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ORBITAL STABILITY OF SOLITARY WAVES OF THE NONLINEAR SCHRÖDINGER-KDV EQUATION

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Abstract This paper concerns the orbital stability of solitary waves of the system of KdV equation coupling with nonlinear Schrödinger equation. By applying the abstract results of Grillakis et al. [1–2] and detailed spectral analysis, we obtain the stability of the solitary waves.

Key Words Solitary wave; stability; nonlinear Schrödinger-KdV equation.
Classification 35Q55, 35B35.

1. Introduction

In this paper, we consider the following nonlinear Schrödinger-KdV system

$$\begin{cases} i\varepsilon_t + \varepsilon_{xx} - n\varepsilon = 0 \\ n_t + n_x + \beta(n^2)_x + \alpha n_{xxx} = -(|\varepsilon|^2)_x \end{cases} \quad x \in \mathbf{R} \tag{1.1}$$

with n a real function and ε a complex function. The problem (1.1) arises in laser and plasma physics. The local and global existence of initial value problem for (1.1) was considered in [3].

In this paper, we consider the stability of solitary waves of (1.1). By applying the abstract theory of Grillakis et al. [1–2] and detailed spectral analysis, we obtain the sufficient conditions for the stability of the solitary waves.

For the other types of equations, such as nonlinear Schrödinger equation, KdV equation and BO equation, the orbital stability of solitary waves were considered in [1–2, 4–8].

This paper is organized as follows: in Section 2, we state the results of the existence of solitary waves; in Section 3, we state the assumptions and the stability results; in Section 4, we obtain the sufficient conditions for the stability.

2. The Existence of Solitary Waves

Consider the following nonlinear Schrödinger-KdV system

$$\begin{cases} i\varepsilon_t + \varepsilon_{xx} - n\varepsilon = 0 \\ n_t + n_x + \beta(n^2)_x + \alpha n_{xxx} = -(|\varepsilon|^2)_x \end{cases} \quad x \in \mathbf{R} \tag{2.1}$$

Let

$$\varepsilon(t,x) = e^{-i\omega t}e^{iq(x-vt)}\widehat{\varepsilon}_{\omega,v}(x-vt) \tag{2.2}$$

$$n(t,x) = n_{\omega,v}(x-vt) \tag{2.3}$$

be the solitary waves of (2.1), where ω, q, v are real numbers, $\widehat{\varepsilon}_{\omega,v}$ and $n_{\omega,v}$ are real functions. Put (2.2)–(2.3) into (2.1), we obtain

$$\widehat{\varepsilon}''_{\omega,v} + i(2q-v)\widehat{\varepsilon}'_{\omega,v} + (\omega + qv - q^2 - n_{\omega,v})\widehat{\varepsilon}_{\omega,v} = 0 \tag{2.4}$$

$$-(v-1)n_{\omega,v} + \beta n^2_{\omega,v} + \alpha n''_{\omega,v} + \widehat{\varepsilon}^2_{\omega,v} = 0 \tag{2.5}$$

(2.4) implies

$$2q = v \tag{2.6}$$

Let $\widehat{\varepsilon}_{\omega,v} = c_1 \mathrm{sech} c_2 x$ satisfy (2.4) with constants c_1, c_2 determined later, then we have

$$\widehat{\varepsilon}''_{\omega,v} = (c_2^2 - 2c_2^2\mathrm{sech}^2 c_2 x)\widehat{\varepsilon}_{\omega,v} = \Big(-\omega - \frac{v^2}{4} + n_{\omega,v}\Big)\widehat{\varepsilon}_{\omega,v} \tag{2.7}$$

Suppose $n_{\omega,v} \to 0$, as $x \to \infty$, by (2.7), we have

$$n_{\omega,v} = -2c_2^2\mathrm{sech}^2 c_2 x + c_2^2 + \omega + \frac{v^2}{4} = -2c_2^2\mathrm{sech}^2 c_2 x \tag{2.8}$$

$$c_2^2 = -\omega - \frac{v^2}{4} \tag{2.9}$$

Put (2.8), (2.9) into (2.5), we have

$$2(v-1)c_2^2\mathrm{sech}^2 c_2 x + 4\beta c_2^4 \mathrm{sech}^4 c_2 x = 4\alpha c_2^4(2\mathrm{sech}^2 c_2 x - 3\mathrm{sech}^4 c_2 x) - c_1^2 \mathrm{sech}^2 c_2 x \tag{2.10}$$

It follows from (2.6), (2.9) and (2.10) that

$$\begin{cases} \alpha = -\dfrac{1}{3}\beta, \quad \beta > 0 \\ q = \dfrac{v}{2} \\ c_2 = \sqrt{-\omega - \dfrac{v^2}{4}} \\ c_1 = \sqrt{2\Big(-\omega - \dfrac{v^2}{4}\Big)\Big(1 - v - \dfrac{4}{3}\beta\Big(-\omega - \dfrac{v^2}{4}\Big)\Big)}, \quad v < 1 \end{cases}$$

thus

$$\begin{cases} \widehat{\varepsilon}_{\omega,v}(x) = \sqrt{2\Big(-\omega-\frac{v^2}{4}\Big)\Big(1-v-\frac{4}{3}\beta\Big(-\omega-\frac{v^2}{4}\Big)\Big)}\operatorname{sech}\Big(\frac{\sqrt{-4\omega-v^2}}{2}x\Big) \\ n_{\omega,v}(x) = \frac{4\omega+v^2}{2}sech^2\Big(\frac{\sqrt{-4\omega-v^2}}{2}x\Big) \\ q=\frac{v}{2}, \quad \alpha=-\frac{1}{3}\beta, \quad v<1 \end{cases} \tag{2.11}$$

Finally, we have

Theorem 1 *For any real constants ω, v, α, β satisfy*

$$\beta > 0 \quad and \quad -\frac{v^2}{4}-\frac{3}{4\beta}(1-v) < \omega < -\frac{v^2}{4} \tag{2.12}$$

there exist solitary waves of (2.1) *in the form of* (2.2)–(2.3), *with $n_{\omega,v}, \widehat{\varepsilon}_{\omega,v}, q, v, \alpha$ and β satisfying* (2.11).

3. Main Results

Rewrite (2.1) as

$$\begin{cases} i\varepsilon_t + \varepsilon_{xx} - n\varepsilon = 0 \\ n_t + n_x + \beta(n^2)_x + \alpha n_{xxx} = -(|\varepsilon|^2)_x \end{cases} \quad x \in \mathbf{R} \tag{3.1}$$

Let $\vec{u} = (\varepsilon, n)$. The function space in which we shall work is $X = H^1_{\text{complex}}(\mathbf{R}) \times H^1_{\text{real}}(\mathbf{R})$, with real inner product

$$(\vec{u}_1, \vec{u}_2) = \operatorname{Re}\int_{\mathbf{R}} (n_1 n_2 + n_{1x} n_{2x} + \varepsilon_1 \overline{\varepsilon}_2 + \varepsilon_{1x}\overline{\varepsilon}_{2x})dx \tag{3.2}$$

The dual space of X is $X^* = H^{-1}_{\text{complex}}(\mathbf{R}) \times H^{-1}_{\text{real}}(\mathbf{R})$, there is a natural isomorphism $I : X \to X^*$ defined by

$$\langle I\vec{u}_1, \vec{u}_2\rangle = (\vec{u}_1, \vec{u}_2) \tag{3.3}$$

where $\langle\cdot,\cdot\rangle$ denotes the pairing between X and X^*.

$$\langle \vec{f}, \vec{u}\rangle = \operatorname{Re}\int_{\mathbf{R}} (f_2 n + f_1\overline{\varepsilon})dx \tag{3.4}$$

By (3.2)–(3.4), it is obvious

$$I = \begin{pmatrix} 1-\frac{\partial^2}{\partial x^2} & \\ & 1-\frac{\partial^2}{\partial x^2} \end{pmatrix}$$

Let T_1, T_2 be one-parameter groups of unitary operator on X defined by

$$T_1(s_1)\vec{u}(\cdot) = \vec{u}(\cdot - s_1) \quad \text{for} \quad \vec{u}(\cdot) \in X, \quad s_1 \in \mathbf{R} \tag{3.5}$$

$$T_2(s_2)\vec{u}(\cdot) = (\varepsilon(\cdot)e^{-is_2}, n(\cdot)), \quad \text{for} \quad \vec{u}(\cdot) \in X, \quad s_2 \in \mathbf{R} \tag{3.6}$$

Obviously

$$T_1'(0) = \begin{pmatrix} -\dfrac{\partial}{\partial x} & \\ & -\dfrac{\partial}{\partial x} \end{pmatrix}$$

$$T_2'(0) = \begin{pmatrix} -i & \\ & 0 \end{pmatrix}$$

It follows from Theorem 1 and (3.1) that there exist solitary waves $T_1(vt)T_2(\omega t)$ $(\varepsilon_{\omega,v}(x), n_{\omega,v}(x))$ of (3.1), with $n_{\omega,v}, \varepsilon_{\omega,v}$ defined by

$$\begin{cases} \varepsilon_{\omega,v}(x) = e^{i\frac{v}{2}x}\sqrt{2\Big(-\omega - \dfrac{v^2}{4}\Big)\Big(1 - v - \dfrac{4}{3}\beta\Big(-\omega - \dfrac{v^2}{4}\Big)\Big)}\,\mathrm{sech}\Big(\dfrac{\sqrt{-4\omega - v^2}}{2}x\Big) \\ n_{\omega,v}(x) = \dfrac{4\omega + v^2}{2}\mathrm{sech}^2\Big(\dfrac{\sqrt{-4\omega - v^2}}{2}x\Big) \end{cases} \tag{3.7}$$

Denote

$$\Phi_{\omega,v}(x) = (\varepsilon_{\omega,v}(x), n_{\omega,v}(x)) \tag{3.8}$$

$$\varepsilon_{\omega,v}(x) = e^{i\frac{v}{2}x}\widehat{\varepsilon}_{\omega,v}(x) \tag{3.9}$$

In this and the following sections, we shall consider the orbital stability of solitary waves $T_1(vt)T_2(\omega t)\Phi_{\omega,v}(x)$ of (3.1). Note that the equation (3.1) is invariant under $T_1(\cdot)$ and $T_2(\cdot)$, we define the orbital stability as follows:

Definition *The solitary wave $T_1(vt)T_2(\omega t)\Phi_{\omega,v}(x)$ is orbitally stable if for all $\varepsilon > 0$ there exists $\delta > 0$ with the following property. If $\|\vec{u}_0 - \Phi_{\omega,v}\|_X < \delta$ and $u(t)$ is a solution of (3.1) in some interval $[0, t_0)$ with $\vec{u}(0) = \vec{u}_0$, then $\vec{u}(t)$ can be continued to a solution in $0 \le t < +\infty$, and*

$$\sup_{0<t<+\infty} \inf_{s_1 \in \mathbf{R}} \inf_{s_2 \in \mathbf{R}} \|\vec{u}(t) - T_1(s_1)T_2(s_2)\Phi_{\omega,v}\|_X < \varepsilon \tag{3.10}$$

Otherwise $T_1(vt)T_2(\omega t)\Phi_{\omega,v}(x)$ is called orbitally unstable.

So long as ω, v are fixed, we write $\varepsilon, \widehat{\varepsilon}, n$ for $\varepsilon_{\omega,v}, \widehat{\varepsilon}_{\omega,v}, n_{\omega,v}$. Define

$$E(\vec{u}) = \int_{\mathbf{R}} \Big(|\varepsilon_x|^2 + n|\varepsilon|^2 + \frac{1}{2}n^2 + \frac{\beta}{3}n^3 - \frac{\alpha}{2}n_x^2\Big)dx \tag{3.11}$$

It is easy to verify that $E(\vec{u})$ is invariant under T_1 and T_2, and formally conserved under the flow of (3.1). Namely

$$E(T_1(s_1)T_2(s_2)\vec{u}) = E(\vec{u}), \quad \text{for any } s_1, s_2 \in \mathbf{R} \tag{3.12}$$

and for any $t \in \mathbf{R}$, $\vec{u}(t)$ is a flow of (3.1)

$$E(\vec{u}(t)) = E(\vec{u}(0)) \tag{3.13}$$

Note that the equation (3.1) can be written as the following Hamiltonian system

$$\frac{d\vec{u}}{dt} = JE'(\vec{u}) \tag{3.14}$$

with a skew-symmetrically linear operator J defined by

$$J = \begin{pmatrix} \frac{-i}{2} & \\ & -\frac{\partial}{\partial x} \end{pmatrix} \tag{3.15}$$

E' is the Frechet derivative of E. As in [2] and [1], we define

$$B_1 = \begin{pmatrix} -2i\frac{\partial}{\partial x} & \\ & 1 \end{pmatrix}$$

such that $T_1'(0) = JB_1$,

$$B_2 = \begin{pmatrix} 2 & \\ & 0 \end{pmatrix}$$

such that $T_2'(0) = JB_2$,

$$Q_1(\vec{u}) = \frac{1}{2}\langle B_1\vec{u}, \vec{u}\rangle = \frac{1}{2}\int_{\mathbf{R}} n^2 dx + \operatorname{Im}\int_{\mathbf{R}} (\varepsilon_x \bar{\varepsilon}) dx \tag{3.16}$$

$$Q_2(\vec{u}) = \frac{1}{2}\langle B_2\vec{u}, \vec{u}\rangle = \int_{\mathbf{R}} |\varepsilon|^2 dx \tag{3.17}$$

As in [2] and [1], by (3.13)–(3.18), we can prove that

$$Q_1(T_1(s_1)T_2(s_2)\vec{u}) = Q_1(\vec{u}), \quad Q_2(T_1(s_1)T_2(s_2)\vec{u}) = Q_2(\vec{u}) \tag{3.18}$$

for any $s_1, s_2 \in \mathbf{R}$.

And for any $t \in \boldsymbol{R}$, $\vec{u}(t)$ is a flow of (3.1)

$$Q_1(\vec{u}(t)) = Q_1(\vec{u}(0)), \quad Q_2(\vec{u}(t)) = Q_2(\vec{u}(0)) \tag{3.19}$$

Furthermore

$$E'(\Phi_{\omega,v}) - vQ_1'(\Phi_{\omega,v}) - \omega Q_2'(\Phi_{\omega,v}) = 0 \tag{3.20}$$

where E', Q_1', Q_2' are the Frechet derivatives of E, Q_1 and Q_2 and defined by

$$E'(\vec{u}) = \begin{pmatrix} -2\varepsilon_{xx} + 2n\varepsilon \\ |\varepsilon|^2 + n + \beta n^2 + \alpha n_{xx} \end{pmatrix}$$

$$Q_1'(\vec{u}) = \begin{pmatrix} -2i\varepsilon_x \\ n \end{pmatrix}, \quad Q_2'(\vec{u}) = \begin{pmatrix} 2\varepsilon \\ 0 \end{pmatrix}$$

Define an operator from X to X^*.

$$H_{\omega,v} = E''(\Phi_{\omega,v}) - vQ_1''(\Phi_{\omega,v}) - \omega Q_2''(\Phi_{\omega,v}) \tag{3.21}$$

with $\vec{\psi} = (\psi_1, \psi_2) \in X$, and

$$H_{\omega,v}\vec{\psi} = \begin{pmatrix} -2\psi_{1xx} + 2n\psi_1 + 2\varepsilon\psi_2 + 2iv\psi_{1x} - 2\omega\psi_1 \\ \varepsilon\overline{\psi}_1 + \overline{\varepsilon}\psi_1 + \psi_2 + \alpha\psi_{2xx} + 2\beta n\psi_2 - v\psi_2 \end{pmatrix} \tag{3.22}$$

Observe that $H_{\omega,v}$ is self-adjoint in the sense that $H^*_{\omega,v} = H_{\omega,v}$. This means that $I^{-1}H_{\omega,v}$ is a bounded self-adjoint operator on X. The 'spectrum' of $H_{\omega,v}$ consists of the real numbers λ such that $H_{\omega,v} - \lambda I$ is not invertible. We claim that $\lambda = 0$ belongs to the spectrum of $H_{\omega,v}$.

By (3.12), (3.18), (3.20) and (3.21), it is easy to prove that

$$H_{\omega,v}T_1'(0)\Phi_{\omega,v}(x) = 0 \tag{3.23}$$

$$H_{\omega,v}T_2'(0)\Phi_{\omega,v}(x) = 0 \tag{3.24}$$

Let

$$Z = \{k_1T_1'(0)\Phi_{\omega,v}(x) + k_2T_2'(0)\Phi_{\omega,v}(x)/k_1, k_2 \in \mathbf{R}\} \tag{3.25}$$

By (3.23) and (3.24), Z is contained in the kernel of $H_{\omega,v}$.

Assumption 1 (Spectral decomposition of $H_{\omega,v}$)

The space X is decomposed as a direct sum

$$X = N + Z + P \tag{3.26}$$

where Z is defined above, N is a finite-dimensional subspace such that

$$\langle H_{\omega,v}\vec{u}, \vec{u}\rangle < 0 \quad \text{for} \quad 0 \neq \vec{u} \in N \tag{3.27}$$

and P is a closed subspace such that

$$\langle H_{\omega,v}\vec{u}, \vec{u}\rangle \geq \delta\|\vec{u}\|_X^2 \quad \text{for} \quad \vec{u} \in P \tag{3.28}$$

with some constant $\delta > 0$ independent of $\vec{u}$.

We define $d(\omega, v) : \mathbf{R} \times \mathbf{R} \to \mathbf{R}$ by

$$d(\omega, v) = E(\Phi_{\omega,v}) - vQ_1(\Phi_{\omega,v}) - \omega Q_1(\Phi_{\omega,v}) \tag{3.29}$$

We define $d''(\omega, v)$ to be the Hessian of function d. It is a symmetric bilinear form.

The global existence of solutions for the initial value problem (3.1) in $H^1(\mathbf{R}) \times H^1(\mathbf{R})$ was obtained in [3]. Let us now state our main results about stability of solitary wave $T_1(vt)T_2(\omega t)\Phi_{\omega,v}(x)$.

Theorem 2 *Under the condition of Theorem* 1, *if*

$$\beta \le 3, \quad v \ge 0, \quad and \quad -\frac{v^2}{4} - \frac{1}{6}(1-v) < \omega < -\frac{v^2}{4} \tag{3.30}$$

The solitary waves $T_1(vt)T_2(\omega t)\Phi_{\omega,v}(x)$ of (3.1) *with the expression* (3.7)-(3.9) *are orbitally stable.*

4. The Proof of Theorem 2

In virtue of (3.11)-(3.19), we can apply the abstract stability theory of Shatah [1] to (3.1). It is obvious that $d(\omega, v)$ is non-degenerate at (ω, v). Let $p(d'')$ be the number of positive eigenvalues of its Hessian at (ω, v). Let $n(H_{\omega,v})$ be the number of negative eigenvalues of $H_{\omega,v}$. To prove Theorem 2, it is sufficient to prove that under the condition of (2.11) and (3.30), Assumption 1 holds and $n(H_{\omega,v}) = p(d'')$, which will be proved in the following.

First we prove that Assumption 1 holds and $n(H_{\omega,v}) = 1$.

For any $\vec{\psi} \in X$, rewrite it as

$$\vec{\psi} = (e^{i\frac{v}{2}x} z_1, z_2) \tag{4.1}$$

$$\text{with } z_1 = y_1 + iy_2,\ y_1 = \operatorname{Re} z_1,\ y_2 = \operatorname{Im} z_1$$

then by (3.22) we have

$$\begin{aligned}
\langle H_{\omega,v}\vec{\psi}, \vec{\psi}\rangle &= \operatorname{Re}\int_{\mathbf{R}} \left\{ -2\left[z_1'' + \left(\omega + \frac{v^2}{4} - n\right) z_1\right] \overline{z}_1 + 2\varepsilon z_2 e^{-i\frac{v}{2}x}\overline{z}_1 \right\} dx \\
&\quad + \operatorname{Re}\int_{\mathbf{R}} [\varepsilon e^{-i\frac{v}{2}x}\overline{z}_1 z_2 + \overline{\varepsilon} e^{i\frac{v}{2}x} z_1 z_2 + (1-v)z_2^2 + \alpha z_{2xx} z_2 + 2\beta n z_2^2] dx \\
&= 4\operatorname{Re}\int_{\mathbf{R}} (\varepsilon z_2 e^{-i\frac{v^2}{2}x}\overline{z}_1) dx + \langle L_4 z_2, z_2\rangle + \langle L_2 y_1, y_1\rangle + \langle L_2 y_2, y_2\rangle \\
&= \langle L_3 z_2, z_2\rangle + \langle L_1 y_1, y_1\rangle + \langle L_2 y_2, y_2\rangle
\end{aligned}$$

$$+\int_{\mathbf{R}}\left(\frac{2\widehat{\varepsilon}y_1}{\sqrt{1-v+\frac{4\beta}{3}(\omega+\frac{v^2}{4})}}+\sqrt{1-v+\frac{4\beta}{3}\left(\omega+\frac{v^2}{4}\right)}z_2\right)^2dx \tag{4.2}$$

with

$$L_1=2\left[-\frac{\partial^2}{\partial x^2}-\omega-\frac{v^2}{4}+3n\right] \tag{4.3}$$

$$L_2=2\left[-\frac{\partial^2}{\partial x^2}-\omega-\frac{v^2}{4}+n\right] \tag{4.4}$$

$$L_3=-\frac{\beta}{3}\frac{\partial^2}{\partial x^2}+2\beta n+(1-v)-\left(1-v+\frac{4\beta}{3}\left(\omega+\frac{v^2}{4}\right)\right)$$

$$=\frac{\beta}{3}\left[-\frac{\partial^2}{\partial x^2}+4\left(-\omega-\frac{v^2}{4}\right)+6n\right] \tag{4.5}$$

$$L_4=-\frac{\beta}{3}\frac{\partial^2}{\partial x^2}+2\beta n+(1-v) \tag{4.6}$$

Note that

$$L_1=-2\frac{\partial^2}{\partial x^2}+2\left(-\omega-\frac{v^2}{4}\right)+M_1(x) \tag{4.7}$$

$$L_2=-2\frac{\partial^2}{\partial x^2}+2\left(-\omega-\frac{v^2}{4}\right)+M_2(x) \tag{4.8}$$

$$L_3=-\frac{\beta}{3}\frac{\partial^2}{\partial x^2}+\frac{4\beta}{3}\left(-\omega-\frac{v^2}{4}\right)+M_3(x) \tag{4.9}$$

with

$$M_1(x)\to 0 \quad \text{as} \quad |x|\to+\infty \tag{4.10}$$

$$M_2(x)\to 0 \quad \text{as} \quad |x|\to+\infty \tag{4.11}$$

$$M_3(x)\to 0 \quad \text{as} \quad |x|\to+\infty \tag{4.12}$$

Thus, by Weyl's theorem on the essential spectrum [9], we have

$$\sigma_{\text{ess}}(L_1)=\left[2\left(-\omega-\frac{v^2}{4}\right),+\infty\right) \tag{4.13}$$

$$\sigma_{\text{ess}}(L_1)=\left[2\left(-\omega-\frac{v^2}{4}\right),+\infty\right) \tag{4.14}$$

$$\sigma_{\text{ess}}(L_3)=\left[\frac{4\beta}{3}\left(-\omega-\frac{v^2}{4}\right),+\infty\right) \tag{4.15}$$

Denote $c_2^2 = -\omega - \frac{v^2}{4}$, (2.12) assures $c_2^2 > 0$. It follows from (2.2) (2.4), (2.5) and (3.9) that

$$L_1\widehat{\varepsilon}_x = 0 \tag{4.16}$$

$$L_2\widehat{\varepsilon} = 0 \tag{4.17}$$

$$L_3 n_x = 0 \tag{4.18}$$

By (3.7), (3.9) and (4.16), we see that $\widehat{\varepsilon}_x$ has a simple zero at $x = 0$, then Sturm-Liouville theorem implies that 0 is the second eigenvalue of L_1, and L_1 has exactly one strictly negative eigenvalue. Note that

$$L_1(\widehat{\varepsilon}^2) = -6\Big(-\omega - \frac{v^2}{4}\Big)(\widehat{\varepsilon}^2) \tag{4.19}$$

Thus, the first negative eigenvalue of L_1 is $-6\Big(-\omega - \frac{v^2}{4}\Big)$, with an eigenvalue of L_1 is $\widehat{\varepsilon}^2$. By (3.7) and (4.18), we see that n_x has a simple zero at $x = 0$, then Sturm-Liouville theorem implies that 0 is the second eigenvalue of L_3, and L_3 has exactly one strictly negative eigenvalue. Note that

$$L_3(2n\widehat{\varepsilon}) = -\frac{5}{3}\beta\Big(-\omega - \frac{v^2}{4}\Big)(2n\widehat{\varepsilon}) \tag{4.20}$$

Thus, the first negative eigenvalue of L_3 is $-\frac{5}{3}\beta\Big(-\omega - \frac{v^2}{4}\Big)$, with an eigenvalue of L_3 being $2n\widehat{\varepsilon}$. Also by (3.7) and (3.9), (4.17) implies 0 is the first simple eigenvalue of L_2.

In virtue of (4.5)–(4.20), as in [4, 6], we have the following lemmas.

Lemma 1 *For any real functions $y_1 \in H^1(\mathbf{R})$ satisfying*

$$\langle y_1, \widehat{\varepsilon}^2\rangle = \langle y_1, \widehat{\varepsilon}_x\rangle = 0 \tag{4.21}$$

then

$$\langle L_1 y_1, y_1\rangle \ge 2\Big(-\omega - \frac{v^2}{4}\Big)\|y_1\|_{L^2}^2 \tag{4.22}$$

and there exists a positive number $\delta_1 > 0$ such that

$$\langle L_1 y_1, y_1\rangle \ge \delta_1\Big(-\omega - \frac{v^2}{4}\Big)\|y_1\|_{H^1}^2 \tag{4.23}$$

Lemma 2 *For any real function $y_2 \in H^1(\mathbf{R})$ satisfying*

$$\langle y_2, \widehat{\varepsilon}\rangle = 0 \tag{4.24}$$

there exists a positive number $\delta_2 > 0$ *such that*

$$\langle L_2 y_2, y_2\rangle \geq \delta_2\Big(-\omega - \frac{v^2}{4}\Big)\|y_2\|_{H^1}^2 \tag{4.25}$$

Lemma 3 *For any real functions* $z_2 \in H^1(\mathrm{R})$ *satisfying*

$$\langle z_2, 2n\widehat{\varepsilon}\rangle = \langle z_2, n_x\rangle = 0 \tag{4.26}$$

then

$$\langle L_3 z_2, z_2\rangle \geq \beta\Big(-\omega - \frac{v^2}{4}\Big)\|z_2\|_{L^2}^2 \tag{4.27}$$

and there exists a positive number $\delta_3 > 0$ *such that*

$$\langle L_3 z_2, z_2\rangle \geq \delta_3\beta\Big(-\omega - \frac{v^2}{4}\Big)\|z_2\|_{H^1}^2 \tag{4.28}$$

For any $\vec{\psi} \in X$, from (4.1), we can simply denote $\vec{\psi}$ by

$$\vec{\psi} = (y_1, y_2, z_2) \tag{4.29}$$

Choose

$$\begin{aligned} &y_1^- = \widehat{\varepsilon}^2, \quad y_2^- = 0, \quad z_2^- = 2n\widehat{\varepsilon} \\ &\vec{\psi}^- = (y_1^-, y_2^-, z_2^-) \end{aligned} \tag{4.30}$$

then

$$\langle H_{\omega,v}\vec{\psi}^-, \vec{\psi}^-\rangle = -\frac{5}{3}\beta\Big(-\omega - \frac{v^2}{4}\Big)\langle 2n\widehat{\varepsilon}, 2n\widehat{\varepsilon}\rangle - 6\Big(-\omega - \frac{v^2}{4}\Big)\langle \widehat{\varepsilon}^2, \widehat{\varepsilon}^2\rangle < 0 \tag{4.31}$$

Also notice that the kernel of $H_{\omega,v}$ is spanned by the following two vectors

$$\vec{\psi}_{0,1} = (\widehat{\varepsilon}_x, 0, n_x) \tag{4.32}$$

$$\vec{\psi}_{0,2} = (0, \widehat{\varepsilon}, 0) \tag{4.33}$$

$$Z = \{k_1\vec{\psi}_{0,1} + k_2\vec{\psi}_{0,2}/k_1, k_2 \in \mathrm{R}\} \tag{4.34}$$

Let

$$\begin{aligned} P = \{\vec{p} \in X/\vec{p} = (p_1, p_2, p_3), &\langle p_3, 2n\widehat{\varepsilon}\rangle + \langle p_1, \widehat{\varepsilon}^2\rangle = 0 \\ &\langle p_3, n_x\rangle + \langle p_1, \widehat{\varepsilon}_x\rangle = 0, \langle p_2, \widehat{\varepsilon}\rangle = 0\} \end{aligned} \tag{4.35}$$

$$N = \{k\vec{\psi}^-/k \in \mathrm{R}\} \tag{4.36}$$

Obviously (3.27) holds.

For any $\vec{u} \in X$, $\vec{u} = (y_1, y_2, z_2)$, choose $a_1 = (\langle z_2, 2n\hat{\varepsilon}\rangle + \langle y_1, \hat{\varepsilon}^2\rangle)/(\langle 2n\hat{\varepsilon}, 2n\hat{\varepsilon}\rangle + \langle \hat{\varepsilon}^2, \hat{\varepsilon}^2\rangle)$, $b_1 = (\langle n_x, z_2\rangle + \langle \hat{\varepsilon}_x, y_1\rangle)/(\langle n_x, n_x\rangle + \langle \hat{\varepsilon}_x, \hat{\varepsilon}_x\rangle)$, $b_2 = (\langle \hat{\varepsilon}, y_2\rangle)/(\langle \hat{\varepsilon}, \hat{\varepsilon}\rangle)$, then $\vec{u}$ can be uniquely represented by

$$\vec{u} = a_1\vec{\psi}^- + b_1\vec{\psi}_{0,1} + b_2\vec{\psi}_{0,2} + \vec{p} \tag{4.37}$$

with $\vec{p} \in P$, which implies (3.26).

For subspace P, it remains to prove (3.28).

Lemma 4 *For any $\vec{p} \in P$, defined by (4.35), under the condition of (2.12) and (3.30), there exists a constant $\delta > 0$ such that*

$$\langle H_{\omega,v}\vec{p}, \vec{p}\rangle \geq \delta\|\vec{p}\|_X \tag{4.38}$$

with δ independent of $\vec{p}$.

Proof For any $\vec{p} \in P$, by (4.35), define

$$\vec{\psi}_1 = \left(\frac{\hat{\varepsilon}^2}{\langle \hat{\varepsilon}^2, \hat{\varepsilon}^2\rangle}, 0, \frac{2n\hat{\varepsilon}}{\langle 2n\hat{\varepsilon}, 2n\hat{\varepsilon}\rangle}\right) \tag{4.39}$$

$$\vec{\psi}_2 = \left(\frac{\hat{\varepsilon}_x}{\langle \hat{\varepsilon}_x, \hat{\varepsilon}_x\rangle}, 0, \frac{n_x}{\langle n_x, n_x\rangle}\right) \tag{4.40}$$

Choose $a = -\langle p_3, 2n\hat{\varepsilon}\rangle$, $b = -\langle p_3, n_x\rangle$, then $\vec{p}$ can be uniquely represented by

$$\vec{p} = a\vec{\psi}_1 + b\vec{\psi}_2 + \vec{p'} \tag{4.41}$$

with $\vec{p'} = (\tilde{p}_1, p_2, \tilde{p}_3)$, $\tilde{p}_1, p_2$ and $\tilde{p}_3$ satisfying Lemmas 1, 2, 3. Denote

$$\phi_1 = \frac{\hat{\varepsilon}^4}{\langle \hat{\varepsilon}^3, \hat{\varepsilon}^3\rangle} - \frac{\hat{\varepsilon}^2}{\langle \hat{\varepsilon}^2, \hat{\varepsilon}^2\rangle} \tag{4.42}$$

$$\phi_2 = \frac{\hat{\varepsilon}^2\hat{\varepsilon}_x}{\langle \hat{\varepsilon}\hat{\varepsilon}_x, \hat{\varepsilon}\hat{\varepsilon}_x\rangle} - \frac{\hat{\varepsilon}_x}{\langle \hat{\varepsilon}_x, \hat{\varepsilon}_x\rangle} \tag{4.43}$$

such that

$$\langle \phi_1, \phi_1\rangle = \frac{\langle \hat{\varepsilon}^4, \hat{\varepsilon}^4\rangle}{\langle \hat{\varepsilon}^3, \hat{\varepsilon}^3\rangle^2} - 2\frac{1}{\langle \hat{\varepsilon}^2, \hat{\varepsilon}^2\rangle} + \frac{1}{\langle \hat{\varepsilon}^2, \hat{\varepsilon}^2\rangle} = \frac{1}{14}\frac{1}{\langle \hat{\varepsilon}^2, \hat{\varepsilon}^2\rangle} \tag{4.44}$$

$$\langle \phi_1, \hat{\varepsilon}^2\rangle = \langle \phi_1, \hat{\varepsilon}_x\rangle = \langle \phi_1, \phi_2\rangle = 0 \tag{4.45}$$

$$\langle \phi_2, \phi_2\rangle = \frac{\langle \hat{\varepsilon}^2\hat{\varepsilon}_x, \hat{\varepsilon}^2\hat{\varepsilon}_x\rangle}{\langle \hat{\varepsilon}\hat{\varepsilon}_x, \hat{\varepsilon}\hat{\varepsilon}_x\rangle^2} - 2\frac{1}{\langle \hat{\varepsilon}_x, \hat{\varepsilon}_x\rangle} + \frac{1}{\langle \hat{\varepsilon}_x, \hat{\varepsilon}_x\rangle} = \frac{3}{7}\frac{1}{\langle \hat{\varepsilon}_x, \hat{\varepsilon}_x\rangle} \tag{4.46}$$

$$\langle \phi_2, \hat{\varepsilon}^2\rangle = \langle \phi_2, \hat{\varepsilon}_x\rangle = \langle \phi_1, \phi_2\rangle = 0 \tag{4.47}$$

Thus, by (4.2), (4.39)–(4.47) and Lemmas 1–3, we have

$$\langle H_{\omega,v}\vec{p}, \vec{p}\rangle = \langle L_1\tilde{p}_1, \tilde{p}_1\rangle + \langle L_2p_2, p_2\rangle + \langle L_3\tilde{p}_3, \tilde{p}_3\rangle$$

$$+\int_{\mathbf{R}}\left(\frac{2\widetilde{\varepsilon}\widetilde{p}_1}{\sqrt{1-v+\frac{4\beta}{3}(\omega+\frac{v^2}{4})}}+\sqrt{1-v+\frac{4\beta}{3}\Big(\omega+\frac{v^2}{4}\Big)}\widetilde{p}_3\right)^2dx$$

$$+\Big\langle 2a\Big[1-v+\frac{4\beta}{3}\Big(\omega+\frac{v^2}{4}\Big)+\frac{32}{5}\Big(-\omega-\frac{v^2}{4}\Big)\Big]\phi_1,\widetilde{p}_1\Big\rangle$$

$$+\Big\langle 2b\Big[1-v+\frac{4\beta}{3}\Big(\omega+\frac{v^2}{4}\Big)+\frac{16}{5}\Big(-\omega-\frac{v^2}{4}\Big)\Big]\phi_2,\widetilde{p}_1\Big\rangle$$

$$+\Big[1-v+\frac{4\beta}{3}\Big(\omega+\frac{v^2}{4}\Big)+\frac{32}{5}\Big(-\omega-\frac{v^2}{4}\Big)\Big]a^2\frac{1}{\langle 2n\widehat{\varepsilon},2n\widehat{\varepsilon}\rangle}$$

$$+\Big[1-v+\frac{4\beta}{3}\Big(\omega+\frac{v^2}{4}\Big)+\frac{2}{5}\Big(-\omega-\frac{v^2}{4}\Big)\Big]a^2\frac{1}{\langle\widehat{\varepsilon}^2,\widehat{\varepsilon}^2\rangle}$$

$$+\Big[1-v+\frac{4\beta}{3}\Big(\omega+\frac{v^2}{4}\Big)+\frac{16}{5}\Big(-\omega-\frac{v^2}{4}\Big)\Big]b^2\Big(\frac{1}{\langle n_x,n_x\rangle}+\frac{1}{\langle\widehat{\varepsilon}_x,\widehat{\varepsilon}_x\rangle}\Big) \tag{4.48}$$

Denote $y=1-v+\frac{4\beta}{3}\Big(\omega+\frac{v^2}{4}\Big)>0$, $\sigma=-\omega-\frac{v^2}{4}>0$ then

$$\begin{aligned}\langle H_{\omega,v}\vec{p},\vec{p}\rangle &= \langle L_1\widetilde{p}_1,\widetilde{p}_1\rangle+\langle L_2p_2,p_2\rangle+\langle L_3\widetilde{p}_3,\widetilde{p}_3\rangle\\
&\quad+\int_{\mathbf{R}}\Big(\frac{2\widetilde{\varepsilon}\widetilde{p}_1}{\sqrt{y}}+\sqrt{y}\widetilde{p}_3\Big)^2dx-2\sigma\langle\widetilde{p}_1,\widetilde{p}_1\rangle\\
&\quad+\frac{1}{2\sigma}\int_{\mathbf{R}}\Big(2\sigma\widetilde{p}_1+a\Big(y+\frac{32}{5}\sigma\Big)\phi_1+b\Big(y+\frac{16}{5}\sigma\Big)\phi_2\Big)^2dx\\
&\quad+\frac{a^2}{\sigma}\Big[\frac{5}{32}y^2+\Big(2y-\frac{25}{96}\beta y\Big)\sigma+\frac{2}{5}\sigma^2-\frac{1}{28}\Big(y+\frac{32}{5}\sigma\Big)^2\Big]\frac{1}{\langle\widehat{\varepsilon}^2,\widehat{\varepsilon}^2\rangle}\\
&\quad+\frac{b^2}{\sigma}\Big[\frac{5}{16}y^2+2\sigma y+\frac{16}{5}\sigma^2-\frac{3}{14}\Big(y+\frac{16}{5}\sigma\Big)^2\Big]\frac{1}{\langle\widehat{\varepsilon}_x,\widehat{\varepsilon}_x\rangle}\end{aligned} \tag{4.49}$$

Denote $\delta_4=\frac{5}{32}y^2+\Big(2y-\frac{25}{96}\beta y\Big)\sigma+\frac{2}{5}\sigma^2-\frac{1}{28}\Big(y+\frac{32}{5}\sigma\Big)^2$, $\delta_5=\frac{5}{16}y^2+2\sigma y+\frac{16}{5}\sigma^2-\frac{3}{14}\Big(y+\frac{16}{5}\sigma\Big)^2$. (3.30) assures $\delta_4>0$, $\delta_5>0$.

(1) If
$$2\sigma\|\widetilde{p}_1\|_{L^2}^2\geq\frac{1}{\sigma}\Big\|a\Big(y+\frac{32}{5}\sigma\Big)\phi_1+b\Big(y+\frac{16}{5}\sigma\Big)\phi_2\Big\|_{L^2}^2 \tag{4.50}$$

then

$$\frac{1}{2\sigma}\int_{\mathbf{R}}\Big(2\sigma\widetilde{p}_1+a\Big(y+\frac{32}{5}\sigma\Big)\phi_1+b\Big(y+\frac{16}{5}\sigma\Big)\phi_2\Big)^2dx\geq\sigma\|\widetilde{p}_1\|_{L^2}^2 \tag{4.51}$$

(2) If
$$2\sigma\|\widetilde{p}_1\|_{L^2}^2\leq\frac{1}{\sigma}\Big\|a\Big(y+\frac{32}{5}\sigma\Big)\phi_1+b\Big(y+\frac{16}{5}\sigma\Big)\phi_2\Big\|_{L^2}^2 \tag{4.52}$$

then

$$\frac{\delta_4 a^2}{\sigma}\frac{1}{\langle\widehat{\varepsilon}^2,\widehat{\varepsilon}^2\rangle}+\frac{\delta_5 b^2}{\sigma}\frac{1}{\langle\widehat{\varepsilon}_x,\widehat{\varepsilon}_x\rangle}$$

$$\geq\frac{\delta_4 a^2}{2\sigma}\frac{1}{\langle\widehat{\varepsilon}^2,\widehat{\varepsilon}^2\rangle}+\frac{\delta_5 b^2}{2\sigma}\frac{1}{\langle\widehat{\varepsilon}_x,\widehat{\varepsilon}_x\rangle}+\delta_6\sigma\|\widetilde{p}_1\|^2_{L^2} \tag{4.53}$$

where $\delta_6=\dfrac{14}{(y+\frac{32}{5}\sigma)^2+6(y+\frac{16}{5}\sigma)^2}\min\{\delta_4,\delta_5\}>0$, denote $\delta_7=\min\{1,\delta_6\}>0$. Thus, for any $\vec{p}\in P$, it follows from (4.49)–(4.53), Lemmas 1–3, that

$$\begin{aligned}\langle H_{\omega,v}\vec{p},\vec{p}\rangle \geq\ &\langle L_1\widetilde{p}_1,\widetilde{p}_1\rangle+\langle L_2p_2,p_2\rangle+\langle L_3\widetilde{p}_3,\widetilde{p}_3\rangle\\ &+\int_{\mathbf{R}}\left(\frac{2\widetilde{\varepsilon}\widetilde{p}_2}{\sqrt{y}}+\sqrt{y}\widetilde{p}_1\right)^2dx-2\sigma\langle\widetilde{p}_1,\widetilde{p}_1\rangle\\ &+\frac{\delta_4 a^2}{2\sigma}\frac{1}{\langle\widehat{\varepsilon}^2,\widehat{\varepsilon}^2\rangle}+\frac{\delta_5 b^2}{2\sigma}\frac{1}{\langle\widehat{\varepsilon}_x,\widehat{\varepsilon}_x\rangle}+\delta_7\sigma\langle\widetilde{p}_2,\widetilde{p}_2\rangle\\ \geq\ &\frac{\delta_1\delta_7}{2}\sigma\|\widetilde{p}_1\|^2_{H^1}+\delta_2\sigma\|p_2\|^2_{H^1}+\delta_3\beta\sigma\|\widetilde{p}_3\|^2_{H^1}\\ &+\frac{\delta_4 a^2}{2\sigma}\frac{1}{\langle\widehat{\varepsilon}^2,\widehat{\varepsilon}^2\rangle}+\frac{\delta_5 b^2}{2\sigma}\frac{1}{\langle\widehat{\varepsilon}_x,\widehat{\varepsilon}_x\rangle}\end{aligned} \tag{4.54}$$

Finally, with (4.54) we have

$$\langle H_{\omega,v}\vec{p},\vec{p}\rangle\geq\delta\|\vec{p}\|^2_X \tag{4.55}$$

where $\delta>0$ is independent of $\vec{p}$.

Thus under the condition of (2.12) and (3.30), Assumption 1 holds, and $n(H_{\omega,v})=1$.

In the following, we shall verify that $p(d'')=1$.

Note that (3.20) and (3.29) imply

$$d_\omega(\omega,v)=-Q_2(\Phi_{\omega,v})$$

$$d_v(\omega,v)=-Q_1(\Phi_{\omega,v})$$

$$\begin{aligned}-Q_2(\Phi_{\omega,v})&=-\int_{\mathbf{R}}(\varepsilon\overline{\varepsilon})dx\\ &=\frac{-(4\omega+v^2)(1-v+\frac{4\beta}{3}(\omega+\frac{v^2}{4}))}{2}\int_{\mathbf{R}}\operatorname{sech}^2\left(\frac{\sqrt{-4\omega-v^2}}{2}x\right)dx\\ &=-4\sqrt{-\omega-\frac{v^2}{4}}\left(1-v+\frac{4\beta}{3}\left(\omega+\frac{v^2}{4}\right)\right)\end{aligned}$$

$$
\begin{aligned}
-Q_1(\Phi_{\omega,v}) &= -\frac{1}{2}\int_{\mathbf{R}}(n^2)dx - \operatorname{Im}\int_{\mathbf{R}}(\varepsilon_x\bar{\varepsilon})dx\\
&= -2\Big(-\omega-\frac{v^2}{4}\Big)^2\int_{\mathbf{R}}\operatorname{sech}^4\Big(\frac{\sqrt{-4\omega-v^2}}{2}x\Big)dx - v\Big(-\omega-\frac{v^2}{4}\Big)\\
&\quad\cdot\Big(1-v+\frac{4\beta}{3}\Big(\omega+\frac{v^2}{4}\Big)\Big)\int_{\mathbf{R}}\operatorname{sech}^2\Big(\frac{\sqrt{-4\omega-v^2}}{2}x\Big)dx\\
&= \frac{1}{\sqrt{-\omega-\frac{v^2}{4}}}\Big[-\frac{8}{3}\Big(-\omega-\frac{v^2}{4}\Big)^2 - 2v\Big(-\omega-\frac{v^2}{4}\Big)\\
&\quad\cdot\Big(1-v+\frac{4\beta}{3}\Big(\omega+\frac{v^2}{4}\Big)\Big)\Big]
\end{aligned}
$$

$$
d_{\omega\omega}(\omega,v) = \frac{2}{\sqrt{-\omega-\frac{v^2}{4}}}\Big[1-v-4\beta\Big(-\omega-\frac{v^2}{4}\Big)\Big]
$$

$$
\begin{aligned}
d_{v\omega}(\omega,v) &= d_{\omega v}(\omega,v)\\
&= \frac{v}{\sqrt{-\omega-\frac{v^2}{4}}}\Big[1-v+\frac{4\beta}{3}\Big(\omega+\frac{v^2}{4}\Big)\Big] - 4\Big(\frac{2}{3}\beta v-1\Big)\sqrt{-\omega-\frac{v^2}{4}}
\end{aligned}
$$

$$
\begin{aligned}
d_{vv}(\omega,v) &= 2v\sqrt{-\omega-\frac{v^2}{4}}\Big(2-\frac{2}{3}\beta v\Big)\\
&\quad+\frac{v^2-4(-\omega-\frac{v^2}{4})}{2\sqrt{-\omega-\frac{v^2}{4}}}\Big[1-v+\frac{4\beta}{3}\Big(\omega+\frac{v^2}{4}\Big)\Big]
\end{aligned}
$$

Denote $y = 1-v+\frac{4\beta}{3}\Big(\omega+\frac{v^2}{4}\Big) > 0$, $\sigma = -\omega-\frac{v^2}{4} > 0$, then

$$
\begin{aligned}
\det(d'') &= d_{\omega\omega}d_{vv} - d_{\omega v}d_{v\omega}\\
&= \frac{1}{\sigma}\Big[y\Big(y-\frac{8}{3}\beta\sigma\Big)(v^2-4\sigma) + 4v\sigma\Big(y-\frac{8}{3}\beta\sigma\Big)\Big(2-\frac{2}{3}\beta v\Big)\Big]\\
&\quad-\frac{1}{\sigma}\Big[vy+4\sigma\Big(1-\frac{2}{3}\beta v\Big)\Big]^2\\
&= -4\Big(y^2-\frac{8}{3}\beta\sigma y+4\sigma\Big)\\
&= -4\Big[\Big(y-\frac{4}{3}\beta\sigma\Big)^2+4\sigma\Big(1-\frac{4}{9}\beta^2\sigma\Big)\Big]
\end{aligned}
$$

(3.30) assures $1-\frac{4}{9}\beta^2\sigma > 0$, then

$$
\det(d'') < 0.
$$

Thus d'' has exactly one positive and one negative eigenvalues, whence, $p(d'') = 1$. This completes the proof of Theorem 2.

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