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APPROXIMATE INERTIAL MANIFOLDS OF STRONGLY DAMPED NONLINEAR WAVE EQUATIONS *

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Abstract In this paper we consider a class of strongly damped nonlinear wave equations. By the transformation of unknown functions and decomposition of operators, we construct a family of approximate inertial manifolds, and obtain the estimate of orders of approximation of such manifolds to solution orbits.

Key Words Approximate inertial manifolds; strong damping; nonlinear wave equation.

Classification 35B40, 35P10.

1. Introduction

In this paper, we will study the approximate inertial manifolds of strongly damped nonlinear wave equations with initial boundary value:

$$u_{tt} = \alpha u_{xxt} + \sigma(u_x)_x - f(u) + g(x), \quad x \in (0,1), \quad t \in [0, +\infty) \tag{1.1}$$

$$u(0) = u_0, \quad u_t(0) = u_1 \tag{1.2}$$

$$u(0,t) = u(1,t) = 0 \tag{1.3}$$

where $\sigma(s)$ is a smooth function with the following property

$$\sigma(0) = 0, \quad \sigma'(s) \geq \gamma_0 > 0, \quad \forall s \in \mathrm{R} \tag{1.4}$$

where α, γ_0 are positive constants. As for nonlinear item $f(u)$, we assume that f is smooth and satisfies the following conditions:

(i) $$\lim_{|s|\to\infty} \frac{F(s)}{|s|^2} \geq 0; \tag{1.5}$$

(ii) there exists a positive constant ω such that

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$$\lim_{|s|\to\infty} \frac{s \cdot f(s) - \omega \cdot F(s)}{|s|^2} \geq 0 \tag{1.6}$$

where $F(s) = \int_0^s f(s)ds$.

Let $I = (0,1)$, $L^2(I)$ be the usual Hilbert space of measurable functions which are square integrable on I, with the norm $|v|_0 = \left[\int_I |v|^2 dx\right]^{1/2}$, and inner product $(u,v) = \int_I uv dx$. Denote $A = -\partial_{xx}$, the Laplacian operator on $L^2(I)$, its domain is denoted by $D(A)$. Define $X = D(A) \times L^2(I)$, the norm of X will be denoted by $\|\cdot\|$, $\|(u,v)\| = (\|u\|_2^2 + |v|_0^2)^{1/2}$, where $\|u\|_2^2 = |Au.|_0^2$, $|\cdot|_0$ is the norm of $L^2(I)$.

The problem (1.1)–(1.3) arises when one considers the purely longitudinal motion of a homogeneous bar. This problem is studied in lot of literature. When f and g vanish, the existence and stability of classical solutions were studied by [1], [2]. The existence of solutions $(u, u_t) \in W^{1,\infty} \times W^{1,2}$ was proved by [3].

When f and g do not vanish and $\sigma(s)$ is nonlinear, Berkaliev [4], [5] studied the (E_0, E) attractor and its structure. Recently, in [6] the authors obtained the global existence and uniqueness of solutions $(u, u_t) \in C(0, \infty, X)$ and proved the existence of global compact attractor and its finite dimensionality property. On the other hand, there were many results to the inertial manifolds and approximate inertial manifolds for the nonlinear evolution equations of parabolic type. (See [7] and its references). But for the nonlinear wave equations, it is yet a difficult problem. Recently, K.S. Chueshov [8] studied the approximate inertial manifolds of strongly dampled nonlinear wave equations

$$u_{tt} + \alpha u_t - \Delta u + f(u) = g(x) \tag{1.7}$$

Under some assumption of $f(u)$, the author constructed a family of approximate inertial manifolds and obtained the estimate of orders of approximation of such manifolds to solution orbits. In this paper, by means of the transformation of unknown functions and decomposition of operators, we construct a family of approximate inertial manifolds $M(t)$ for the problem (1.1)–(1.3), our main results are

Theorem 1 *Suppose $\sigma(s)$ and $f(s)$ satisfy* (1.4), (1.5) *and* (1.6) *respectively, and $g \in H^1(I)$. For any $(u_0, u_1) \in X$, there exist a family of approximate inertial manifolds $M_t(p, \dot{p})$ such that*

$$\operatorname{dist}(M_t(p,\dot{p}), v(t)) \leq c\lambda_{N+1}^{-1/2} + c_1 e^{-\beta t} \tag{1.8}$$

when t is sufficiently large, where $v(t) = (u(t), u_t(t))$, and λ_{N+1} is the $(N+1)$th eigenvalue $[(N+1)\pi]^2$, especially $\operatorname{dist}(M_t(p,\dot{p}), A) \leq c\lambda_{N+1}^{1/2} + c_1 e^{-\beta t}$, where A is the global attractor of (1.1)–(1.3) *in X, constant c is independent of N.*

2. Preliminary Results

In [6], the authors showed the following results

Theorem 2 *Suppose* $\sigma(s)$ *and* $f(s)$ *satisfy* (1.4), (1.5)-(1.6) *respectively, and* $g \in H^1(I)$. *Then for any* $(u_0, u_1) \in X$, *there exists a unique global solution* $u(t)$ *satisfying* $(u, u_t) \in C(0, \infty, X)$. *Furthermore, the semigroup* $S(t)$ *associated with the solution of* (1.1)-(1.3) *possesses a global attractor* A.

Theorem 2 shows: if $\|(u_0, u_1)\| \le R$, then there exists a constant $c = c(R)$ such that

$$|u_{xx}|_0, |u_t|_0 \le c(R) \quad \text{for } t \ge t_0 \tag{2.1}$$

where t_0 depends only on R.

Because $S(t)$ is not compact in X, [6] introduces the following decomposition: $S(t) = C(t) + U_1(t)$, where $U_1(t)$ is uniformly precompact for $t > t_0$ for some t_0, and $C(t)$ is a continuous mapping from X into itself such that the following holds:

For every bounded set $B_1 \subset X$,

$$r_c(t) = \sup_{\varphi \in B_1} \|C(t)\varphi\| \to 0 \quad \text{as } t \to +\infty$$

$C(t)\varphi$ and $U_1(t)\varphi$ have the following forms:

$$C(t)(u_0, u_1) = (\overline{\omega}, \overline{v})$$
$$U_1(t)(u_0, u_1) = (\omega, v)$$
$$u(t) = \overline{\omega} + \omega, \quad u_t = \overline{v} + v$$

where $\overline{v}, v, \overline{\omega}, \omega$ are the solutions of the following problems respectively:

$$(\mathrm{P}_1) \qquad \begin{cases} \overline{v}_t - \alpha \overline{v}_{xx} = 0 \\ \overline{v}(x, 0) = u_1 \\ \overline{v}(0, t) = \overline{v}(1, t) = 0 \end{cases}$$

and

$$|\overline{v}(t)|_0^2 \le |u_1|^2 \mathrm{e}^{-\alpha\pi^2 t} \tag{2.2}$$

$$(\mathrm{P}_2) \qquad \begin{cases} v_t - \alpha v_{xx} = (\sigma(u_x))_x - f(u) + g \\ v(x, 0) = 0 \\ v(0, t) = v(1, t) = 0 \end{cases}$$

and

$$|v|_0^2 + |v_x|_0^2 \le c(R), t \ge T, v(x, t) \in L^\infty(0, \infty, H_0^1) \tag{2.3}$$

$$(\mathrm{P}_3) \begin{cases} \overline{\omega}_{tt} - \alpha \overline{\omega}_{xxt} + \alpha^{-1}\sigma'(u_x)(\overline{\omega}_t - \alpha \overline{\omega}_{xx}) = \alpha^{-1}\sigma'(u_x)\overline{v}(x, t) \\ \overline{\omega}(0) = u_0, \quad \overline{\omega}_t(0) = u_1 \\ \overline{\omega}(0, t) = \overline{\omega}(1, t) = 0 \end{cases}$$

and $\overline{\omega}(x, t) \in L^\infty(0, \infty; H^2 \cap H_0^1)$, $\overline{\omega}_t(x, t) \in L^\infty(0, \infty; L^2(I))$, there exists positive constant β_2, such that

$$\|(\overline{\omega}, \overline{\omega}_t)\| \le c(R)\mathrm{e}^{-\beta_2 t} \tag{2.4}$$

$$(\mathrm{P}_4)\begin{cases}(\omega_t-\alpha\omega_{xx})_t+\alpha^{-1}\sigma'(u_x)(\omega_t-\alpha\omega_{xx})=F(x,t)\\ \omega(x,0)=0,\quad \omega_t(x,0)=0\\ \omega(0,t)=\omega(1,t)=0\end{cases}$$

where $F(x,t)=\alpha^{-1}\sigma'(u_x)v(x,t)-f(u)+g$.

Denote

$$\theta(x,t)=\mathrm{e}^{-\alpha^{-1}\int_0^t\sigma'(u_x(x,\tau)d\tau}\int_0^t F(x,\tau)\mathrm{e}^{\alpha^{-1}\int_0^\tau\sigma'(u_x(x,s)ds}d\tau \tag{2.5}$$

[6] showed $\theta(x,t)\in L^\infty(0,\infty,H^1)$, $\theta_t(x,t)\in L^\infty(0,\infty,H^1)$.

Therefore, we have

$$(\mathrm{P}_5)\begin{cases}\omega_t-\alpha\omega_{xx}=\theta(x,t)\\ \omega(x,0)=0,\quad \omega_t(x,0)=0\\ \omega(0,t)=\omega(1,t)=0\end{cases}$$

It is not difficult to obtain $\omega(x,t)\in L^\infty(0,\infty;H^3\cap H_0^1)$, and

$$\|\omega(x,t)\|_{H^3}\le c(R),\quad |\omega_{tx}|_0\le c(R) \tag{2.6}$$

By all the above results, we have the following proposition:

Proposition 3 *Under the assumptions of Theorem 2 if there exists differentiable manifold* $\mu=(\varphi,\psi)$ *in* X *such that* $\|(\varphi-\omega,\psi-v)\|\le c_0\lambda_{N+1}^{-\frac{1}{2}}$ *for* $t\ge t_0$, *then we have*

$$\|(\varphi-u,\psi-u_t)\|\le c_0\lambda_{N+1}^{-\frac{1}{2}}+c_1e^{-\beta t}$$

Proof

$$\begin{aligned}\|(\varphi-u,\psi-u_t)\|&=\|(\varphi-\omega-\overline{\omega},\psi-v-\overline{v})\|\le\|(\varphi-\omega,\psi-v)\|+\|(\overline{\omega},\overline{v})\|\\&\le\|(\varphi-\omega,\psi-v)\|+\|\overline{\omega}\|_2+|\overline{v}|_0\le\|(\varphi-\omega,\psi-v)\|+c(R)\mathrm{e}^{-\beta t}\\&\le c_0\lambda_{N+1}^{-\frac{1}{2}}+c(R)\mathrm{e}^{-\beta t}\end{aligned}$$

where $\beta=\min(\alpha\pi^2,\beta_2)$.

3. Construction of Approximate Inertial Manifolds

By using Proposition 3, we can construct approximate manifolds of the problem (1.1)–(1.3) for (P_2) and (P_5)

Denote $p=P_Nu$, $\dot{p}=P_Nu_t$, $q=Q_Nu$, $\dot{q}=Q_Nu_t$, where P_N is the orthogonal projector with rank N, $Q_N=I-P_N$.

Let $v^p=P_Nv=P_N(u_t-\overline{v})=\dot{p}-\overline{v}^p$, $v^q=Q_Nv$; $\omega^p=P_N\omega=P_N(u-\overline{\omega})=p-\overline{\omega}^p$, $\omega^q=Q_N\omega$, where $\overline{v}^p$, $\overline{\omega}^p$ are the solutions of equations which are the projections of (P_1) and (P_3) under P_N.

Now, we consider the construction of $\varphi_v(p)$.

Decomposing (P_2), we have

$$v_t^p - \alpha v_{xx}^p = P_N[(\sigma(p_x+q_x))_x - f(p+q) + g] \tag{3.1}$$

$$v_t^q - \alpha v_{xx}^q = Q_N[(\sigma(p_x+q_x))_x - f(p+q) + g] \tag{3.2}$$

Let

$$\varphi_v(p) = \alpha^{-1}A^{-1}[Q_N(\sigma'(p_x)p_{xx} - f(p) + g)] \tag{3.3}$$

Denote

$$M^{(1)}(p,\overline{p}) = \dot{p} + \varphi_v(p) \tag{3.4}$$

Proposition 4 *Under the assumptions of Theorem 2, for any $(p,\dot{p}) \in B_1 \subset X$, we have*

$$\mathrm{dist}_{L^2(I)}(M^{(1)}, v) \le c(R)\lambda_{N+1}^{-\frac{1}{2}} + c(R)\mathrm{e}^{-\alpha\pi^2 t}, \quad \text{for } t \ge t_0 \tag{3.5}$$

Proof

$$|\alpha A^{\frac{1}{2}}(\varphi_v - v^q)|_0 \tag{3.6}$$
$$\le \|\sigma'(p_x)p_{xx} - \sigma'(p_x+q_x)(p_{xx}+q_{xx}) - f(p) + f(p+q)\|_{-\frac{1}{2}} + \|v_t^q\|_{-\frac{1}{2}}$$

But

$$|\sigma'(p_x)p_{xx} - \sigma'(p_x+q_x)(p_{xx}+q_{xx})|_0$$
$$\le |\sigma'(p_x) - \sigma'(p_x+q_x))p_{xx}|_0 + |\sigma'(p_x+q_x)q_{xx})|_0 \le C|q_x|_{L^\infty}|p_{xx}|_0 + C|q_{xx}|_0 \le c(R)$$
$$|f(p) - f(p+q)|_0 \le C|q|_0 \le c(R)$$

Hence $\|Q_N[\sigma'(p_x)p_{xx} - \sigma'(p_x+q_x)(p_{xx}+q_{xx}) - f(p) + f(p+q)\|_{-\frac{1}{2}} \le c(R)$.

Because

$$\|v_t^q\|_{-\frac{1}{2}} \le \alpha\|v_{xx}^q\|_{-\frac{1}{2}} + \|\sigma(p_x+q_x)u_{xx} - f(u) + g\|_{-\frac{1}{2}}$$
$$\le \alpha|v_x^q|_0 + |\sigma(p_x+q_x)u_{xx} - f(u) + g|_0$$
$$\le \alpha c(R) + C|u_{xx}|_0 + |f(u)|_0 + |g|_0 \le c(R)$$

We have

$$|\alpha A^{\frac{1}{2}}\varphi_v - \alpha A^{\frac{1}{2}}v^q|_0 \le c(R)$$

It indicates $|\varphi_v - v^q|_0 \le c(R)\lambda_{N+1}^{-\frac{1}{2}}$

$$\mathrm{dist}_{L^2(I)}(M^{(1)}, v) \le |\dot{p} + \varphi_v - v^p - v^q|_0 \le c(R)\lambda_{N+1}^{-\frac{1}{2}} + c(R)\mathrm{e}^{-\alpha\pi^2 t}$$

Next, we construct $\varphi_\omega(p,\dot{p})$, decomposing ($P_4$) into

$$\omega_t^p - \alpha\omega_{xx}^p = P_N\theta(x,t) \tag{3.7}$$

$$\omega_t^q - \alpha\omega_{xx}^q = Q_N\theta(x,t) \tag{3.8}$$

where

$$\theta(x,t) = \mathrm{e}^{-\alpha^{-1}\int_0^t \sigma'(u_x)d\tau}\int_0^t F(x,\tau)\mathrm{e}^{\alpha^{-1}\int_0^\tau \sigma'(u_x)ds}d\tau$$

$$F(x,t) = \alpha^{-1}\sigma'(u_x)v - f(u) + g \tag{3.9}$$

[6] has showed that

$$\|\theta\|_{H^1}^2 + \|\theta_t\|_{H^1}^2 \le c(R) \tag{3.10}$$

$$\|\omega\|_{H^3}^2 + \|\omega_t\|_{H^1}^2 \le c(R) \tag{3.11}$$

Setting

$$\varphi_\omega = \alpha^{-1}A^{-1}[Q_N\theta_1(x,t)] \tag{3.12}$$

where

$$\theta_1(x,t) = \int_0^t F_1(x,\tau)\mathrm{e}^{-\alpha^{-1}\int_\tau^t \sigma'(p_x)ds}d\tau \tag{3.13}$$

$$F_1(x,t) = \alpha^{-1}\sigma'(p_x)\dot{p} - f(p) + g \tag{3.14}$$

We denote the linear semigroup associated with the solution of (2.2) by $T(t)$, that is

$$\overline{v}(t) = T(t)u_1 = \frac{1}{2\sqrt{\pi t}}\int_{-\infty}^{\infty}\mathrm{e}^{-\frac{(x-\xi)^2}{4t}}u_1(\xi)d\xi$$

Therefore, by $u_t = v + \overline{v}$, we have

$$\dot{p} = v^p + P_N T(t)u_1$$

Now we estimate $\|\alpha A(\varphi_\omega - \omega^q)\|_{H^1}$,

$$\|\alpha A(\varphi_\omega - \omega^q)\|_{H^1} \le \|\omega_t^q\|_{H^1} + \|\theta(x,t) - \theta_1(x,t)\|_{H^1} \tag{3.15}$$

By (3.11),

$$\|\omega_t^q\|_{H^1} \le c(R) \tag{3.16}$$

$$\|\theta_1(x,t) - \theta(x,t)\|_{H^1} \le \left\|\int_0^t (F_1(x,\tau) - F(x,\tau))\mathrm{e}^{-\alpha^{-1}\int_\tau^t \sigma'(p_x)ds}d\tau\right\|_{H^1}$$

$$+\left\|\int_0^t F(x,\tau)(\mathrm{e}^{-\alpha^{-1}\int_\tau^t \sigma'(p_x)ds} - \mathrm{e}^{-\alpha^{-1}\int_\tau^t \sigma'(p_x+q_x)ds})d\tau\right\|_{H^1}$$

$$\triangleq \mathrm{I} + \mathrm{II} \tag{3.17}$$

Noting that $\sigma'(s) \ge \gamma_0 > 0$, we obtain

$$\mathrm{I} \le \left\|\int_0^t (\alpha^{-1}\sigma'(p_x)\dot{p} - f(p) + f(p+q) - \alpha^{-1}\sigma'(p_x+q_x)v)\mathrm{e}^{-\alpha^{-1}\gamma_0(t-\tau)}d\tau\right\|_{H^1}$$

$$\leq \left\| \int_0^t [\alpha^{-1}\sigma'(p_x)v^p - \alpha^{-1}\sigma'(p_x+q_x)v + \alpha^{-1}\sigma'(p_x)P_NT(t)u_1 + f(p+q) - f(p)]e^{-\alpha^{-1}\gamma_0(t-\tau)}d\tau \right\|_{H^1}$$

$$\leq \int_0^t \alpha^{-1}[\|(\sigma'(p_x)-\sigma'(p_x+q_x))v^p\|_{H^1} + \|\sigma'(p_x+q_x)v^q\|_{H^1} + \|\sigma'(p_x)P_NT(t)u_1\|_{H^1} + \alpha\|f(p+q)-f(p)\|_{H^1}]e^{-\alpha^{-1}\gamma_0(t-\tau)}d\tau$$

$$\leq C\alpha^{-1}\int_0^t (|q_x|_{L^\infty}\|v^p\|_{H^1} + \|v^q\|_{H^1} + t^{-\frac{1}{2}}|u_1|_0 + \|q\|_{H^1})e^{-\alpha^{-1}\gamma_0(t-\tau)}d\tau$$

$$\leq c(R)\Big[\int_0^t e^{-\alpha^{-1}\gamma_0(t-\tau)}d\tau + t^{-\frac{1}{2}}\int_0^t e^{-\alpha^{-1}\gamma_0(t-\tau)}d\tau\Big]$$

$$\leq c_1(R) + c(R)\alpha\gamma_0^{-1}t^{-\frac{1}{2}}(1-e^{-\alpha^{-1}\gamma_0 t}) \tag{3.18}$$

where we have used $\dot{p} = v^p + P_NT(t)u_1$ and $\|T(t)u_1\|_{H^1} \leq Ct^{-\frac{1}{2}}\|u_1\|_0, \forall t > 0, \|v\|_{H^1}^2 \leq c(R)$ (see (2.3))

$$\mathrm{II} = \left\| \int_0^t F(x,\tau)|(e^{-\alpha^{-1}\int_\tau^t \sigma'(p_x)ds} - e^{-\alpha^{-1}\cdot\int_\tau^t \sigma'(p_x+q_x)ds})d\tau \right\|_{H^1}$$

By using (2.3) and the definition of F, we have

$$\|F(x,t)\|_{L^\infty} \leq \|F(x,t)\|_{H^1} \leq c(R) \tag{3.19}$$

$$\int_0^t e^{-\alpha^{-1}\int_\tau^t \sigma'(p_x+q_x)ds}d\tau \leq \alpha\gamma_0^{-1} \tag{3.20}$$

and

$$\left|\int_0^t \Big[F(x,\tau)e^{-\alpha^{-1}\int_\tau^t \sigma'(p_x)ds}\alpha^{-1}\int_\tau^t \sigma''(p_{xx}+q_{xx})ds\Big]d\tau\right|_0$$

$$\leq \int_0^t \|F(x,\tau)\|_{L^\infty}\cdot e^{-\alpha^{-1}\int_\tau^t \sigma' ds}(\alpha^{-1})\int_\tau^t \|\sigma'\|_{L^\infty}\|u_{xx}\|_0 ds)d\tau$$

$$\leq c(R)\int_0^t (t-\tau)e^{-\alpha^{-1}\int_\tau^t \sigma ds} \leq c(R)\int_0^t (t-\tau)e^{-\alpha^{-1}\gamma_0(t-\tau)}d\tau \leq c_1(R) \tag{3.21}$$

where we have used the uniformly boundedness of $te^{-\alpha^{-1}\gamma_0 t}$ $(\forall t > 0)$.

From (3.19)–(3.21), it is easily to obtain II$\leq C(R)$. Combining (3.18), we have

$$\|\theta_1(x,t) - \theta(x,t)\|_{H^1} \leq c(R) + \alpha\gamma_0^{-1}c(R)t^{-\frac{1}{2}} \leq c_1(R), \quad \forall t \geq 1 \tag{3.22}$$

So that $\|\alpha A(\varphi_\omega - \omega^q)\|_{H^1} \leq c(R)$.

Denote

$$M_t^{(0)}(p,\dot{p}) = p + \varphi_\omega(p,\dot{p}) \tag{3.23}$$

Proposition 5 *Under the assumptions of Theorem 2, for any $(p,\dot{p}) \in B \subset X$, we have* $\mathrm{dist}_{H^2}(M_t^{(0)},\omega) \leq c(R)\lambda_{N+1}^{-\frac{1}{2}} + Ce^{-\beta t}$, *for $t \geq t_1 = \max(1,t_0)$.*

Proof By $\|A(\varphi_\omega - \omega^q)\|_{H^1} \le c(R)$, we obtain

$$\|A(\varphi_\omega - \omega^q)\|_0 \le c(R)\lambda_{N+1}^{-\frac{1}{2}}$$

So

$$\|A(p+\varphi_\omega - \omega^p - \omega^q)\|_2 \le \|A(\varphi_\omega - \omega^q)\|_0 + \|\overline{\omega}\|_2 \le c(R)\lambda_{N+1}^{-\frac{1}{2}} + C\mathrm{e}^{-\beta t}$$

Proof of Theorem 1

Let $M_t = (M_t^{(0)}, M^{(1)})$, by Proposition 3, it is sufficient to estimate $\|(p+\varphi_\omega - \omega, \dot{p} + \varphi_v - v)\|$.

Noting that Proposition 4 and Proposition 5, we have

$$\begin{aligned}\|(p+\varphi_\omega - \omega, \dot{p}+\varphi_v - v)\| &= \|p+\varphi_\omega - \omega\|_2 + \|\dot{p}+\varphi_v - v\|_0 \\ &\le c(R)\lambda_{N+1}^{-\frac{1}{2}} + C\mathrm{e}^{-\beta t} \quad \text{as } t \ge t_1\end{aligned}$$

Therefore, we obtain

$$\mathrm{dist}_X(M, v(t)) \le \|p+\varphi_\omega - \omega\|_2 + \|\dot{p}+\varphi_v - v\|_0 + \|\overline{\omega}\|_2 + \|\overline{v}\|_0 \le c(R)\lambda_{N+1}^{-\frac{1}{2}} + c(R)\mathrm{e}^{-\beta t}$$

Especially, for the global attractor A of (1.1)–(1.3) in X, we have $\mathrm{dist}_X(M, A) \le c(R)\lambda_{N+1}^{-\frac{1}{2}} + c_1\mathrm{e}^{-\beta t}$.

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