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SCATTERING FOR SEMILINEAR WAVE EQUATION WITH SMALL DATA IN HIGH SPACE DIMENSIONS

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Abstract In this paper we study the scattering theory for the semilinear wave equation $u_{tt} - \Delta u = F(u(t,x), Du(t,x))$ in R^n $(n \geq 4)$ with smooth and small data. We show that the scattering operator exists for the nonlinear term $F = F(\lambda) = O(|\lambda|^{1+\alpha})$, where α is an integer and satisfies $\alpha \geq 2, n = 4$; $\alpha \geq 1$, $n \geq 5$.
Key Words Semilinear wave equation; scattering; asymptotic behavior.
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1. Introduction

We consider the scattering problem for the semilinear wave equation

$$u_{tt}(t,x) - \Delta u(t,x) = F(u(t,x), Du(t,x)) \tag{1.1}$$

Here $Du(t,x) = (u_t, \nabla u)(t,x)$. Let $\lambda = (\lambda_0, (\lambda_i), i = 1, \cdots, n+1)$, suppose that in a neighbourhood of $\lambda = 0$, say, for the nonlinear term in (1.1) is a sufficiently smooth function satisfies

$$F(\lambda) = O(|\lambda|^{1+\alpha}) \tag{1.2}$$

where α is an integer ≥ 1. From Li and Yu [1], we know that if there is a relation between α and n as follows

$$\alpha \geq 2, \quad n = 4; \quad \alpha \geq 1, \quad n \geq 5 \tag{1.3}$$

then when initial data is sufficiently small, the Cauchy problem of (1.1) admits a unique global solution to $t \geq 0$. In this paper we establish the scattering theory for (1.1) under the same assumptions on α and n.

In the scattering theory the asymptotic behavior of the solution to (1.1) is compared with that of the solution to the free wave equation

$$u_{tt}(t,x) - \Delta u(t,x) = 0 \tag{1.4}$$

in the energy norm. More precisely, let $E(\mathbf{R}^n)$ be the energy space

$$E(\mathbf{R}^n) = \{(f,g); f \in L^2(\mathbf{R}^n), \nabla f, g \in L^2(\mathbf{R}^n)\}$$

with the norm $\|(f,g)\|_e = \|\nabla f\|_{L^2} + \|g\|_{L^2}$. We show that there exists a suitable normed space Σ dense in $E(\mathbf{R}^n)$ and a neighbourhood $\mathcal{N}$ of the origin in Σ such that for any $(f^-, g^-) \in \mathcal{N}$, there exists a unique solution $u(t,x)$ to (1.1) in $\mathbf{R} \times \mathbf{R}^n$, which behaves asymptotically like the solution $u_0^-(t,x)$ of the Cauchy problem (1.4) with data (f^-, g^-) at $t = 0$ in the sense that

$$\|u(t,\cdot) - u_0^-(t,\cdot)\|_e \to 0 \quad as \quad t \to -\infty$$

and moreover, we also find a unique solution $u_0^+(t,x)$ to (1.4) with corresponding data (f^+, g^+) at $t = 0$ such that

$$\|u(t,\cdot) - u_0^+(t,\cdot)\|_e \to 0 \quad as \quad t \to +\infty$$

Therefore, the scattering operator $S : (f^-, g^-) \to (f^+, g^+)$ can be defined in the set $\mathcal{N}$, provided that the conditions concerning α and n as in (1.3) are satisfied.

For the case that the nonlinear term F doesn't depend on $Du : F(u) = C|u|^p$ with smooth and small data most of the results were known. One can see Reed [2], Strauss [3], Pecher [4, 5], Mochizuki and Motai [6], Morawetz and Strauss [7] and Tsutaya [8]. For the case of large data, we refer the reader to the results by Ginibre and Velo [9]. Recently Kubo and Kubota [10] have considered the asymptotic behavior of the radial solution to (1.1). But for the generalized case as in (1.2), there are few results as the author knows.

In order to establish the scattering theory for (1.1), as usually, we start with considering the following Yang-Feldman equation [11]

$$u(t) = u_0(t) + \int_{-\infty}^{t} \frac{\sin \omega(t-\tau)}{\omega} F(u(\tau), Du(\tau)) d\tau \tag{1.5}$$

in a suitable invariable Sobolev space, here $\omega = (-\Delta)^{\frac{1}{2}}$. However, (1.5) is not useful directly to our problem, because we employ the generalized Sobolev space with weights related to the generators of Lorentz group. More precisely, since we apply a set of partial differential operators which have the weights x and (or) t, it is not clear whether the operators commute with the integral sign in (1.5). Thus one of our main tasks is to investigate the commutation relations between the partial operators and $\int_{-\infty}^{t} \frac{\sin \omega(t-\tau)}{\omega} F(u(\tau), Du(\tau))\, d\tau$. For that purpose we investigate the approximated solution

$$u_\sigma(t) = u_0(t) + \int_{\sigma}^{t} \frac{\sin \omega(t-\tau)}{\omega} F(u(\tau), Du(\tau)) d\tau \tag{1.6}$$

instead of analyzing (1.5) directly. Giving some assumptions of F, making an effective use of another L^2 estimate for the fundamental solution (See (3.4)) together with L^p-L^q inequality due to Hörmander (See (3.2)), we show that there are useful commutation relations between the partial differential operators and $\int_{-\infty}^{t}\frac{\sin\omega(t-\tau)}{\omega}F(u(\tau),Du(\tau))d\tau$. Therefore we find that we can establish the scattering theory for (1.1) in the generalized Sobolev space. Also we notice that these complicated steps are inevitable to study the scattering problem, while they are not inquired in studying the Cauchy problem.

Finally, we remark that the estimates on composite functions are important to solve our problem.

This paper is organized as follows: In Section 2, we introduce some notations and state our main results. In Section 3, we prepare several lemmas and propositions frequently used in Section 4. Finally, in Section 4, we give out the proof of the main theorem. Throughout this paper we denote different constants by the same C from line to line.

2. Notations and Main Results

We first introduce the space $L^{p,q}(\mathbf{R}^n)$, for any $f(x)\in\mathcal{D}'(\mathbf{R}^n)$, we say that $f\in L^{p,q}(\mathbf{R}^n)$, if

$$g(r\xi)=f(r\xi)r^{\frac{n-1}{p}}\in L^p(0,+\infty;L^q(S^{n-1}))$$

where $r=|x|$ and $\xi=(\xi_1,\cdots,\xi_n)$ with $|\xi|=1$, $1\le p,q\le+\infty$, S^{n-1} is the unit sphere in $\mathbf{R}^n$, equipped with the norm

$$\|f(\cdot)\|_{L^{p,q}(\mathbf{R}^n)}=\|f(r\xi)r^{\frac{n-1}{p}}\|_{L^p(0,+\infty;L^q(S^{n-1}))}$$

$L^{p,q}(\mathbf{R}^n)$ is a Banach space. It is easy to see that if $p=q$, then $L^{p,q}(\mathbf{R}^n)$ becomes the usual space $L^p(\mathbf{R}^n)$.

Following S. Klainerman [12], we introduce a set of partial differential operators

$$\Gamma=(L_0;(\partial_a),a=0,1,\cdots,n;(\Omega_{ij}),i,j=1,2,\cdots,n;(L_j),j=1,2,\cdots,n)$$

where

$$L_0=t\partial_t+x_1\partial_1+\cdots+x_n\partial_n,\quad \partial_0=\partial_t,\quad \partial_i=\frac{\partial}{\partial x_i}\quad(i=1,2,\cdots,n)$$

$$\Omega_{ij}=x_i\partial_j-x_j\partial_i\quad(i,j=1,2,\cdots,n),\quad L_j=t\partial_j+x_j\partial_t\quad(j=1,2,\cdots,n)$$

and for any integer $N\ge 0$ we define

$$\|u(t,\cdot)\|_{\Gamma,N,p,q}=\sum_{|k|\le N}\|\Gamma^k u(t,\cdot)\|_{L^{p,q}(\mathbf{R}^n)}$$

for any function $u = u(t,x)$ such that all norms appearing on the right-hand side are bounded, where $1 \le p, q \le \infty$, $k = (k_1, k_2, \cdots, k_\mu)$ is a multi-index, $|k| = k_1 + k_2 + \cdots + k_\mu$, μ is the number of the partial differential operators in $\Gamma : \Gamma = (\Gamma_1, \Gamma_2, \cdots, \Gamma_\mu)$ and $\Gamma^k = \Gamma_1^{k_1}\Gamma_2^{k_2}\cdots\Gamma_\mu^{k_\mu}$, any permutation that in $(\Gamma_1, \Gamma_2, \cdots, \Gamma_\mu)$ only corresponds to an equivalent is uniform. In the special case that $p = q$, we write

$$\|u(t,\cdot)\|_{\Gamma,N,p,q} = \|u(t,\cdot)\|_{\Gamma,N,p}$$

and denote

$$\|u(t,\cdot)\|_{\Gamma,N,p,q,\chi} = \sum_{|k|\le N} \|\chi\Gamma^k u(t,\cdot)\|_{L^{p,q}(\mathbf{R}^n)}$$

where χ is the characteristic function of the set

$$\left\{(t,x) : |x| \le \frac{1+|t|}{2}\right\}$$

We also introduce the space for some integer $S \ge 2n+3$

$$\Sigma = \{(f,g) \in C_0^\infty(\mathbf{R}^n) \times C_0^\infty(\mathbf{R}^n) : \|(f,g)\|_\Sigma < \infty\}$$

here

$$\|(f,g)\|_\Sigma = \sum_{|\alpha|\le s+2}^{\beta\le s+1} \|(\sqrt{1+|x|^2})^\beta \partial_x^\alpha f\|_{L^2(\mathbf{R}^n)} + \sum_{|\alpha|\le s+1}^{\beta\le s} \|(\sqrt{1+|x|^2})^{\beta+1} \partial_x^\alpha g\|_{L^2(\mathbf{R}^n)}$$

Now we can state our main results.

Main Theorem *Let $(f^-, g^-) \in \Sigma$ be given with $\|(f^-,g^-)\|_\Sigma \le \delta$, where δ is sufficiently small, and let $u_0^-(t)$ denote the solution with $(u_0^-(0), u_{0t}^-(0)) = (f^-, g^-)$ of the linear wave equation*

$$u_{0tt}^- - \Delta u_0^- = 0$$

The nonlinear term F satisfies (1.2) and (1.3), under these assumptions we have

(i) *there exists a unique solution $u(t)$ of the perturbed equation (1.1), which is asymptotically equivalent to $u_0^-(t)$ in the sense of energy norm*

$$\|u(t) - u_0^-(t)\|_e \to 0 \quad as \quad t \to -\infty$$

(ii) *there also exists a unique solution $u_0^+(t)$ of the free equation (1.4) with $(u_0^+(0), u_{0t}^+(0)) = (f^+, g^+)$, which is asymptotically equivalent to $u(t)$ in the sense of energy norm*

$$\|u(t) - u_0^+(t)\|_e \to 0 \quad as \quad t \to +\infty$$

Thus we can define the scattering operator $S:(f^-,g^-)\to(f^+,g^+)$ in a whole neighbourhood $\mathcal{N}$ of the origin in Σ.

3. Preliminary Results

In this section we prepare several lemmas and propositions frequently used in Section 4.

Lemma 3.1 *For any function $u=u(t,x)$ for which the norm appearing on the right-hand side below is finite for every $t\in\mathbf{R}$ we have*

$$\|u(t,\cdot)\|_{L^q(\mathbf{R}^n)}\le C(1+|t|)^{-(n-1)(\frac{1}{p}-\frac{1}{q})}\|u(t,\cdot)\|_{\Gamma,s,p} \tag{3.1}$$

$$\|(1+|\cdot|^2)^{\frac{n-1}{2}(\frac{1}{p}-\frac{1}{q})}u(t,\cdot)\|_{L^q(\mathbf{R}^n)}\le C\sum_{|\gamma|+|\theta|\le s}\|D_x^\gamma\Omega^\theta u(t,\cdot)\|_{L^p(\mathbf{R}^n)} \tag{3.2}$$

$$0<s<\frac{n}{p},\quad 1\le p,q\le\infty,\quad \frac{1}{q}\ge\frac{1}{p}-\frac{s}{n}$$

Here γ and θ are multi-index $\gamma=(\gamma_1,\cdots,\gamma_n)$, $\theta=(\theta_{12},\cdots,\theta_{n-1,n})$ with $|\gamma|=\gamma_1+\cdots+\gamma_n$, $|\theta|=\theta_{12}+\cdots+\theta_{n-1,n}$ and $D_x^\gamma=\partial_1^{\gamma_1}\cdots\partial_n^{\gamma_n}$, $\Omega^\theta=\prod\limits_{1\le j<k\le n}\Omega_{jk}^{\theta_{jk}}$.

Proof The proof of (3.1) can be found in Zhou [13]. The inequality (3.2) is proved in Section 3 of Hörmander [14].

Lemma 3.2 *For $n\ge 4$, denote $\omega=(-\Delta)^{\frac{1}{2}}$, for any function $F=F(t,x)$ for which the norm appearing on the right-hand side below is finite for every $\tau\in\mathbf{R}$ we have*

$$\left\|\frac{\sin\omega(t-\tau)}{\omega}F\right\|_{L^2(\mathbf{R}^n)}\le C\|F\|_{q,2,\chi}+C(1+|\tau|)^{-\frac{n-2}{2}}\|F\|_{1,2} \tag{3.3}$$

$$\left\|\frac{\sin\omega(t-\tau)}{\omega}[x_jF]\right\|_{L^2(\mathbf{R}^n)}\le C\|x_jF\|_{q,2,\chi}+C(1+|\tau|)^{-\frac{n-4}{2}}\|F\|_{1,2} \tag{3.4}$$

$$\left\|\frac{\sin\omega t}{\omega}F\right\|_{L^2(\mathbf{R}^n)}\le C\|F\|_{q,2} \tag{3.5}$$

here $\frac{1}{q}=\frac{1}{n}+\frac{1}{2}$ and χ is as in Section 2, C is a positive constant independent of t,F and τ.

Proof The proof of (3.3) and (3.5) can be found in Li and Yu [1]. Here we only give out the proof of (3.4). Making the change of variables $x=(t-\tau)y$, $\xi=\dfrac{\eta}{t-\tau}$ and then proceeding as in the proof for (3.3) due to Li and Yu, we obtain without any difficulty that

$$\left\|\frac{\sin\omega(t-\tau)}{\omega}[x_jF]\right\|_{L^2(\mathbf{R}^n)}$$

$$
\begin{aligned}
&= (2\pi)^{-\frac{n}{2}}\left\|\frac{\sin|\xi|(t-\tau)}{|\xi|}\widehat{x_jF(\tau,\cdot)}\right\|_2 \\
&\le C|t-\tau|^{\frac{n}{2}+1}\sup_{v\in H^1, v\neq 0}\left[\frac{\|\chi(\tau,\cdot)x_jF(\tau,\cdot)\|_{q,2}\|v\|_{p,2}}{\|v\|_{H^1}}|t-\tau|^{-\frac{n}{q}}\right] \\
&\quad + C|t-\tau|^{\frac{n}{2}+1}\sup_{v\in H^1, v\neq 0}\left[\frac{\|F(\tau,\cdot)\|_{1,2}\|(1-\psi)(t-\tau)x_jv(y)\|_{\infty,2}}{\|v\|_{H^1}}|t-\tau|^{-n}\right] \\
&= C|t-\tau|^{\frac{n}{2}+1}(I_1+I_2)
\end{aligned}
\tag{3.6}
$$

where $\frac{1}{p}=\frac{1}{2}-\frac{1}{n}$, $\psi(\tau,y)=\chi(\tau,x)$ is a characteristic function of the set

$$
\left\{(\tau,y): |y|\le \frac{1+|\tau|}{2|t-\tau|}\right\}
$$

By the Sobolev imbedding theorem and the compactness of S^{n-1}, we have

$$
\|v\|_{p,2}\le C\|v\|_p\le \|v\|_{H^1}
$$

Also recalling that $\frac{1}{q}=\frac{1}{n}+\frac{1}{2}$, we obtain

$$
I_1\le C|t-\tau|^{-\frac{n+2}{2}}\|x_jF(\tau,\cdot)\|_{q,2,\chi} \tag{3.7}
$$

On the other hand, taking account of the inequality

$$
\begin{aligned}
\int_{S^{n-1}} r^2v^2(r\xi)d\xi &= -\int_{S^{n-1}}\int_r^\infty \frac{d}{d\lambda}[\lambda^2v^2(r\xi)]d\lambda d\xi \\
&\le 2\int_{S^{n-1}}\int_r^\infty \lambda v^2(r\xi)d\lambda d\xi \\
&\quad + 2\int_{S^{n-1}}\int_r^\infty |\lambda|^2|v(r\xi)|\,|\nabla v(r\xi)|d\lambda d\xi
\end{aligned}
\tag{3.8}
$$

we get

$$
\begin{aligned}
&\|(1-\psi(\tau,\cdot))(t-\tau)y_jv(\cdot)\|_{\infty,2} \\
&\quad\le C|t-\tau|\sup_{r\in\left(\frac{1+|\tau|}{2(t-\tau)},\infty\right)}\left\{\left(\int_{S^{n-1}}\int_r^\infty \frac{1}{\lambda^{n-4}}v^2(\lambda\xi)\lambda^{n-3}d\lambda d\xi\right)^{\frac{1}{2}}\right. \\
&\qquad\left. + \left(\int_{S^{n-1}}\int_r^\infty \frac{1}{\lambda^{n-4}}|v(\lambda\xi)|\,|\nabla v(\lambda\xi)|\lambda^{n-2}d\lambda d\xi\right)^{\frac{1}{2}}\right\} \\
&\quad = C|t-\tau|(I_3+I_4)
\end{aligned}
\tag{3.9}
$$

Therefore, we obtain in view of the assumption $n \geq 4$

$$I_3 \leq C\left(\frac{|t-\tau|}{1+|\tau|}\right)^{\frac{n-4}{2}} \|v\|_{H^1} \tag{3.10}$$

$$I_4 \leq C\left(\frac{|t-\tau|}{1+|\tau|}\right)^{\frac{n-4}{2}} \|v\|_{H^1} \tag{3.11}$$

Combining (3.6)-(3.11) we have completed the proof of (3.4).

Lemma 3.3 *For any multi-index $k = (k_1, \cdots, k_\mu)$, we have*

$$[\Box, \Gamma^k] = \sum_{|i| \leq |k|-1} A_{ki}\Gamma^i \Box$$

$$[\partial_a, \Gamma^k] = \sum_{|i| \leq |k|-1} B_{ki}\Gamma^i \Box$$

where $[,]$ stands for the Possion's bracket and A_{ki}, B_{ki} are some constants and $\Box$ denotes the wave operator.

Lemma 3.4 *Suppose that $F(w)$ is a sufficiently smooth function of $w = (w_1, \cdots, w_M)$ and $F(0) = 0$. Let $u_i(t,x)$ $(i = 1, \cdots, \alpha)$ be functions with compact support in the variable x. For any given integer $s \geq 2n+3$ and any real number r with $1 \leq r \leq 2$, if a vector function $w = w(t,x) = (w_1, \cdots, w_M)$ satisfies*

$$\|w(t,\cdot)\|_{\Gamma,[\frac{s}{2}],\infty} \leq v_0 \tag{3.12}$$

where $[\ ]$ stands for the integral part of a real number and v_0 is a positive constant, and such that all the norms appearing on the right-hand side below are bounded, we have

$$\left\|F(w(t,\cdot))\prod_{i=1}^{\alpha} u_i(t,\cdot)\right\|_{\Gamma,s,r,2}$$

$$\leq C(1+|t|)^{-\frac{n-1}{2}(1-\frac{2}{\alpha p})\alpha}\|w(t,\cdot)\|_{\Gamma,s,2}\prod_{i=1}^{\alpha}\|u_i(t,\cdot)\|_{\Gamma,s,2} \tag{3.13}$$

$$\left\|F(w(t,\cdot))\prod_{i=1}^{\alpha} u_i(t,\cdot)\right\|_{\Gamma,s,r,2,\chi}$$

$$\leq C(1+|t|)^{-\frac{n}{2}(1-\frac{2}{\alpha p})\alpha}\|w(t,\cdot)\|_{\Gamma,s,2}\prod_{i=1}^{\alpha}\|u_i(t,\cdot)\|_{\Gamma,s,2} \tag{3.14}$$

where α is an integer ≥ 1 and $\frac{1}{p} = \frac{1}{r} - \frac{1}{2}$.

Lemma 3.5 *Suppose that $F(w)$ is a sufficiently smooth function of $w = (w_1, \cdots, w_M)$ satisfying if*

$$|w| < v_0$$

then

$$F(w) = O(|w|^{\alpha+1})$$

where α is an integer ≥ 1. For any integer $s \geq 2n+3$, if the vector function $w^- = (w_1^-, \cdots, w_M^-)$ and $w^= = (w_1^=, \cdots, w_M^=)$ satisfy (3.12) and have compact support in x, and if all norms appearing on the right-hand side below are bounded, we have

$$\|F(w^-) - F(w^=)\|_{\Gamma,s,r,2} \leq C(1+|t|)^{-\frac{n-1}{2}(1-\frac{2}{\alpha p})\alpha} \|w^\sim\|^\alpha_{\Gamma,s,2} \|w^*\|_{\Gamma,s,2} \tag{3.15}$$

$$\|F(w^-) - F(w^=)\|_{\Gamma,s,r,2,\chi} \leq C(1+|t|)^{-\frac{n}{2}(1-\frac{2}{\alpha p})\alpha} \|w^\sim\|^\alpha_{\Gamma,s,2} \|w^*\|_{\Gamma,s,2} \tag{3.16}$$

where $1 \leq r \leq 2$, r and p satisfies $\dfrac{1}{p} = \dfrac{1}{r} - \dfrac{1}{2}$, and

$$w^* = w^- - w^=$$

$$\|w^\sim\|_{\Gamma,s,2} = \|w^-\|_{\Gamma,s,2} + \|w^=\|_{\Gamma,s,2}$$

Proof The proof of Lemma 3.3, Lemma 3.4 and Lemma 3.5 can be found in Li and Yu [1].

Now we shall prove that L_j, L_0, Ω_{ij} commute with the integral sign in (1.5). Put

$$I_\sigma(t) = \int_\sigma^t \frac{\sin\omega(t-\tau)}{\omega} F(u(\tau), Du(\tau)) d\tau$$

we have

Proposition 3.1 *The following equalities hold if F is assumed as in the main theorem*

$$L_j I_\sigma(t) = \frac{\sin\omega(t-\sigma)}{\omega}[x_j F(\sigma)] + \int_\sigma^t \frac{\sin\omega(t-\tau)}{\omega} L_j F(\tau) d\tau \tag{3.17}$$

$$\Omega_{ij} I_\sigma(t) = \int_\sigma^t \frac{\sin\omega(t-\tau)}{\omega} \Omega_{ij} F(\tau) d\tau \tag{3.18}$$

$$L_0 I_\sigma(t) = \sigma \frac{\sin\omega(t-\sigma)}{\omega} F(\sigma) + 2\int_\sigma^t \frac{\sin\omega(t-\tau)}{\omega} F(\tau) d\tau + \int_\sigma^t \frac{\sin\omega(t-\tau)}{\omega} L_0 F(\tau) d\tau \tag{3.19}$$

$$L_j \partial_t I_\sigma(t) = \cos(t-\sigma)[x_j F(\sigma)] - \int_\sigma^t \frac{\sin\omega(t-\tau)}{\omega} \partial_j F(\tau) d\tau + \int_\sigma^t \cos\omega(t-\tau) L_j F(\tau) d\tau \tag{3.20}$$

$$\Omega_{ij}\partial_t I_\sigma(t) = \int_\sigma^t \cos\omega(t-\tau)\Omega_{ij}F(\tau)d\tau \tag{3.21}$$

$$\begin{aligned} L_0\partial_t I_\sigma(t) &= \sigma\cos\omega(t-\sigma)F(\sigma) + \int_\sigma^t \cos\omega(t-\tau)F(\tau)d\tau \\ &\quad + \int_\sigma^t \cos\omega(t-\tau)L_0F(\tau)d\tau \end{aligned} \tag{3.22}$$

$$\begin{aligned} L_j\partial_k I_\sigma(t) &= \frac{\sin\omega(t-\sigma)}{\omega}\partial_k[x_jF(\sigma)] + \delta_{jk}\int_\sigma^t \cos\omega(t-\tau)F(\tau)d\tau \\ &\quad + \int_\sigma^t \frac{\sin\omega(t-\tau)}{\omega}(\partial_k L_jF(\tau))d\tau \end{aligned} \tag{3.23}$$

$$\begin{aligned} \Omega_{ij}\partial_k I_\sigma(t) &= \delta_{jk}\int_\sigma^t \frac{\sin\omega(t-\tau)}{\omega}\partial_iF(\tau)d\tau - \delta_{ik}\int_\sigma^t \frac{\sin\omega(t-\tau)}{c\omega}\partial_jF(\tau)d\tau \\ &\quad + \int_\sigma^t \frac{\sin\omega(t-\tau)}{\omega}\partial_k\Omega_{ij}F(\tau)d\tau \end{aligned} \tag{3.24}$$

$$\begin{aligned} L_0\partial_k I_\sigma(t) &= \sigma\frac{\sin\omega(t-\sigma)}{\omega}\partial_kF(\sigma) + \int_\sigma^t \frac{\sin\omega(t-\tau)}{\omega}\partial_kF(\tau)d\tau \\ &\quad + \int_\sigma^t \frac{\sin\omega(t-\tau)}{\omega}\partial_k L_0F(\tau)d\tau \end{aligned} \tag{3.25}$$

Proof The proof of this proposition can be easily obtained by applying Fourier transform and carrying out integrations by part a few times with respect to τ. Here we omit it. For details one also can see Hidano [15].

Proposition 3.2 *Assume that $u \in X_{s,E}$, we have*

$$\|F(u(\tau), Du(\tau))\|_{\Gamma,s,q,2,\chi} \le C(1+|\tau|)^{-\frac{n}{2}(1-\frac{2}{\alpha n})\alpha}E^{\alpha+1} \tag{3.26}$$

$$\|F(u(\tau), Du(\tau))\|_{\Gamma,s,1,2} \le C(1+|\tau|)^{-\frac{n}{2}(1-\frac{2}{\alpha})\alpha}E^{\alpha+1} \tag{3.27}$$

$$\|F(u(\tau), Du(\tau))\|_{\Gamma,s,2} \le C(1+|\tau|)^{-\frac{n-1}{2}\alpha}E^{\alpha+1} \tag{3.28}$$

$$\begin{aligned} &\|(1+|\cdot|^2)^{\frac{1}{2}}\Gamma^{s-1}F(u(\tau), Du(\tau))\|_{L^2(\mathbb{R}^n)} \\ &\quad \le C(1+|\tau|)^{-(n-1)(\frac{1}{2}-\frac{1}{n\alpha-\alpha})\alpha}E^{\alpha+1} \end{aligned} \tag{3.29}$$

where $X_{s,E} = \{v : D_s(v) \le E\}$ *and* $D_s(v) = \sum_{i=0}^{2}\sup_{t\in\mathbb{R}}\|D^iv\|_{\Gamma,s,2}$.

Proof The proof of (3.26)–(3.28) can be easily obtained by using Lemma 3.2 and Lemma 3.4. Here we only prove (3.29), in fact it only needs to prove

$$\begin{aligned} &\|(1+|\cdot|^2)^{\frac{1}{2}}(|u|^\alpha\Gamma^{s-1}u + |Du|^\alpha\Gamma^{s-1}Du)\|_{L^2(\mathbb{R}^n)} \\ &\quad \le C(1+|\tau|)^{-(n-1)(\frac{1}{2}-\frac{1}{n\alpha-\alpha})\alpha}E^{\alpha+1} \end{aligned} \tag{3.30}$$

First, following the Hölder inequality and Lemma 3.1, we have

$$\begin{aligned}&\|(1+|\cdot|)|u|^{\alpha}\Gamma^{s-1}u\|_{L^2(\mathbb{R}^n)}\\ &\le C\|(1+|\cdot|^2)^{\frac{1}{2}}\Gamma^{s-1}u\|_{L^{\frac{2(n-1)}{n-3}}(\mathbb{R}^n)}\||u|^{\alpha}\|_{L^{n-1}(\mathbb{R}^n)}\\ &\le C(1+|\tau|)^{-(n-1)(\frac{1}{2}-\frac{1}{n\alpha-\alpha})\alpha}E^{\alpha+1}\end{aligned}\tag{3.31}$$

Proceeding as above, we also have the following estimate

$$\begin{aligned}&\|(1+|\cdot|^2)^{\frac{1}{2}}|Du|^{\alpha}\Gamma^{s-1}Du\|_{L^2(\mathbb{R}^n)}\\ &\le C(1+|\tau|)^{-(n-1)(\frac{1}{2}-\frac{1}{n\alpha-\alpha})\alpha}E^{\alpha+1}\end{aligned}\tag{3.32}$$

Thus we have finished the proof of Proposition 3.2.

Then combining Proposition 3.1 with Proposition 3.2, we have

$$L_jI_{-\infty}(t)=\int_{-\infty}^{t}\frac{\sin\omega(t-\tau)}{\omega}L_jF(\tau)d\tau\tag{3.33}$$

$$\Omega_{ij}I_{-\infty}(t)=\int_{-\infty}^{t}\frac{\sin\omega(t-\tau)}{\omega}\Omega_{ij}F(\tau)d\tau\tag{3.34}$$

$$L_0I_{-\infty}(t)=2\int_{-\infty}^{t}\frac{\sin\omega(t-\tau)}{\omega}F(\tau)d\tau+\int_{-\infty}^{t}\frac{\sin\omega(t-\tau)}{\omega}L_0F(\tau)d\tau\tag{3.35}$$

We still set

$$(L_jI)_{\sigma}(t)=\int_{-\sigma}^{t}\frac{\sin\omega(t-\tau)}{\omega}L_jF(\tau)d\tau\tag{3.36}$$

$$(\Omega_{ij}I)_{\sigma}(t)=\int_{-\sigma}^{t}\frac{\sin\omega(t-\tau)}{\omega}\Omega_{ij}F(\tau)d\tau\tag{3.37}$$

$$(L_0I)_{\sigma}(t)=2\int_{\sigma}^{t}\frac{\sin\omega(t-\tau)}{\omega}F(\tau)d\tau+\int_{\sigma}^{t}\frac{\sin\omega(t-\tau)}{\omega}L_0F(\tau)d\tau\tag{3.38}$$

In the same manner we can prove for $s=2$

$$\Gamma^sI_{-\infty}=\sum_{|\alpha|\le s}\int_{-\infty}^{t}\frac{\sin\omega(t-\tau)}{\omega}\Gamma^{\alpha}F(\tau)d\tau\tag{3.39}$$

$$\begin{aligned}\Gamma^sDI_{-\infty}=&\sum_{|\alpha|\le s}^{|\beta|=1}\int_{-\infty}^{t}\frac{\sin\omega(t-\tau)}{\omega}D_x^{\beta}\Gamma^{\alpha}F(\tau)d\tau\\ &+\sum_{|\alpha|\le s}\int_{-\infty}^{t}\cos\omega(t-\tau)\Gamma^{\alpha}F(\tau)d\tau\end{aligned}\tag{3.40}$$

$$\Gamma^s D^2 I_{-\infty} = \sum_{|\alpha|\le s}^{|\beta|=1} \int_{-\infty}^{t} \frac{\sin\omega(t-\tau)}{\omega} D_x^\beta \Gamma^\alpha DF(\tau)d\tau + \sum_{|\alpha|\le s} \int_{-\infty}^{t} \cos\omega(t-\tau)\Gamma^\alpha DF(\tau)d\tau \tag{3.41}$$

Therefore by induction we have the Proposition 3.3

Proposition 3.3 *If $u \in X_{s,E}$, we have the following equalities:*

$$\Gamma^s I_{-\infty} = \sum_{|\alpha|\le s} C_\alpha \int_{-\infty}^{t} \frac{\sin\omega(t-\tau)}{\omega} \Gamma^\alpha F(u(\tau), Du(\tau))d\tau \tag{3.42}$$

$$\Gamma^s DI_{-\infty} = \sum_{|\alpha|\le s}^{|\beta|=1} C_{\alpha\beta} \int_{-\infty}^{t} \frac{\sin\omega(t-\tau)}{\omega} D_x^\beta \Gamma^\alpha F(u(\tau), Du(\tau))d\tau + \sum_{|\alpha|\le s} C_\alpha \int_{-\infty}^{t} \cos\omega(t-\tau)\Gamma^\alpha F(u(\tau), Du(\tau))d\tau \tag{3.43}$$

$$\Gamma^s D^2 I_{-\infty} = \sum_{|\alpha|\le s}^{|\beta|=1} C_{\alpha,\beta} \int_{-\infty}^{t} \frac{\sin\omega(t-\tau)}{\omega} D_x^\beta \Gamma^\alpha DF(u(\tau), Du(\tau))d\tau + \sum_{|\alpha|\le s} C_\alpha \int_{-\infty}^{t} \cos\omega(t-\tau)\Gamma^\alpha DF(u(\tau), Du(\tau))d\tau \tag{3.44}$$

here $C_\alpha, C_{\alpha\beta}$ are constants.

4. The Proof of the Main Theorem

In this section we shall complete the proof of the main theorem by using Lemma 4.1 and Lemma 4.2.

Lemma 4.1 *Assume that $(f, g) \in \Sigma$ and $s \ge 2n+3$, then the solution of the Cauchy problem*

$$u_{0tt} - \Delta u_0 = 0 \tag{4.1}$$

$$t = 0 : u_0 = f, \quad u_{0t} = g \tag{4.2}$$

fulfills the estimate

$$D_s(u_0) \le C\|(f, g)\|_\Sigma$$

Lemma 4.2 *Let (f^-, g^-), $u_0^-(t)$ and F be assumed as in the main theorem, then there exists a unique solution $u(t)$ of the integral equation*

$$u(t) = u_0^-(t) + \int_{-\infty}^{t} \frac{\sin\omega(t-\tau)}{\omega} F(u(\tau), Du(\tau))d\tau$$

with $u(t) \in X_{s,E}$. Furthermore, $u(t)$ is the solution of (1.1) with Cauchy data given at $t = -\infty$, and also has the property

$$\|u(t) - u_0^-(t)\|_e \to 0 \quad as \quad t \to -\infty$$

Proof of the main theorem In fact, following Lemma 4.2 we have shown the part (i) of the main theorem. Hence we only need to prove the part (ii). Define now

$$u_0^+(t) = u(t) + \int_t^\infty \frac{\sin\omega(t-\tau)}{\omega} F(u(\tau), Du(\tau))d\tau$$

and

$$u_0^+(0) = f^+, \quad u_0^+(0) = g^+$$

Then we show in exactly the same manner as above $u_0^+(t)$ is a unique solution to (1.4) with the property

$$\|u(t) - u_0^+(t)\|_e \to 0 \quad as \quad t \to +\infty$$

So that we can define the scattering operator $S : (f^-, g^-) \to (f^+, g^+)$, it exists in a whole neighbourhood $\mathcal{N}$ of the origin in Σ. Thus we have finished the proof of the main theorem.

Now we turn to prove Lemma 4.1 and Lemma 4.2.

Proof of Lemma 4.1 By Lemma 3.3, we have

$$\Box L_j u_0 = 0 \tag{4.3}$$

$$t = 0 : \quad L_j u_0 = x_j g, \quad (L_j u_0) = \partial_j f + x_j \Delta f \tag{4.4}$$

$$\Box \Omega_{jk} u_0 = 0 \tag{4.5}$$

$$t = 0 : \Omega_{jk} u_0 = x_j \partial_k f - x_k \partial_j f, \quad (\Omega_{jk} u_0)_t = x_j \partial_x g - x_k \partial_j g \tag{4.6}$$

$$\Box L_0 u_0 = 0 \tag{4.7}$$

$$t = 0 : L_0 u_0 = \sum_{i=1}^n x_i \partial_i f, \quad (L_0 u_0)_t = g + \sum_{i=1}^n x_i \partial_i g \tag{4.8}$$

Following (4.3)–(4.8), we have

$$L_j u_0(t) = \cos\omega t(x_j g) + \frac{\sin\omega t}{\omega}(\partial_j f + x_j \Delta f) \tag{4.9}$$

$$\Omega_{jk} u_0(t) = \cos\omega t(x_j \partial_k f - x_k \partial_j f) + \frac{\sin\omega t}{\omega}(x_j \partial_k g - x_k \partial_j g) \tag{4.10}$$

$$L_0 u_0(t) = \sum_{i=1}^n \cos\omega t(x_i \partial_i f) + \frac{\sin\omega t}{\omega} g + \sum_{i=1}^n \frac{\sin\omega t}{\omega}(x_i \partial_i g) \tag{4.11}$$

Therefore, by (3.5) know that

$$\|u_0(t)\|_{\Gamma,1,2} \le C\|(f,g)\|_\Sigma$$

In a completely similar way we can get the estimates for $\|u_0(t)\|_{\Gamma,s,2}$, $\|Du_0(t)\|_{\Gamma,s,2}$ and $\|D^2u_0(t)\|_{\Gamma,s,2}$. Here we omit the details.

Proof of Lemma 4.2 Put

$$u_\sigma(t) = u_0(t) + \int_\sigma^t \frac{\sin\omega(t-\tau)}{\omega} F(u(\tau), D(u(\tau))d\tau$$

for any $\sigma \in (-\infty, 0)$. In view of Lemma 3.3, Lemma 3.4 and Lemma 4.1, it is easily verified that $u_\sigma(t)$ is a unique solution to (1.1) with $(u_\sigma(\sigma), u_{t\sigma}(\sigma)) = (u_0^-(\sigma), u_{0t}^-(\sigma))$. By virtue of Proposition 3.3, clearly, we have $u_\sigma \to u$ as $\sigma \to -\infty$ which is the solution of (1.5) and $u \in \bigcap_{k=0}^{s+2} C^k(\mathbf{R}, H^{s+2-k}(\mathbf{R}^n))$. We want to apply the contraction mapping principle and first remark that $u_0^-(t) \in X_{s,E}$. Now we consider $w^-, w^= \in X_{s,E}$ and estimate

$$\begin{aligned}
&\left\|\frac{\sin\omega(t-\omega)}{\omega}\Gamma^s(F(w^-, Dw^-) - F(w^=, Dw^=))\right\|_{L^2(\mathbf{R}^n)} \\
&\le \|F(w^-, Dw^-) - F(w^=, Dw^=)\|_{\Gamma,s,q,2,\chi} \\
&\quad + C(1+|t|)^{-\frac{n-2}{2}}\|(F(w^-, Dw^-) - F(w^=, Dw^=))\|_{\Gamma,s,1,2} \\
&\le C(1+|t|)^{-\frac{n}{2}(1-\frac{2}{\alpha n})\alpha} E^\alpha \|w^- - w^=\|_{X_{s,E}} \\
&\quad + C(1+|t|)^{-\frac{n-2}{2}}(1+|t|)^{-\frac{n-1}{2}(1-\frac{1}{\alpha})\alpha} E^\alpha \|w^- - w^=\|_{X_{s,E}} \\
&\le C(1+|t|)^{-\frac{(n-1)\alpha-1}{2}} E^\alpha \|w^- - w^=\|_{X_{s,E}}
\end{aligned} \tag{4.12}$$

and

$$\begin{aligned}
&\|F(w^-, Dw^-) - F(w^=, Dw^=)\|_{\Gamma,s,2} \\
&\qquad \le C(1+|t|)^{-\frac{(n-1)\alpha}{2}} E^\alpha \|w^- - w^=\|_{X_{s,E}}
\end{aligned} \tag{4.13}$$

$$\begin{aligned}
&\|DF(w^-, Dw^-) - DF(w^=, Dw^=)\|_{\Gamma,s,2} \\
&\qquad \le C(1+|t|)^{-\frac{(n-1)\alpha}{2}} E^\alpha \|w^- - w^=\|_{X_{s,E}}
\end{aligned} \tag{4.14}$$

Here we have already used Lemma 3.2, Lemma 3.4 and $\alpha \ge 1$, hence, we obtain

$$\begin{aligned}
&\int_{-\infty}^t \left\|\frac{\sin\omega(t-\tau)}{\omega}\Gamma^s(F(w^-, Dw^-) - F(w^=, Dw^=))\right\|_{L^2(\mathbf{R}^n)} d\tau \\
&\le C\int_{-\infty}^t (1+|\tau|)^{-\frac{(n-1)\alpha-1}{2}} E^\alpha \|w^- - w^=\|_{X_{s,E}} d\tau \\
&\le CE^\alpha \|w^- - w^=\|_{X_{s,E}}
\end{aligned} \tag{4.15}$$

$$\int_{-\infty}^t \|F(w^-, Dw^-) - F(w^=, Dw^=)\|_{\Gamma,s,2} d\tau$$

$$\leq C\int_{-\infty}^{t}(1+|\tau|)^{-\frac{(n-1)\alpha-1}{2}}E^{\alpha}\|w^{-}-w^{=}\|_{X_{s,E}}d\tau$$
$$\leq CE^{\alpha}\|w^{-}-w^{=}\|_{X_{s,E}} \tag{4.16}$$

$$\int_{-\infty}^{t}\|DF(w^{-},Dw^{-})-DF(w^{=},Dw^{=})\|_{\Gamma,s,2}d\tau$$
$$\leq C\int_{-\infty}^{t}(1+|\tau|)^{-\frac{(n-1)\alpha-1}{2}}E^{\alpha}\|w^{-}-w^{=}\|_{X_{s,E}}d\tau$$
$$\leq CE^{\alpha}\|w^{-}-w^{=}\|_{X_{s,E}} \tag{4.17}$$

If we denote the transformation which maps w^{-} into the right-hand side of the considered integral equation by T we have shown

$$\|T(w^{-})-T(w^{=})\|_{X_{s,E}}\leq CE^{\alpha}\|w^{-}-w^{=}\|_{X_{s,E}} \tag{4.18}$$

as well as

$$\|T(w^{-})\|_{X_{s,E}}\leq C\delta+CE^{\alpha+1} \tag{4.19}$$

Assume now that

$$C\delta\leq\frac{1}{2}\delta_1,\quad 2C\delta_1^{\alpha}\leq\frac{1}{2}$$

if $D_s(w^{-}),D_s(w^{=})\leq\delta_1$ we conclude that

$$\|T(w^{-})-T(w^{=})\|_{X_{s,E}}\leq\frac{1}{2}\|w^{-}-w^{=}\|_{X_{s,E}}$$

and

$$\|T(w^{-})\|_{X_{s,E}}\leq C\delta_1$$

The contraction mapping principle shows the existence of the unique solution in X_{s,δ_1}.

Now we begin to compare with the asymptotic behavior between $u(t)$ and $u_0^{-}(t)$ as $t\to-\infty$. In fact, we have

$$\|u(t)-u_0^{-}(t)\|_e\leq\int_{-\infty}^{t}\|F(u,Du)\|_{L^2(\mathbb{R}^n)}d\tau$$
$$\leq\int_{-\infty}^{t}(1+|\tau|)^{-\frac{(n-1)\alpha-1}{2}}\|u\|_{X_{s,E}}d\tau$$
$$\leq(1+|t|)^{-\frac{(n-1)\alpha-1}{2}+1}\|u\|_{X_{s,E}} \tag{4.20}$$

by the assumption of α we can see

$$-\frac{(n-1)\alpha-1}{2}+1<0$$

Thus, we obtain

$$\|u(t) - u_0^-(t)\|_e \to 0 \quad (t \to -\infty)$$

So we have completed the proof of Lemma 4.2.

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