0 \le \gamma < \infty \text{ if } n=2$$

$u_0(x) \in W^{2,p}(\Omega) \cap W^{2,2}(\Omega) \cap W_0^{1,p}(\Omega)$, then there exists a unique solution $u \in C([0,\infty); W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega))$, where $\max\left\{1, \frac{n}{2}\right\} < p < \infty$. Clearly the condition $\frac{n}{2} < p$, which is necessary if one uses the methods of [5–7], is very harsh. For example, according to this condition for the most important case $p=2$ the values of n only can be $n \le 3$. So these results are no satisfactory. However up to now for the case $n \ge 2p$ the existence of global $W^{2,p}$ solution of the problem (1)–(3) is still open.

In this paper by using completely different method from [1–7] we study the problem (1)–(3) in arbitrary dimensions. We only assume that $\partial\Omega$ is sufficiently smooth, $f(s) \in C^1$ and satisfies (H), $u_0(x) \in W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$, then for any $T>0$ we obtain a unique global solution $u \in W^{1,\infty}(0,T; W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega))$, where $2 \le p < \infty$. So we have generalized and improved the known results essentially.

In this paper we always assume $\Omega \subset \mathbf{R}^n$ be a sufficiently smooth bounded domain, $\|\cdot\|_p$ denotes $L^p(\Omega)$ norm, $\|\cdot\| \equiv \|\cdot\|_2$, $\|\cdot\|_{k,p}$ denotes $W^{k,p}(\Omega)$ norm and $(u,v) = \int_\Omega u(x)v(x)dx$; C, C_i, M, M_i and E_i all denote the constants independent of u.

2. Global $W^{2,2}$ Solutions

Let $\{w_j(x)\}$ be a system of eigenfunctions of the problem $\Delta w_j + \lambda w_j = 0$ in Ω, $w_j|_{\partial\Omega} = 0$ construct approximate solutions of the problem (1)–(3) as follows

$$u_m(x,t) = \sum_{j=1}^m \alpha_{jm}(t) w_j(x), \quad m = 1, 2, \cdots \tag{4}$$

According to Galerkin method $\alpha_{jm}(t)$ satisfies

$$(u_{mt}, w_s) - (\Delta u_{mt}, w_s) = (\operatorname{div} f(u_m), w_s) \tag{5}$$

$$\alpha_{jm}(0) = a_{jm},\ s,\ j = 1, 2, \cdots, m \tag{6}$$

Lemma 1 *Assume that $f(s) \in C^1$, $u_0(x) \in W_0^{1,2}(\Omega)$, and choose a_{jm} such that $u_m(x,0) \xrightarrow{W^{1,2}} u_0(x)$, then we have*

$$\|u_m\|^2 + \|\nabla u_m\|^2 \equiv \|u_m(0)\|^2 + \|\nabla u_m(0)\|^2 \le E_1 \quad (0 \le t < \infty) \tag{7}$$

Proof Multiplying (5) by $\alpha_{sm}(t)$ and summing it for s we obtain

$$\frac{d}{dt}[\|u_m\|^2 + \|\nabla u_m\|^2] = -2(f(u_m), \operatorname{div} u_m)$$

$$= -2\int_\Omega \operatorname{div} F(u_m)dx = -2\int_{\partial\Omega} F(u_m)dS = 0$$

where $F(u) = \int_0^u f(s)ds$, it follows (7).

Lemma 2 *Assume that $f(s) \in C^1$ and satisfies* (H), $u_0(x) \in W^{2,2}(\Omega) \cap W_0^{1,2}(\Omega)$, *and choose a_{jm} such that $u_m(x,0) \xrightarrow{W^*} u_0(x)$, then $\forall T > 0$ we have*

$$\|\nabla u_m\|^2 + \|\Delta u_m\|^2 \le E_2 \quad (0 \le t \le T) \tag{8}$$

Proof Multiplying (5) by $\lambda_s \alpha_{sm}(t)$ and summing it for s, from (H), Lemma 1 and Sobolev embedding theorem it follows that

$$\begin{aligned}\frac{d}{dt}[\|\nabla u_m\|^2 + \|\Delta u_m\|^2] &= -2(\operatorname{div} f(u_m), \Delta u_m)\\ &\le 2\|f'(u_m)\|_q \|\nabla u_m\|_p \|\Delta u_m\| \le M_1 \|\Delta u_m\|^2\end{aligned}$$

here and in following Lemma 3, Theorem 2 and Lemma 4, $p = \frac{2n}{n-2}$ if $n \ge 3$, $2 \le p < \infty$ if $n = 2$, $p = \infty$ if $n = 1$, $\frac{1}{p} + \frac{1}{q} = \frac{1}{2}$. Integrating with respect to t from 0 to t, by Gronwall inequality it follows (8).

Lemma 3 *Under the conditions of Lemma 2 we have further*

$$\|\nabla u_{mt}\| + \|\Delta u_{mt}\| \le E_3 \quad (0 \le t \le T) \tag{9}$$

Proof Multiplying (5) by $\lambda_s \alpha'_{sm}(t)$ and summing it for s, from (H) and Lemma 2 we get

$$\|\nabla u_{mt}\|^2 + \|\Delta u_{mt}\|^2 \le \|f'(u_m)\|_q \|\nabla u_m\|_p \|\Delta u_{mt}\| \le M_2 \|\Delta u_{mt}\|$$

it yields (9).

From Lemmas 1–3 we can obtain the following

Theorem 1 *Suppose that $f(s) \in C^1$ and satisfies* (H), $u_0(x) \in W^{2,2}(\Omega) \cap W_0^{1,2}(\Omega)$, *then $\forall T > 0$ problem* (1)–(3) *has at least one solution $u(x,t) \in W^{1,\infty}(0,T; W^{2,2}(\Omega) \cap W_0^{1,2}(\Omega))$.*

Theorem 2 *Under the conditions of Theorem 1, the $W^{2,2}$ solution of the problem* (1)–(3) *is unique.*

Proof Let u and v be any two $W^{2,2}$ solutions, $w = u - v$, then

$$w_t - \Delta w_t = \operatorname{div} f(u) - \operatorname{div} f(v) \tag{10}$$

Multiplying (10) by w and integrating on Ω we obtain

$$\frac{d}{dt}[\|w\|^2 + \|\nabla w\|^2] = -2(f(u) - f(v), \nabla w)$$

$$\leq 2\|\tilde{f}'\|_q\|w\|_p\|\nabla w\| \leq M_3\|\nabla w\|^2$$

where $\tilde{f}' = f'(u + \theta(v - u))$, $0 < \theta < 1$. Integrating with respect to t from 0 to t we can obtain

$$\|w\|^2 + \|\nabla w\|^2 = 0, \quad w = 0$$

3. Global $W^{2,p}$ Solutions $(2 < p < \infty)$

Lemma 4 *Assume that $f(s) \in C^1$ and satisfies* (H), *u is the global $W^{2,2}$ solution of the problem* (1)–(3), *then we have further*

$$u_{tt} \in L^\infty(0, T; W_0^{1,2}(\Omega)) \tag{11}$$

Proof First rewrite (5) as follows

$$(u_{mt}, w_s) - (\Delta u_{mt}, w_s) = -(f(u_m), \nabla w_s) \tag{5'}$$

Differentiating (5′) with respect to t, multiplying the obtained equality by $\alpha''_{sm}(t)$ and summing it for s, from Lemma 1 and Lemma 3 we obtain

$$\|u_{mtt}\|^2 + \|\nabla u_{mtt}\|^2 \leq \|f'(u_m)\|_q\|u_{mt}\|_p\|\nabla u_{mtt}\| \leq M_4\|\nabla u_{mtt}\|$$

it follows that

$$\|u_{mtt}\| + \|\nabla u_{mtt}\| \leq E_4 \quad (0 \leq t \leq T)$$

so (11) holds.

From Theorems 7.2 and 7.4 of [8] we can get the following two Lemmas.

Lemma 5 *Assume that $v(x) \in W_0^{1,2}(\Omega)$ is the unique solution of the equation*

$$-\Delta v = \operatorname{div} f(x) \tag{12}$$

i.e.

$$\int_\Omega (\nabla v \cdot \nabla \varphi + f(x) \cdot \nabla \varphi)dx = 0, \quad \forall \varphi \in W_0^{1,2}(\Omega)$$

$\partial\Omega \in C^1$, $f(x) \in L^p(\Omega)$, then $v(x) \in W_0^{1,p}(\Omega)$, $\forall 2 < p < \infty$.

Lemma 6 *Assume that $v(x) \in W_0^{1,2}(\Omega)$ is the unique solution of the equation*

$$v - \Delta v = \operatorname{div} f(x) \tag{13}$$

i.e.

$$\int_\Omega (v\varphi + \nabla v \cdot \nabla \varphi + f(x) \cdot \nabla \varphi)dx = 0, \quad \forall \varphi \in W_0^{1,2}(\Omega)$$

$f(x) \in W^{m+1,p}(\Omega)$, $\partial\Omega \in C^{m+1,1}$, *then* $v(x) \in W^{m+2,p}(\Omega)$, $\forall m \ge 0$, $1 < p < \infty$.

Rewrite (13) as following equivalent integral equation

$$v = (I-\Delta)^{-1}\mathrm{div}\, f(x) \tag{14}$$

then in Lemma 6 the equation (13) can be replaced by the equation (14).

Let $u(x,t)$ be the unique $W^{2,2}$ solution of the problem (1)–(3), then u satisfies

$$u_t = (I-\Delta)^{-1}\mathrm{div}\, f(u) \tag{15}$$

$$\Delta u_t = \mathrm{div}\,[\mathrm{grad}\,(I-\Delta)^{-1}\mathrm{div}\, f(u)] \tag{16}$$

and

$$u = u_0 + \int_0^t (I-\Delta)^{-1}\mathrm{div}\, f(u)d\tau \tag{17}$$

Remark By Theorem 1, Lemma 4, we have $u_t \in L^\infty(0,T;W^{2,2}(\Omega)) \cap C([0,T]; W^{1,2}(\Omega))$, so for any fixed $t \in (0,T]$, (15)–(17) all have meaning.

Theorem 3 *Suppose that* $f(s) \in C^1$ *and satisfies* (H), $u_0(x) \in W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$ $(2 < p < \infty)$, *then* $\forall T > 0$ *problem* (1)–(3) *has a unique solution* $u(x,t) \in W^{1,\infty}(0,T;W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega))$.

Proof Since $u_0(x) \in W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega) \subset W^{2,2}(\Omega) \cap W_0^{1,2}(\Omega)$, by Theorem 1, Theorem 2 and the remark, $\forall T > 0$ the problem (1)–(3) has a unique solution $u(x,t) \in W^{1,\infty}(0,T;W^{2,2}(\Omega) \cap W_0^{1,2}(\Omega))$. Look upon $f(u)$ as a known function, then for any fixed $t \in (0,T]$, $u_t(.,t) \in W^{2,2}(\Omega) \cap W_0^{1,2}(\Omega)$ is the unique solution of the linear integral equation (15) and linear differential equation (16).

Step 1 By Sobolev embeding theorem $u \in W^{1,\infty}(0,T;W^{1,q}(\Omega) \cap L^r(\Omega))$, where $2 \le q \le \dfrac{2n}{n-2}$ if $n \ge 3$, $2 \le q < \infty$ if $n = 2$, $2 \le q \le \infty$ if $n = 1$; $2 \le r \le \dfrac{2n}{n-4}$ if $n > 4$, $2 \le r < \infty$ if $n = 4$, $2 \le r \le \infty$ if $n \le 3$.

Clearly the following inequality holds

$$\|f(u)\|_{1,\overline{p}_1} \le C_0\|f'(u)\|_{r_1}\|\nabla u\|_{q_1} + \|f(u)\|_{\overline{p}_1}, \quad \frac{1}{r_1} + \frac{1}{q_1} = \frac{1}{\overline{p}_1} \tag{18}$$

(i) $n \le 3$, choose $r_1 = \infty$, $q_1 = \dfrac{2n}{n-2}$ if $n = 3$, q_1 be an arbitrarily large positive number if $n = 2$, $q_1 = \infty$ if $n = 1$, then we have $\overline{p}_1 = q_1$ and

$$\|f'(u)\|_{r_1}\|\nabla u\|_{q_1} \le C_1, \quad \|f(u)\|_{p_1} \le C_2 \tag{19}$$

(ii) $n = 4$, choose $q_1 = \dfrac{2n}{n-2}$, r_1 be an arbitrarily large positive number, then (19) also holds, where $\overline{p}_1 = \dfrac{2(n-\delta)}{n-2}$, δ is an arbitrarily small positive number.

(iii) $n > 4$, choose $q_1 = \dfrac{2n}{n-2}$, $r_1 = \dfrac{n(n-2)}{n-4}$, then again $\|f'(u)\|_{r_1}\|\nabla u\|_{q_1} \le C_1$. In view of $\overline{p}_1\dfrac{n}{n-2} < \dfrac{2n}{n-4}$, so by (H), $\|f(u)\|_{\overline{p}_1} \le C_2$, where $\overline{p}_1 = \dfrac{2n(n-2)}{n^2-2n-4} > 2$.

Thus for all n we always have $f(u) \in L^\infty(0,T;W^{1,\overline{p}_1}(\Omega))$, where $\overline{p}_1 = \infty$ if $n = 1$, $\overline{p}_1 =$ an arbitrarily large positive number if $n = 2$, $\overline{p}_1 = \dfrac{2(n-\delta)}{n-2}$, $\delta = 0$ if $n = 3$, δ is an arbitrarily small positive number if $n = 4$, $\overline{p}_1 = \dfrac{2n(n-2)}{n^2-2(n+2)}$ if $n > 4$. So by (15) and Lemma 6 $u_t \in L^\infty(0,T;W^{2,\overline{p}_1}(\Omega))$. And by (16) and Lemma 5, $u_t \in L^\infty(0,T;W_0^{1,\overline{p}_1}(\Omega))$, so $u_t \in L^\infty(0,T;W^{2,\overline{p}_1}(\Omega)\cap W_0^{1,\overline{p}_1}(\Omega))$. From (17) $u \in W^{1,\infty}(0,T;W^{2,p_1}(\Omega)\cap W_0^{1,p_1}(\Omega))$, $p_1 = \min\{\overline{p}_1, p\}$. So when $\overline{p}_1 \ge p$, in particular, when $n \le 2$, we obtain $u \in W^{1,\infty}(0,T;W^{2,p}(\Omega)\cap W_0^{1,p}(\Omega))$. Therefore in the following we only need consider the case $n \ge 3$ and $\overline{p}_1 < p$.

Step 2 Assume $n \ge 3$ and $p_1 = \overline{p}_1 < p$.

First it follows from $u \in W^{1,\infty}(0,T;W^{2,p_1}(\Omega))$ that $u \in W^{1,\infty}(0,T;W^{1,q}(\Omega)\cap L^r(\Omega))$, where $2 \le q \le \dfrac{np_1}{n-p_1}$ if $p_1 < n$, $2 \le q < \infty$ if $p_1 = n$, $2 \le q \le \infty$ if $p_1 > n$; $2 \le r \le \dfrac{np_1}{n-2p_1}$ if $2p_1 < n$, $2 \le r < \infty$ if $2p_1 = n$, $2 \le r \le \infty$ if $2p_1 > n$.

Consider the following inequality

$$\|f(u)\|_{1,\overline{p}_2} \le C_0\|f'(u)\|_{r_2}\|\nabla u\|_{q_2} + \|f(u)\|_{\overline{p}_2}, \quad \frac{1}{r_2}+\frac{1}{q_2}=\frac{1}{\overline{p}_2} \tag{20}$$

(i) $2p_1 > n$, choose $r_2 = \infty$; $q_2 = \dfrac{np_1}{n-p_1}$ if $p_1 < n$, q_2 be an arbitrarily large positive number if $p_1 = n$, $q_2 = \infty$ if $p_1 > n$, then we have $\overline{p}_2 = q_2$ and

$$\|f'(u)\|_{r_2}\|\nabla u\|_{q_2} \le C_1, \quad \|f(u)\|_{\overline{p}_2} \le C_2 \tag{21}$$

(ii) $2p_1 = n$, choose $q_2 = \dfrac{np_1}{n-p_1} = n$, r_2 be an arbitrarily large positive number, then (21) also holds.

(iii) $2p_1 < n$, choose $q_2 = \dfrac{np_1}{n-p_1}$, $r_2 = \dfrac{p_1 n(n-2)}{2(n-2p_1)}$, then $\|f'(u)\|_{r_2}\|\nabla u\|_{q_2} \le C_1$, and from $\overline{p}_2\dfrac{n}{n-2} < \dfrac{np_1}{n-2p_1}$, $\|f(u)\|_{\overline{p}_2} \le C_2$ follows.

So for all cases we always have $f(u) \in L^\infty(0,T;W^{1,\overline{p}_2}(\Omega))$, where $\overline{p}_2 = \infty$ if $p_1 > n$, $\overline{p}_2$ can be an arbitrarily large positive number if $p_1 = n$, $\overline{p}_2 = \dfrac{p_1(n-\delta)}{n-p_1}$, $\delta = 0$ if $2p_1 > n$ and $p_1 < n$, δ is an arbitrarily small positive number if $2p_1 = n$, $\overline{p}_2 = \dfrac{p_1 n(n-2)}{n^2-(n+2)p_1}$ if $2p_1 < n$. So from (15)–(17) and Lemma 5–Lemma 6 we obtain $u_t \in L^\infty(0,T;W^{2,\overline{p}_2}(\Omega)\cap W_0^{1,\overline{p}_2}(\Omega))$, $u \in W^{1,\infty}(0,T;W^{2,p_2}(\Omega)\cap W_0^{1,p_2}(\Omega))$, $p_2 = \min\{\overline{p}_2, p\}$. Thus if $\overline{p}_2 \ge p$, in particular if $p_1 \ge n$, then $u \in W^{1,\infty}(0,T;W^{2,p}(\Omega)\cap W_0^{1,p}(\Omega))$. If $\overline{p}_2 < p$, we again obtain $\overline{p}_3$ by a similar way. And so on and so forth we can obtain $\overline{p}_1, \overline{p}_2, \cdots$, satisfying $\overline{p}_{k+1} = \infty$ if $p > \overline{p}_k > n$, $\overline{p}_{k+1}$ can be an arbitrarily large positive number if $p > \overline{p}_k = n$. For these two cases we again obtain $u \in W^{1,\infty}(0,T; W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega))$.

If $\overline{p}_k < \min\{p, n\}$, then

$$\overline{p}_{k+1} = \frac{(n-\delta)\overline{p}_k}{n - \overline{p}_k}, \delta = \begin{cases} 0 \quad \text{if} \quad \overline{p}_k < n \quad \text{and} \quad 2\overline{p}_k > n \\ \text{an arbitrarily small positive number, if } 2\overline{p}_k = n \end{cases} \tag{22}$$

or

$$\overline{p}_{k+1} = \frac{n(n-2)\overline{p}_k}{n^2 - (n+2)\overline{p}_k}, \quad \text{if } 2\overline{p}_k < n, \quad k = 0, 1, 2, \cdots \tag{23}$$

$$\overline{p}_0 = 2 \quad \text{Clearly } \overline{p}_{k+1} > \overline{p}_k$$

(1) If $2\overline{p}_0 = 4 \ge n$, then $2\overline{p}_k > n$, $\forall k \ge 1$, so we have (22), $\forall k \ge 0$. Note that

$$\frac{\overline{p}_{k+1}}{\overline{p}_k} = \frac{n-\delta}{n-\overline{p}_k} > \frac{n-\delta}{n-\overline{p}_{k-1}} = \frac{\overline{p}_k}{\overline{p}_{k-1}}$$

so

$$\overline{p}_k = \frac{\overline{p}_k}{\overline{p}_{k-1}} \cdot \frac{\overline{p}_{k-1}}{\overline{p}_{k-2}} \cdots \frac{\overline{p}_1}{\overline{p}_0} \overline{p}_0 > 2\left(\frac{\overline{p}_1}{2}\right)^k$$

hence there exists a k_0 such that $\overline{p}_{k_0-1} < \min\{p, n\}$ and $\overline{p}_{k_0} \ge \min\{p, n\}$. If $p \le n$, then $\overline{p}_{k_0} \ge p$ and $u \in W^{1,\infty}(0,T; W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega))$; If $p > n$, then $\overline{p}_{k_0+1} = \infty$ or an arbitrarily large positive number, so $p_{k_0+1} = \min\{\overline{p}_{k_0+1}, p\} = p$, $u \in W^{1,\infty}(0,T; W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega))$.

(2) $2\overline{p}_0 = 4 < n$

(i) $2\overline{p}_k < n, \forall k \ge 1$, then (23) holds, $\forall k \ge 0$, and again we have

$$\frac{\overline{p}_{k+1}}{\overline{p}_k} = \frac{n(n-2)}{n^2 - (n+2)\overline{p}_k} > \frac{n(n-2)}{n^2 - (n+2)\overline{p}_{k-1}} = \frac{\overline{p}_k}{\overline{p}_{k-1}}, \quad \overline{p}_k > 2\left(\frac{\overline{p}_1}{2}\right)^k$$

so there must exists a k_0 such that $\overline{p}_{k_0-1} < \min\left\{p, \frac{n}{2}\right\} = p$, and $\overline{p}_{k_0} \ge p$, thereby $u \in W^{1,\infty}(0,T; W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega))$.

(ii) There exists a k_0 such that $2\overline{p}_{k_0-1} < n$, $2\overline{p}_{k_0} \ge n$ and $\overline{p}_{k_0} < \min\{p, n\}$, then

$$\frac{\overline{p}_{k+1}}{\overline{p}_k} = \frac{n(n-2)}{n^2 - (n+2)\overline{p}_k}, \quad k = 0, 1, \cdots, k_0 - 1$$

$$\frac{\overline{p}_{k+1}}{\overline{p}_k} = \frac{n-\delta}{n-\overline{p}_k}, k = k_0, k_0+1, \cdots$$

Note that

$$\frac{\overline{p}_{k_0+1}}{\overline{p}_{k_0}} = \frac{n-\delta}{n-\overline{p}_{k_0}} > \frac{n(n-2)}{n^2 - (n+2)\overline{p}_{k_0-1}} = \frac{\overline{p}_{k_0}}{\overline{p}_{k_0-1}}$$

so for all $k \geq 1$ again we have

$$\frac{\overline{p}_{k+1}}{\overline{p}_k} > \frac{\overline{p}_k}{\overline{p}_{k-1}}.$$

The other proof is similar to that of (1).

References

[1] Goldstein J.A. and Wichnoski B., On the Benjamin-Bona-Mahony equation in higher dimensions, *Nonlinear Analysis,* 4 (1980), 665–675.

[2] Avrin J. and Goldstein J.A., Global existence for the Benjamin-Bona-Mahony equation in arbitrary dimensions, *Nonlinear Analysis,* 9 (1985), 861–865.

[3] Calvert B., The equation $A(t,u(t))' + B(t,u(t)) = 0$, *Math. Proc. Cam. Phill. Soc.*, 79 (1976), 545–567.

[4] Guo Boling, Initial boundary value problem for one class of system of multidimensional inhomogeneous GBBM equation, *Chin. Ann. of Math.*, B8 (1987), 226–238.

[5] Chen Yunmei, Remark on the global existence for the GBBM equation in arbitrary dimensions, *Appl. Anal.*, 30(1-3) (1988), 1–15.

[6] Goldstein J.A., Kajikiya R. and Oharn S., On some nonlinear dispersive equations in several space variables, *Differential and Integral Equation*, 3(4) (1990), 617–632.

[7] Guo Boling and Miao Cangxing, On inhomogeneous GBBM equation, *J. Partial Differential Equation*, 8(3) (1995), 193–204.

[8] Rodrigues J.F., Obstacle Problems in Mathematical Physics. North-Holland-Amstedan, New York, 1987.