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ON THE EXISTENCE AND STABILITY OF POSITIVE SOLUTIONS FOR SOME PAIRS OF DIFFERENTIAL EQUATIONS*

Guo Zongming
(Department of Mathematics, Henan Normal University, Xinxiang 453002, China)
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Abstract In this paper, we are concerned with the existence and stability of the positive solutions of a semilinear elliptic system

$$
\begin{aligned}
&-\Delta u(x) = a(x)v^{\delta}(x) + e(x) \\
&-\Delta v(x) = b(x)u^{\mu}(x) + m(x) \quad \text{in } \Omega \\
&u = v = 0 \quad \text{on } \partial\Omega
\end{aligned}
$$

where $\Omega \subset \mathbf{R}^N$ is a bounded domain with smooth boundary $\partial\Omega$. It is shown that under the suitable conditions on δ, μ, there exist a stable and an unstable positive solutions for this system if e and m are sufficiently small in L^∞.

Key Words Positive solutions; multiple solutions; stability; semilinear differential systems.

Classification 25J35, 35B32.

1. Introduction

Let Ω be a bounded domain in $\mathbf{R}^N$ $(N \geq 2)$ with smooth boundary $\partial\Omega$ and $e, m \in L^\infty(\Omega)$, $e, m \geq 0$ in Ω and $e, m \not\equiv 0$ in Ω. In this paper we are concerned with the existence and stability for positive solutions of a semilinear differential system

$$-\Delta u(x) = a(x)v^{\delta}(x) + e(x) \tag{1.1}$$

$$-\Delta v(x) = b(x)u^{\mu}(x) + m(x) \quad \text{in} \quad \Omega \tag{1.2}$$

$$u = v = 0 \quad \text{on} \quad \partial\Omega \tag{1.3}$$

where $a, b \in C^0(\overline{\Omega})$ with

$$\widetilde{a} := \min_{\overline{\Omega}} a(x) > 0, \quad \widetilde{b} := \min_{\overline{\Omega}} b(x) > 0$$

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By a positive solution (u,v) of the problem (1.1)–(1.3) we mean that $(u,v) \in C^2(\Omega) \times C^2(\Omega)$ and $u > 0$, $v > 0$ in Ω.

When $e \equiv m \equiv 0$ in Ω, the existence of at least one positive solution of (1.1)–(1.3) has been studied recently by the author [1] under the suitable assumptions on δ and μ. In this paper, we shall show that under the same assumptions on δ and μ as in [1] and that $\|e\|_{L^\infty}$, $\|m\|_{L^\infty}$ are sufficiently small, the problem (1.1)–(1.3) has at least two positive solutions. Moreover, we also study the stability of such solutions. The existence of positive solutions of such kind of problems has been studied by many authors, see, for example, [2–6]. In all of these papers, the solutions were obtained by means of variational principles. The advantage of such methods is that the systems with some kinds of general nonlinearities can be handled. The shortcomings of such approaches are that they can not be easily used to discuss the systems with more than two equations; and to discuss the stability of solutions, meanwhile can not be easily used to discuss the existence and stability of the solutions of the corresponding parabolic systems. In this paper, we use the degree theory to study the existence of positive solutions of (1.1)–(1.3). Meanwhile, we also deal with the stability of such solutions. We should mention that our methods in this paper can be used to deal with the existence and stability of positive periodic solutions of the corresponding parabolic differential system

$$
\begin{aligned}
&\frac{\partial u}{\partial t} - \Delta u = a(t,x)v^\delta + e(t,x)\\
&\frac{\partial v}{\partial t} - \Delta v = b(t,x)u^\mu + m(t,x) \quad \text{in} \quad \mathbf{R}_+ \times \Omega\\
&u = v = 0 \quad \text{on} \quad \mathbf{R}_+ \times \partial\Omega\\
&u(t) = u(t+T), \quad v(t) = v(t+T) \quad \text{in} \quad \overline{\Omega}\\
&u > 0, \quad v > 0 \quad \text{on} \quad \mathbf{R}_+ \times \Omega
\end{aligned}
$$

Our methods of this paper can also be used to deal with the systems with more than two differential equations. The systems with more general nonlinearities need further discussion.

2. Existence Results

In this section, we first describe our existence theorem and then use the Leray-Schauder degree to show the existence of positive solutions. Throughout the rest of this paper, we set $E = C^0(\Omega) \times C^0(\Omega)$, denote by$\|\cdot\|_q$ the norm of $L^q(\Omega)$. We denote K the cone of non-negative functions in E.

Theorem 2.1 *Let $a, b : \Omega \to (0, +\infty)$ be bounded continuous functions such that*

$$\tilde{a} = \min_{x\in\overline{\Omega}} a(x) > 0, \quad \tilde{b} = \min_{x\in\overline{\Omega}} b(x) > 0$$

Suppose that $\delta > 1$, $\mu > 1$ and

$$\max\left\{\frac{2(1+\delta)}{\mu\delta - 1} - (N-2), \frac{2(1+\mu)}{\mu\delta - 1} - (N-2)\right\} \geq 0$$

Let $e, m \in L^\infty(\Omega)$, $e, m \geq 0$ in Ω and $e, m \not\equiv 0$ in Ω. If $\|e\|_\infty$ and $\|m\|_\infty$ are sufficiently small, then there exist at least two positive solutions for (1.1)–(1.3) in $C^2(\Omega) \times C^2(\Omega)$.

We denote by λ_1 and ϕ_1 the first eigenvalue and the first eigenfunction of $-\Delta$ in Ω with the Dirichlet boundary condition. It is well-known that the inverse operator $(-\Delta)^{-1} : E \to E$ of $(-\Delta)$ is compact. We first note that by the maximum principle to find a positive solution of (1.1)–(1.3) is equivalent to finding a nontrivial solution (u, v) of

$$\begin{aligned} -\Delta u &= a(x)|v|^\delta + e(x) \\ -\Delta v &= b(x)|u|^\mu + m(x) \quad \text{in} \quad \Omega \\ u = v &= 0 \quad \text{on} \quad \partial\Omega \end{aligned}$$

Let $f_1(x, v) = a(x)|v|^\delta + e(x)$, $f_2(x, u) = b(x)|u|^\mu + m(x)$ and $A(u, v) = (-\Delta)^{-1}(f_1, f_2) = ((-\Delta)^{-1}f_1, (-\Delta)^{-1}f_2)$. The positive solutions for (1.1)–(1.3) are fixed points of the operator A from E into itself. We first investigate the Leray-Schauder degree of $I - A$ near 0 and then in a large ball in E, where I is the identity map from E to E.

Lemma 2.2 *For any given $c > 0$, there exist positive numbers $r < c$ and $\bar{\delta}$ such that for $e, m \in L^\infty(\Omega)$ with $\|e\|_\infty + \|m\|_\infty \leq \bar{\delta}$,*

$$\deg_E(I - A, B_r(0), 0) = 1$$

where $B_r(0)$ denotes the ball centered at 0 in E with radius r.

Proof We know that (1.1)–(1.3) can be equivalently written in the form

$$(u, v) = A(u, v)$$

Set $q > N + 1$. Thus, according to the standard regularity of $-\Delta$, there exists $C > 0$ such that

$$\begin{aligned} \|u\|_{C^0} &\leq C\|a(x)|v|^\delta + e(x)\|_q \\ \|v\|_{C^0} &\leq C\|b(x)|u|^\mu + m(x)\|_q \end{aligned}$$

We make the homotopy $H : [0, 1] \times E \to E$ by

$$H(s, u, v) = (u, v) - (-\Delta)^{-1}[s(a(x)|v|^\delta + e(x)), s(b(x)|u|^\mu + m(x))]$$

We shall show that there exists $r > 0$ independent of s such that $H(s,\cdot) \neq 0$ in $\partial B_r(0)$ for all $s \in [0,1]$. Let $C_1 = \max\{\|a(x)\|_\infty, \|b(x)\|_\infty\}$. Let $c > 0$. Take $0 < r < 1$ satisfying $r < c$ and $\xi^{2\delta} \leq \frac{\lambda_1^2}{2C_1^2}\xi^2$ and $\xi^{2\mu} \leq \frac{\lambda_1^2}{2C_1^2}\xi^2$ for all $\xi \in \mathbf{R}$ with $|\xi| \leq r$. (Here we are using the facts that $\delta > 1$, $\mu > 1$.) Thus, if (u,v) satisfies that $H(s,u,v) = 0$, then

$$\int_\Omega |\nabla u|^2 = s\int_\Omega (a(x)|v|^\delta u + e(x)u)dx \tag{2.1}$$

$$\int_\Omega |\nabla v|^2 = s\int_\Omega (b(x)|u|^\mu v + m(x)v)dx \tag{2.2}$$

It follows from (2.1) that

$$\begin{aligned}\int_\Omega |\nabla u|^2 - s\int_\Omega (a(x)|v|^\delta u + e(x)u)dx \\ &\geq \int_\Omega |\nabla u|^2 - \frac{1}{2\lambda_1}\int_\Omega a^2(x)v^{2\delta}dx - \frac{\lambda_1}{2}\int_\Omega u^2dx - \int_\Omega e(x)udx \\ &\geq \frac{\lambda_1}{2}\int_\Omega u^2dx - \frac{1}{2\lambda_1}\int_\Omega a^2(x)v^{2\delta}dx - \int_\Omega eudx\end{aligned}$$

It follows from (2.2) that

$$\begin{aligned}\int_\Omega |\nabla v|^2 - s\int_\Omega (b(x)|u|^\mu v + m(x)v)dx \\ &\geq \int_\Omega |\nabla v|^2 - \frac{1}{2\lambda_1}\int_\Omega b^2(x)u^{2\mu}dx - \frac{\lambda_1}{2}\int_\Omega v^2dx - \int_\Omega m(x)vdx \\ &\geq \frac{\lambda_1}{2}\int_\Omega v^2dx - \frac{1}{2\lambda_1}\int_\Omega b^2(x)u^{2\mu}dx - \int_\Omega mvdx\end{aligned}$$

Therefore,

$$\begin{aligned}0 &\geq \frac{\lambda_1}{2}\int_\Omega (u^2+v^2)dx - \frac{1}{2\lambda_1}\int_\Omega (a^2(x)v^{2\delta} + b^2(x)u^{2\mu})dx - \int_\Omega (eu+mv)dx \\ &\geq \frac{\lambda_1}{2}\int_\Omega (u^2+v^2)dx - \frac{\lambda_1}{4}\int_\Omega (u^2+v^2)dx - \int_\Omega (eu+mv)dx \\ &\geq \frac{\lambda_1}{2}\int_\Omega (u^2+v^2)dx - \max\{\|e\|_\infty, \|m\|_\infty\}\int_\Omega (u+v)dx \\ &\geq \frac{\lambda_1}{2}\int_\Omega (|u|^{\mu q} + |v|^{\delta q})dx - \max\{\|e\|_\infty, \|m\|_\infty\}\int_\Omega (u+v)dx \\ &\geq \frac{\lambda_1}{2}\Big(\frac{(\|u\|_{C^0} + \|v\|_{C^0})}{CC_1} - |\Omega|^{1/q}\frac{(\|e\|_\infty + \|m\|_\infty)}{C_1}\Big)^q \\ &\quad - |\Omega|(\|e\|_\infty + \|m\|_\infty)(\|u\|_0 + \|v\|_0) > 0\end{aligned}$$

if $\frac{\lambda_1}{2}\Big(\frac{r}{CC_1} - |\Omega|^{1/q}\frac{\bar{\delta}}{C_1}\Big)^q - |\Omega|\bar{\delta}r > 0$; where $\|u\|_{C^0} + \|v\|_{C^0} = r$. We also use the fact that $\delta q > 2$ and $\mu q > 2$ here. This contradiction implies the nonexistence of the solutions of (2.1)–(2.2) on $\partial B_r(0)$ for $s \in [0,1]$. From the homotopy invariance of the Leray-Schauder degree, it follows that

$$\deg_E(I - A, B_r(0), 0) = \deg_E(I, B_r(0), 0) = 1$$

Lemma 2.3 *For each $(e, m) \in L^\infty(\Omega) \times L^\infty(\Omega)$ with $e \geq 0$, $m \geq 0$ in Ω, there exists $R > r$ such that*

$$\deg_E(I - A, B_R(0), 0) = 0$$

Proof Let us consider the problem

$$-\Delta u = a(x)(|v|^\delta + s|v| + s) + e(x) \tag{2.3}$$

$$-\Delta v = b(x)(|u|^\mu + s|u| + s) + m(x) \tag{2.4}$$

$$u = v = 0 \quad \text{on} \quad \partial\Omega \tag{2.5}$$

where $s \geq 0$.

Since $e \geq 0$ and $m \geq 0$ in Ω, it follows from the strong maximum principle that if $(u(s,\cdot), v(s,\cdot))$ is a nontrivial solution of (2.3)–(2.5), $u(s,\cdot), v(s,\cdot) > 0$ in Ω.

Let $g_s(x, u, v) = (a(x)(|v|^\delta + s|v| + s) + e(x),\ b(x)(|u|^\mu + s|u| + s) + m(x))$ and $F_s(u, v) = (-\Delta)^{-1} \circ g_s$. We first show that there exists $S > 0$ such that there is no nontrivial solution of (2.3)–(2.5) when $s \geq S$. Since any nontrivial solution of (2.3)–(2.5) is a positive solution of (2.3)–(2.5), we only need to prove that there is no positive solution of (2.3)–(2.5) when $s \geq S$. Multiplying (2.3) and (2.4) by the first eigenfunction ϕ_1 of $-\Delta$ under the Dirichlet boundary condition, we easily see that

$$\lambda_1 \int_\Omega u\phi_1 dx \geq s \int_\Omega a(x) v\phi_1 dx \geq \tilde{a}s \int_\Omega v\phi_1 dx \tag{2.6}$$

and

$$\lambda_1 \int_\Omega v\phi_1 dx \geq s\tilde{b} \int_\Omega u\phi_1 dx \tag{2.7}$$

Thus, (2.6) and (2.7) imply that

$$\lambda_1^2 \int_\Omega v\phi_1 dx \geq \tilde{a}\tilde{b}s^2 \int_\Omega v\phi_1 dx$$

This is clearly impossible for s sufficiently large. This means that there exists $S > 0$ such that there is no nontrivial solution of (2.3)–(2.5) when $s \geq S$.

Now we will show that for $s \in [0, S]$, there exists $R > 0$ independent of s such that, if $(u(s,\cdot), v(s,\cdot))$ is a nontrivial solution of (2.3)–(2.5), then we have $\|u(s,\cdot)\|_\infty + \|v(s,\cdot)\|_\infty < R$. Since a nontrivial solution of (2.3)–(2.5) must be a positive solution of (2.3)–(2.5), we only need to establish the *a priori* bounds for positive solutions of (2.3)–(2.5). The proof is based on the similar argument to that in the proof of Theorem A of

[1]. The main method is the blow up argument. For convenience, we only sketch the proof. Suppose that there are sequences $\{s_k\}$ with $s_k \in [0, S]$ and $\{(u_k(s_k,\cdot), v_k(s_k,\cdot))\}$ of positive solutions of (2.3)–(2.5) with $s = s_k$ such that

$$t_k = \max_{x\in\overline{\Omega}} u_k(s_k, x) \to +\infty \quad \text{or} \quad \xi_k = \max_{x\in\overline{\Omega}} v_k(s_k, x) \to +\infty$$

as $k \to +\infty$, we will derive contradiction.

Define

$$\alpha_1 = \frac{2(1+\delta)}{\delta\mu - 1} \quad \text{and} \quad \alpha_2 = \frac{2(1+\mu)}{\delta\mu - 1}$$

then $\alpha_1 > 0$ and $\alpha_2 > 0$. Set

$$\gamma_k = t_k^{\frac{1}{\alpha_1}} + \xi_k^{\frac{1}{\alpha_2}}$$

then $\gamma_k \to +\infty$, as $k \to +\infty$. Choose $x_k, \tilde{x}_k \in \Omega$ such that $t_k = u_k(s_k, x_k)$, $\xi_k = v_k(s_k, \tilde{x}_k)$. Let $Q_k = \gamma_k d(x_k, \partial\Omega)$ and $\tilde{Q}_k = \gamma_k d(\tilde{x}_k, \partial\Omega)$. Here $d(x, \partial\Omega)$ denotes the distance between x and $\partial\Omega$. Then there are three cases to be considered here: (i) there exists a subsequence of $\{(Q_k, \tilde{Q}_k)\}$ (still denoted by $\{(Q_k, \tilde{Q}_k)\}$) such that $Q_k \to +\infty$ and $\tilde{Q}_k \to +\infty$ as $k \to +\infty$; (ii) one of the Q_k and $\tilde{Q}_k$ approaches to $+\infty$ as $k \to +\infty$; (iii) $\{(Q_k, \tilde{Q}_k)\}$ are uniformly bounded. According to the three cases above, we can make the suitable transformations as in [1] and by the blow up argument as in [1] to obtain the contradictions respectively. Note that since $s_k \in [0, S]$ for all k, there is no any difficulty when we use the arguments of [1] here.

The arguments above imply that there exists $R > 0$ independent of $s \geq 0$ such that for any nontrivial solution $(u(s,\cdot), v(s,\cdot))$ of (2.3)–(2.5)

$$\|u(s,\cdot)\|_\infty + \|v(s,\cdot)\|_\infty < R$$

Since $\deg(I - F_s, B_R(0), 0) = 0$ for $s = S$, the assertion follows from the homotopy invariance of the Leray-Schauder degree.

Proof of Theorem 2.1 By Lemmas 2.2 and 2.3 and the degree's excision property we find that $\deg_E(I - A, B_R(0)\backslash\overline{B}_r(0), 0) = -1$. This implies that the problem (1.1)–(1.3) has at least two positive solutions.

3. Stability and Instability of the Positive Solutions

We discuss the stability and the instability for the positive solutions of the problem (1.1)–(1.3) in the present section. Our main result is

Theorem 3.1 *Let $a(x), b(x), \delta, \mu$ satisfy the conditions in Theorem 2.1. Let $e, m \in L^\infty(\Omega)$, $e, m \geq 0$ in Ω and $e, m \not\equiv 0$ in Ω. If $\|e\|_\infty$ and $\|m\|_\infty$ are sufficiently small, then there exist at least one stable and one unstable positive solutions of (1.1)–(1.3) in $C^2(\Omega) \times C^2(\Omega)$.*

By the stability of (u, v) we mean the stability of (u, v) as the solution of the corresponding parabolic system of (1.1)–(1.3). The following lemma can be found in [7].

Lemma 3.2 *Assume $\alpha, \beta, \gamma, \psi \in L^\infty(\Omega)$ with $\beta, \gamma > 0$ in Ω. Then there exists a real eigenvalue $\widetilde{\lambda}$ of the problem*

$$-\Delta h = \alpha h + \beta k + \lambda h \tag{3.1}$$

$$-\Delta k = \psi k + \gamma h + \lambda k \quad \text{in} \quad \Omega \tag{3.2}$$

$$h = k = 0 \quad \text{on} \quad \partial\Omega \tag{3.3}$$

(known as the principal eigenvalue) such that (3.1)–(3.3) has no eigenvalues λ with $\operatorname{Re}\lambda < \widetilde{\lambda}$; $\widetilde{\lambda}$ has an eigenfunction in K and $\widetilde{\lambda}$ is the only real λ to which there corresponds an eigenfunction in K to (3.1)–(3.3). Here K is as that in Section 2.

It follows from Lemma 3.2 and Henry [8] that (u, v) is stable if the principal eigenvalue $\widetilde{\lambda}$ of the problem

$$-\Delta h = \delta a(x) v^{\delta-1} k + \lambda h \tag{3.4}$$

$$-\Delta k = \mu b(x) u^{\mu-1} h + \lambda k \quad \text{in} \quad \Omega \tag{3.5}$$

$$h = k = 0 \quad \text{on} \quad \partial\Omega \tag{3.6}$$

is positive and (u, v) is unstable if $\widetilde{\lambda} < 0$.

Lemma 3.3 *Let (u, v) be a positive solution of (1.1)–(1.3) and $A(u, v)$ be as that in Section 2. Then*

(i) *If $\widetilde{\lambda} > 0$, then* $\operatorname{index}_K(A, (u, v)) = 1$,

(ii) *If $\widetilde{\lambda} = 0$, then* $\operatorname{index}_K(A, (u, v)) = 0$.

Proof Suppose that $\widetilde{\lambda} > 0$. We shall prove that

$$\deg_E(I - A'(u, v), B_r(0), 0) = 1$$

for any $r > 0$. We consider the problem

$$-\Delta h = s\delta a(x) v^{\delta-1} k \tag{3.7}$$

$$-\Delta k = s\mu b(x) u^{\mu-1} h \quad \text{in} \quad \Omega \tag{3.8}$$

$$h = k = 0 \quad \text{on} \quad \partial\Omega \tag{3.9}$$

for $s \in [0, 1]$. We shall prove that this problem has no non-trivial solution. In fact, suppose that this problem has a solution (h_s, k_s) (possibly complex valued), then by Kato's inequality (See Kato [9], Lemma 3)

$$-\Delta |h_s| \le s\delta a(x) v^{\delta-1} |k_s|$$

$$-\Delta|k_s| \le s\mu b(x)u^{\mu-1}|h_s| \quad \text{in } \Omega$$
$$|h_s| = |k_s| = 0 \quad \text{on } \partial\Omega$$

This means that $sA'(u,v)(|h_s|,|k_s|) \ge (|h_s|,|k_s|)$. Thus,

$$r(sA'(u,v)) \ge 1 \tag{3.10}$$

where $r(T(s))$ denotes the spectral radius of $T(s)$. On the other hand, by Lemma 3.2, we have that there exists (h_1,k_1) with $h_1 > 0$, $k_1 > 0$ on Ω and $\|h_1\|_\infty + \|k_1\|_\infty = 1$ such that

$$-\Delta h_1 = \delta a(x)v^{\delta-1}k_1 + \tilde{\lambda}h_1$$
$$-\Delta k_1 = \mu b(x)u^{\mu-1}h_1 + \tilde{\lambda}k_1 \quad \text{in } \Omega$$
$$h_1 = k_1 = 0 \quad \text{on } \partial\Omega$$

This implies that

$$sA'(u,v)(h_1,k_1) < (h_1,k_1)$$

since $\tilde{\lambda} > 0$. Therefore, as before,

$$r(sA'(u,v)) < 1$$

This contradicts (3.10). This contradiction implies that there is no non-trivial solution of the problem (3.7)–(3.9). By the homotopy invariance of the Leray-Schauder degree, we know that

$$\deg_E(I - A'(u,v), B_r(0), 0) = 1$$

for any $r > 0$.

We know from above that $I - A'(u,v)$ is invertible. Thus

$$\operatorname{index}_K(A(u,v),(u,v)) = \deg_E(I - A'(u,v), B_r(0), 0) = 1$$

Here we are using Theorem 1 in [10] and that (u,v) is demi-interior to K (in the sense of [10]). We next prove the statement (ii). Let (h_1,k_1) be the principal eigenfunction corresponding to $\tilde{\lambda}$. We set $E_1 = \{s(h_1,k_1) : s \in \mathbf{R}\}$, here (h_1,k_1) spans the kernel of the operator $I - A'(u,v)$ (since $\tilde{\lambda} = 0$). Then there is a closed subspace E_2 of $L^\infty(\Omega) \times L^\infty(\Omega)$ which is invariant under $A'(u,v)$ such that $L^\infty(\Omega) \times L^\infty(\Omega) = E_1 \oplus E_2$. Denote by P_i the projection from $L^\infty(\Omega) \times L^\infty(\Omega)$ onto E_i for $i = 1,2$, P_i is a bounded operator, that is, there exists $M > 0$ satisfying

$$\|P_i w\|_\infty \le M\|w\|_\infty, \quad i = 1,2$$

for all $w \in L^\infty(\Omega) \times L^\infty(\Omega)$. We choose $\varepsilon > 0$ as satisfying

$$\{s(h_1, k_1) : |s| < \varepsilon\} \times \{w \in E_2 : \|w\|_\infty < \varepsilon\} \subset B_\delta(0)$$

and put $U_\varepsilon((u, v)) = (u, v) + \{s(h_1, k_1) : |s| < \varepsilon\} \times \{w \in E_2 : \|w\|_\infty < \varepsilon\}$.

Now we show that $(\mu(\mu-1)b(x)u^{\mu-2}h_1^2, \delta(\delta-1)a(x)v^{\delta-2}k_1^2) \notin E_2$. We know that $(\mu(\mu-1)b(x)u^{\mu-2}h_1^2, \delta(\delta-1)a(x)v^{\delta-2}k_1^2) \in E$. In fact, if $\delta \geq 2$ and $\mu \geq 2$, we easily know that this is true. We only consider the cases that $\delta < 2$ or $\mu < 2$ or that both of them are less than 2. The regularity of $-\Delta$ implies that $u, v, h_1, k_1 \in C^1(\overline{\Omega})$. By Lemma 2.3 in [11] we know that there exist l_i $(i = 1, \cdots, 8)$ with $l_{2k} \geq l_{2k-1} > 0$ $(k = 1, 2, 3, 4)$ such that

$$l_1 d(x, \partial\Omega) \leq u(x) \leq l_2 d(x, \partial\Omega)$$
$$l_3 d(x, \partial\Omega) \leq v(x) \leq l_4 d(x, \partial\Omega)$$
$$l_5 d(x, \partial\Omega) \leq h_1(x) \leq l_6 d(x, \partial\Omega)$$
$$l_7 d(x, \partial\Omega) \leq k_1(x) \leq l_8 d(x, \partial\Omega)$$

These imply that $u^{\mu-2}h_1^2$ and $v^{\delta-2}k_1^2$ are bounded in the neighborhood of $\partial\Omega$. Thus, $u^{\mu-2}h_1^2, v^{\delta-2}k_1^2 \in L^\infty(\Omega)$. Assume that $(\mu(\mu-1)b(x)u^{\mu-2}h_1^2, \delta(\delta-1)a(x)v^{\delta-2}k_1^2) \in E_2$. Then we shall see later that there is $(w_1, w_2) \in L^\infty(\Omega) \times L^\infty(\Omega)$ such that

$$\begin{aligned} &-\Delta w_1 - \delta a(x)v^{\delta-1}w_2 = \delta(\delta-1)a(x)v^{\delta-2}k_1^2 \\ &-\Delta w_2 - \mu b(x)u^{\mu-1}w_1 = \mu(\mu-1)b(x)u^{\mu-2}h_1^2 \quad \text{in} \quad \Omega \\ &w_1 = w_2 = 0 \quad \text{on} \quad \partial\Omega \end{aligned}$$

Then

$$\begin{aligned} 0 &< \int_\Omega [\mu(\mu-1)b(x)u^{\mu-2}h_1^3 + \delta(\delta-1)a(x)v^{\delta-2}k_1^3]dx \\ &= \int_\Omega w_2(-\Delta h_1 - \delta a(x)v^{\delta-1}k_1)dx + \int_\Omega w_1(-\Delta k_1 - \mu b(x)u^{\mu-1}h_1)dx = 0 \end{aligned}$$

This contradiction implies that $(\mu(\mu-1)b(x)u^{\mu-2}h_1^2, \delta(\delta-1)a(x)v^{\delta-2}k_1^2) \notin E_2$. Therefore, there exists $\beta \neq 0$ satisfying

$$\begin{pmatrix} \mu(\mu-1)b(x)u^{\mu-2}h_1^2 \\ \delta(\delta-1)a(x)v^{\delta-2}k_1^2 \end{pmatrix} - P_2\begin{pmatrix} \mu(\mu-1)b(x)u^{\mu-2}h_1^2 \\ \delta(\delta-1)a(x)v^{\delta-2}k_1^2 \end{pmatrix} = \beta\begin{pmatrix} h_1 \\ k_1 \end{pmatrix} \tag{3.11}$$

Now we prove that for any $(f(x), g(x)) \in E_2$, there exists a solution $(h, k) \in L^\infty(\Omega) \times L^\infty(\Omega)$ for the problem

$$-\Delta h - \delta a(x)v^{\delta-1}k = g(x) \tag{3.12}$$

$$-\Delta k - \mu b(x)u^{\mu-1}h = f(x) \quad \text{in} \quad \Omega \tag{3.13}$$

$$h = k = 0 \quad \text{on} \quad \partial\Omega \tag{3.14}$$

In fact, consider the homotopy

$$H(s,x,h,k) = (h,k) - (-\Delta)^{-1}[s(\delta a(x)v^{\delta-1}k + g(x)), s(\mu b(x)u^{\mu-1}h + f(x))]$$

We shall prove that there exists $M > 0$ independent of $s \in [0,1]$ such that if (h,k) satisfies that $H(s,x,h,k) = 0$, then $\|h\|_\infty + \|k\|_\infty \leq M$. In fact, suppose that there exist sequences $\{s_n\} \subset (0,1)$ and $\{(h_n,k_n)\}$ satisfying $H(s_n,x,h_n,k_n) = 0$ and $\|h_n\|_\infty + \|k_n\|_\infty \to +\infty$ as $n \to +\infty$, then (h_n,k_n) satisfies

$$-\Delta h_n = s_n(\delta a(x)v^{\delta-1}k_n + g(x)) \tag{3.15}$$

$$-\Delta k_n = s_n(\mu b(x)u^{\mu-1}h_n + f(x)) \quad \text{in } \Omega \tag{3.16}$$

$$h_n = k_n = 0 \quad \text{on } \partial\Omega \tag{3.17}$$

Let $z_n = h_n/(\|h_n\|_\infty + \|k_n\|_\infty)$ and $w_n = k_n/(\|h_n\|_\infty + \|k_n\|_\infty)$. Then $\|z_n\|_\infty + \|w_n\|_\infty = 1$. This implies that $\|z_n\|_\infty \geq 1/2$ or $\|w_n\|_\infty \geq 1/2$. Since $\|z_n\|_\infty \leq 1$, $\|w_n\|_\infty \leq 1$, $s_n \to \tilde{s} \in [0,1]$ and

$$-\Delta z_n = s_n\left(\delta a(x)v^{\delta-1}w_n + \frac{g(x)}{\|h_n\|_\infty + \|k_n\|_\infty}\right) \tag{3.18}$$

$$-\Delta w_n = s_n\left(\mu b(x)u^{\mu-1}z_n + \frac{f(x)}{\|h_n\|_\infty + \|k_n\|_\infty}\right) \tag{3.19}$$

$$z_n = w_n = 0 \quad \text{on} \quad \partial\Omega \tag{3.20}$$

By the regularity of $(-\Delta)^{-1}$ we have that $z_n \to \tilde{z}$, $w_n \to \tilde{w}$ in $C^1(\Omega)$ as $n \to \infty$, $\tilde{z}$ and $\tilde{w}$ satisfy

$$-\Delta\tilde{z} = \tilde{s}\delta a(x)v^{\delta-1}\tilde{w}$$

$$-\Delta\tilde{w} = \tilde{s}\mu b(x)u^{\mu-1}\tilde{z} \quad \text{in} \quad \Omega$$

$$\tilde{z} = \tilde{w} = 0 \qquad \text{on } \partial\Omega$$

Since $\tilde{s} \in [0,1]$, we have from above that $\tilde{z} \equiv \tilde{w} \equiv 0$ in Ω. This contradicts the fact that $\|\tilde{z}\|_\infty \geq 1/2$ or $\|\tilde{w}\|_\infty \geq 1/2$.

Let $G(s,x,h,k) = (s(\delta a(x)v^{\delta-1}k + g(x)), s(\mu b(x)u^{\mu-1}h + f(x)))$. Then according to the homotopy invariance of the Leray-Schauder degree, we find that

$$\deg_E(I - (-\Delta)^{-1} \circ G(1,x,\cdot), B_M(0), 0) = 1$$

This implies that there exists a solution of (3.12)–(3.14). We also know that if there exist two solutions $(\hat{h}_i, \hat{k}_i)$ $(i = 1,2)$ for (3.12)–(3.14), then $(\hat{h}_1, \hat{k}_1) - (\hat{h}_2, \hat{k}_2) = s(h_1, k_1)$.

This implies that for $(f(x), g(x)) \in E_2$, there exists a unique solution $(h, k) \in E_2$ for (3.12)–(3.14). Moreover, by the similar idea to above we also have

$$\|h\|_\infty + \|k\|_\infty \le C(\|f\|_\infty + \|g\|_\infty)$$

In fact, we first multiply both sides of (3.12)–(3.13) by $1/(\|f\|_\infty + \|g\|_\infty)$, then obtain $\frac{\|h\|_\infty}{\|f\|_\infty + \|g\|_\infty} + \frac{\|k\|_\infty}{\|f\|_\infty + \|g\|_\infty} \le C$ by the same idea as that in the proof of (3.18)–(3.20).

Fix $(u_1, v_1) \in E_2$ with $\|u_1\|_\infty + \|v_1\|_\infty < \varepsilon$ and define

$$\begin{aligned}\Psi_{(u,v)}(s) = & \int_\Omega \big(-\Delta(u + u_1 + sh_1) - a(x)|v + v_1 + sk_1|^{\delta-1}(v + v_1 + sk_1) - h\big)\beta k_1 dx \\ & + \int_\Omega \big(-\Delta(v + v_1 + sk_1) - b(x)|u + u_1 \\ & + sh_1|^{\mu-1}(u + u_1 + sh_1) - m\big)\beta h_1 dx\end{aligned}$$

for $s \in \mathrm{R}$ with $|s| < \varepsilon$. Then we have

$$\begin{aligned}\Psi_{(u,v)}(0) = & \int_\Omega \big(-\Delta(u + u_1) - a(x)|v + v_1|^{\delta-1}(v + v_1) - h\big)\beta k_1 dx \\ & + \int_\Omega \big(-\Delta(v + v_1) - b(x)|u + u_1|^{\mu-1}(u + u_1) - m\big)\beta h_1 dx \\ \Psi'_{(u,v)}(0) = & \int_\Omega \big(-\Delta h_1 - \delta a(x)|v + v_1|^{\delta-1}k_1\big)\beta k_1 dx \\ & + \int_\Omega \big(-\Delta k_1 - \mu b(x)|u + u_1|^{\mu-1}h_1\big)\beta h_1 dx \\ \Psi''_{(u,v)}(0) = & \, \delta(\delta - 1)\int_\Omega a(x)|v + v_1|^{\delta-3}(v + v_1)\beta k_1^3 dx \\ & + \mu(\mu - 1)\int_\Omega b(x)|u + u_1|^{\mu-3}(u + u_1)\beta h_1^3 dx\end{aligned}$$

Consequently, we find

$$\Psi_{(u,v)}(s) = \Psi_{(u,v)}(0) + \Psi'_{(u,v)}(0)s + \Psi''_{(u,v)}(0)s^2 + o(s^2) \tag{3.21}$$

For $\rho \in [0, 1]$, set

$$g_\rho(x, u, v) = \begin{pmatrix} b(x)u^\mu + m(x) \\ a(x)v^\delta + h(x) \end{pmatrix} + \rho \begin{pmatrix} h_1 \\ k_1 \end{pmatrix}$$

We claim that for each $0 < \rho \le 1$, the compact mapping $(-\Delta)^{-1} \circ g_\rho$ has no fixed point in $U_\varepsilon((u, v))$ and that (u, v) is the unique fixed point of $(-\Delta)^{-1} \circ g$ in $U_\varepsilon((u, v))$

Suppose that

$$\begin{pmatrix} -\Delta(v+v_1+\tau k_1) \\ -\Delta(u+u_1+\tau h_1) \end{pmatrix} = \begin{pmatrix} b(x)|u+u_1+\tau h_1|^{\mu-1}(u+u_1+\tau h_1)+m(x) \\ a(x)|v+v_1+\tau k_1|^{\delta-1}(v+v_1+\tau k_1)+h(x) \end{pmatrix} + \rho\beta\begin{pmatrix} h_1 \\ k_1 \end{pmatrix} \tag{3.22}$$

for some $\rho \in [0,1]$, $(u_1,v_1) \in E_2$ and $\tau \in \mathbf{R}$ with $\|u_1\|_\infty + \|v_1\|_\infty < \varepsilon$ and $|\tau| < \varepsilon$. Then by (3.21) and that

$$f(x+x_1) = f(x) + f'(x)x_1 + \frac{1}{2}f''(x)x_1^2 + o(x_1^2)$$

if x_1 is sufficiently small, we have

$$\begin{pmatrix} -\Delta v_1 - \mu b(x)u^{\mu-1}u_1 \\ -\Delta u_1 - \delta a(x)v^{\delta-1}v_1 \end{pmatrix} = \begin{pmatrix} \frac{1}{2}\mu(\mu-1)b(x)u^{\mu-2}(u_1+\tau h_1)^2 + \xi_2(u_1,v_1,h_1,k_1,\tau) \\ \frac{1}{2}\delta(\delta-1)a(x)v^{\delta-2}(v_1+\tau k_1)^2 + \xi_1(u_1,v_1,h_1,k_1,\tau) \end{pmatrix} + \rho\beta\begin{pmatrix} h_1 \\ k_1 \end{pmatrix} \tag{3.23}$$

where $\|\xi_i\|_\infty \le C(\|u_1\|_\infty^2 + \|v_1\|_\infty^2 + \tau^2)$ $(i=1,2)$. We also use the fact that the principal eigenvalue $\widetilde{\lambda} = 0$. Project both sides of the equality on E_2, it follows that

$$\begin{pmatrix} -\Delta v_1 - \mu b(x)u^{\mu-1}u_1 \\ -\Delta u_1 - \delta a(x)v^{\delta-1}v_1 \end{pmatrix} = \frac{1}{2}P_2\begin{pmatrix} \mu(\mu-1)b(x)u^{\mu-2}(u_1+\tau h_1)^2 \\ \delta(\delta-1)a(x)v^{\delta-2}(v_1+\tau k_1)^2 \end{pmatrix} + P_2\begin{pmatrix} \xi_2(u_1,v_1,h_1,k_1,\tau) \\ \xi_1(u_1,v_1,h_1,k_1,\tau) \end{pmatrix} \tag{3.24}$$

and hence

$$\begin{pmatrix} v_1 \\ u_1 \end{pmatrix} = \begin{pmatrix} -\Delta & -\delta a(x)v^{\delta-1} \\ -\mu b(x)u^{\mu-1} & -\Delta \end{pmatrix}^{-1}\left\{\frac{1}{2}P_2\begin{pmatrix} \mu(\mu-1)b(x)u^{\mu-2}(u_1+\tau h_1)^2 \\ \delta(\delta-1)a(x)v^{\delta-2}(v_1+\tau k_1)^2 \end{pmatrix} + P_2\begin{pmatrix} \xi_2(u_1,v_1,h_1,k_1,\tau) \\ \xi_1(u_1,v_1,h_1,k_1,\tau) \end{pmatrix}\right\} \tag{3.25}$$

From the arguments above, we have that there exists $C_1 > 0$ satisfying

$$\|u_1\|_\infty + \|v_1\|_\infty \le C_1((\|u_1\|_\infty + \|v_1\|_\infty)^2 + \tau^2)$$

Since we may choose ε so small that $C_1\varepsilon < 1/2$, we obtain from the inequality above that

$$\|u_1\|_\infty + \|v_1\|_\infty \le 2C_1\tau^2 \tag{3.26}$$

for each $(u_1, v_1) \in E_2$ satisfying (3.22). Then we have from (3.21), (3.22), (3.23) and the inequality above that

$$0 \le \rho\beta^2 \int_\Omega (h_1^2 + k_1^2)dx = \Psi_{(u,v)}(\tau)$$

$$= -\frac{1}{2}\left[\int_\Omega \left(\begin{pmatrix}\mu(\mu-1)u^{\mu-2}h_1^2 \\ \delta(\delta-1)v^{\delta-2}k_1^2\end{pmatrix}^T - P_2\begin{pmatrix}\mu(\mu-1)u^{\mu-2}h_1^2 \\ \delta(\delta-1)v^{\delta-2}k_1^2\end{pmatrix}^T\right)\begin{pmatrix}\beta h_1 \\ \beta k_1\end{pmatrix} dx\right]\tau^2$$

$$+ \xi_3(\tau) = -\frac{1}{2}\left[\int_\Omega \begin{pmatrix}\beta h_1 \\ \beta k_1\end{pmatrix}^T \begin{pmatrix}\beta h_1 \\ \beta k_1\end{pmatrix} dx\right]\tau^2 + \xi_3(\tau)$$

where $\xi_3 \in o(\tau^2)$. Here we use the fact that $\int_\Omega [(-\Delta v_1 - \mu b(x)u^{\mu-1}u_1)h_1 + (-\Delta u_1 - \delta a(x)v^{\delta-1}v_1)k_1]dx = 0$. Then we have from the inequality above that $(u_1, v_1) \in E_2$ with $\|u_1\|_\infty + \|v_1\|_\infty < \varepsilon$ satisfies (3.22) only when $\rho = 0$ and $\tau = 0$. (3.26) implies $(u_1, v_1) = 0$. Thus we have shown that $(-\Delta)^{-1} \circ g_\rho$ has no fixed point in $U_\varepsilon((u, v))$ for $0 < \rho \le 1$ and (u, v) is the unique fixed point of $(-\Delta)^{-1} \circ g$ in $U_\varepsilon((u, v))$. Then according to the homotopy invariance of the Leray-Schauder degree, we find that

$$\deg_E(I - (-\Delta)^{-1} \circ g, U_\varepsilon((u, v)), 0) = \deg_E(I - (-\Delta)^{-1} \circ g_1, U_\varepsilon((u, v)), 0) = 0$$

This implies that $\text{index}_K(A, (u, v)) = 0$.

Proof of Theorem 3.1 By Lemma 2.2, there is $r > 0$ sufficiently small such that the problem (1.1)–(1.3) possesses a positive solution $(\tilde{u}, \tilde{v})$ in $B_r(0)$. Let $\tilde{\lambda}$ and (h_1, k_1) be as before, it holds that

$$-\Delta h_1 - \delta a(x)\tilde{v}^{\delta-1}k_1 = \tilde{\lambda} h_1$$
$$-\Delta k_1 - \mu b(x)\tilde{u}^{\mu-1}h_1 = \tilde{\lambda} k_1$$

Assume that $\tilde{\lambda} \le 0$. Then multiplying the first eigenfunction ϕ_1 of the first eigenvalue λ_1 of the problem

$$-\Delta u = \lambda u, \quad u = 0 \quad \text{on} \quad \partial\Omega$$

in both sides of the equations above and integrating them on Ω, we have that

$$\lambda_1 \int_\Omega h_1\phi_1 dx = \delta \int_\Omega a(x)\tilde{v}^{\delta-1}k_1\phi_1 dx + \tilde{\lambda}\int_\Omega h_1\phi_1 dx \tag{3.27}$$

$$\lambda_1 \int_\Omega k_1\phi_1 dx = \mu \int_\Omega b(x)\tilde{u}^{\mu-1}h_1\phi_1 dx + \tilde{\lambda}\int_\Omega k_1\phi_1 dx \tag{3.28}$$

We have from (3.27) that

$$\lambda_1 \int_\Omega h_1\phi_1 dx \le \delta\|a\|_\infty\|\tilde{v}\|_\infty^{\delta-1}\int_\Omega k_1\phi_1 dx \tag{3.29}$$

We also have from (3.28) that

$$\lambda_1 \int_\Omega k_1\phi_1 dx \le \mu\|b\|_\infty \|\widetilde{u}\|_\infty^{\mu-1} \int_\Omega h_1\phi_1 dx \tag{3.30}$$

Combining (3.29) with (3.30), we have that

$$\lambda_1^2 \le \mu\delta\|a\|_\infty\|b\|_\infty\|\widetilde{v}\|_\infty^{\delta-1}\|\widetilde{u}\|_\infty^{\mu-1}$$

This is a contradiction since r is sufficiently small. This contradiction implies that $\widetilde{\lambda} > 0$, and then $(\widetilde{u}, \widetilde{v})$ is stable. On the other hand, taking R sufficiently large, we have by the proof of Theorem 2.1 that

$$\deg_E(I - A, B_R(0)\backslash B_r(0), 0) = -1$$

We can see from the proof of Lemma 3.3 that if (u, v) is a positive solution of (1.1)–(1.3) in $B_R(0)$ satisfying $\widetilde{\lambda} \ge 0$, then (u, v) is isolated. Then by Lemma 3.3, we conclude that there exists $(\widetilde{u}_1, \widetilde{v}_1) \in B_R(0)\backslash B_r(0)$ with $\widetilde{\lambda} < 0$. Therefore, (u_2, v_2) is unstable. This completes the proof.

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