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$C^{1,\alpha}$ REGULARITY OF VISCOSITY SOLUTIONS OF FULLY NONLINEAR PARABOLIC PDE UNDER NATURAL STRUCTURE CONDITIONS

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Abstract In this paper, we concern the fully nonlinear parabolic equations $u_t + F(x,t,u,Du,D^2u) = 0$. Under the natural structure conditions as that in [1], we obtain the $C^{1,\alpha}$ estimates of the viscosity solutions.

Key Words Fully nonlinear parabolic equations; $C^{1,\alpha}$ regularily; viscosity solutions.

Classification 35K55, 35K60.

In [1], Chen Yazhe established the interior $C^{1,\alpha}$ regularity of viscosity solutions to fully nonlinear elliptic equations under natural structure conditions. We shall extend this result to parabolic equations.

Consider the Dirichlet problem of fully nonlinear parabolic equations

$$u_t + F(x,t,u,Du,D^2u) = 0 \qquad \text{in } Q_T = \Omega \times (0,T] \tag{1}$$

$$u = \varphi(x,t) \qquad \text{on } \partial^* Q_T = (\partial\Omega \times (0,T]) \cup (\Omega \times \{0\}) \tag{2}$$

$F(x,t,r,p,X)$ is a continuous function on $\Omega \times (0,T] \times \boldsymbol{R} \times \boldsymbol{R}^N \times \boldsymbol{M}^N$, where $\boldsymbol{M}^N$ denotes the space of $N \times N$ symmetric matrices equipped with usual order, and satisfies

$$\lambda \operatorname{Tr}(Y) \le F(x,t,r,p,X) - F(x,t,r,p,X+Y) \le \Lambda \operatorname{Tr}(Y) \tag{3}$$
$$\forall Y \in \mathrm{M}^N,\ Y \ge 0$$

$$F(x,t,r,p,X) - F(x,t,s,p,X) \ge 0$$
$$\forall r \ge s. \tag{4}$$

Let's give the definition of viscosity solutions.

Definition *Let u be an upper (resp. lower) semi-continuous function in Q_T. u is said to be a viscosity subsolution (resp. supersolution) of (1) if for all $\varphi(x,t) \in C^{2,1}(Q_T)$ at each local maximum (resp. minimum) point $(x_0,t_0) \in Q_T$ of $u - \varphi$, we have*

$$\varphi_t(x_0,t_0) + F(x_0,t_0,u(x_0,t_0),D\varphi(x_0,t_0),D^2\varphi(x_0,t_0)) \le 0$$

$$(resp.\ \varphi_t(x_0,t_0)+F(x_0,t_0,u(x_0,t_0),D\varphi(x_0,t_0),D^2\varphi(x_0,t_0))\geq 0)$$

$u\in C(Q_T)$ is said to be a viscosity solution of (1) *if it is both a viscosity subsolution and a viscosity supersolution of* (1).

Under natural structure conditions and some property of the boundary, the author in [1] used Perron's method and obtained the existence of viscosity solution for the Dirichlet problem of elliptic equations based on the comparison principle. He also proved the Lipschitz continuity of the viscosity solution. We can obtain these for parabolic equations. The details are described in [2]. In this paper, we shall derive the $C^{1,\alpha}$ estimates for the viscosity solution u of (1)-(2) with $u\in C^1$ under the following additional conditions

$$|F(x,t,r,p,X)-F(y,t,r,q,X)| \leq \mu_1(|r|,|p|)\{1+[\mu(|x-y|)+|p-q|^{1/2}\mu(|p-q|)]\|X\|\} \quad (5)$$

$$|F(x,t,r,p,0)|\leq \mu_2(|r|,|p|) \quad (6)$$

where $\mu(\cdot)$ is nondecreasing, $\mu(\sigma)\to 0$ as $\sigma\to 0$, $\mu(\sigma)/\sigma\geq 1$ in $(0,+\infty)$ and $\int_{0+}\mu(\sigma)/\sigma d\sigma<\infty$, $\mu_i(s,t)$ is nondecreasing with respect to s and $t (i=1,2)$.

Theorem 1 *Assume that F satisfies* (3)-(6). *Let u be a Lipschitz continuous viscosity solution of* (1). *Then Du is Hölder continuous.*

We require the parabolic analogue of Proposition 3.1 in [1]. It takes the following form ([2]):

Lemma 2 *Let u and v be, respectively, a viscosity subsolution and a viscosity supersolution of* (1). *$\Psi(x,y,t)\in C^{2,1}(\Omega\times\Omega\times(0,T])$. If $u(x,t)-v(y,t)-\Psi(x,y,t)$ attains its maximum in $\Omega\times\Omega\times(0,T]$, then there exists $(\bar{x},\bar{y},\bar{t})\in\Omega\times\Omega\times(0,T]$ and $X,Y\in \mathbf{M}^N$ such that*

$$u(\bar{x},\bar{t})-v(\bar{y},\bar{t})-\Psi(\bar{x},\bar{y},\bar{t})=\max\{u(x,t)-v(y,t)-\Psi(x,y,t)\} \quad (7)$$

$$\Psi_\tau(\bar{x},\bar{y},\bar{t})+F(\bar{x},\bar{t},u(\bar{x},\bar{t}),D_x\Psi(\bar{x},\bar{y},\bar{t}),X)-F(\bar{y},\bar{t},v(\bar{y},\bar{t}),-D_y\Psi(\bar{x},\bar{y},\bar{t}),-Y)\leq 0 \quad (8)$$

and

$$\begin{pmatrix} X & 0\\ 0 & Y\end{pmatrix}\leq D^2\Psi(\bar{x},\bar{y},\bar{t}) \quad (9)$$

Before proving Theorem 1, we need some results in [3].

Let's introduce the Pucci's extremal operators:

$$\mu^-(D^2u)=\lambda\sum_{e_i>0}e_i+\Lambda\sum_{e_i<0}e_i$$

$$\mu^+(D^2u)=\Lambda\sum_{e_i>0}e_i+\lambda\sum_{e_i<0}e_i$$

where e_i are the eigenvalues of Hessian matrix D^2u.

For any continuous function f, we denote by $\underline{S}(f)$ (resp. $\overline{S}(f)$) the class of viscosity subsolutions (resp. super-solutions) of $u_t - \mu^+(D^2u) = f$ (resp. $u_t - \mu^-(D^2u) = f$).

Let

$$S^*(f) = \bar{S}(-|f|) \cap \underline{S}(|f|)$$

We call $u(x,t)$ has tangent paraboloid from below of aperture h at (x_0, t_0), if for $t \le t_0$,

$$u(x,t) \ge u(x_0,t_0) + \langle P, x - x_0\rangle + C\Big(-\frac{1}{2}|x-x_0|^2 + t_0 - t\Big)$$

where $|u(x_0,t_0)| + |p| + C \le h$, $C > 0$.

Denote

$$D_h(u) = \{(x,t) : u \text{ has tangent paraboloid from below of aperture } h \text{ at } (x,t)\}$$
$$\hat{D}_h(u) = D_h(-u)$$

Lemma 3(Theorem 4.11 in [3]) *Let $u \in S^*(1)$ and $|u| \le 1$ in $Q^* = B_{9\sqrt{N}}(0) \times (0,10]$, $K = (-1,1)^N \times (0,1]$, then*

$$|D_h(u) \cap K| \ge 1 - h^{-2\nu}$$
$$|\hat{D}_h(u) \cap K| \ge 1 - h^{-2\nu} \quad \text{for } h > 1$$

where ν depends only on λ, Λ, N.

Corollary 4 *Let $u \in S^*(A)$ in $B_{9\sqrt{N}R}(x_0) \times (t_0 - R^2, t_0 + 9R^2]$, where A is a positive constant. Then for $h > 1$ there exists a point (x_1, t_1) with $|x_0 - x_1| \le h^{-\gamma/n+2}R$, $0 < t_0 - t_1 \le h^{-2\gamma/n+2}R^2$ such that for $t \le t_1$ and for some $\hat{a}$ with $|\hat{a}| \le \mathrm{Lip}\ u$*

$$|u(x,t) - u(x_1,t_1) - \langle \hat{a}, x - x_1\rangle| \le \frac{h}{R^2}\Big(\underset{Q_R(x_0,t_0)}{\mathrm{OSC}}\ u + AR^2 + R^2\Big)(|x-x_1|^2 + t_1 - t) \quad (10)$$

Proof Let

$$v(y,\tau) = \frac{u(x_0 + Ry, t_0 - R^2 + R^2\tau) - u(x_0,t_0)}{\underset{Q_R(x_0,t_0)}{\mathrm{OSC}}\ u + (1+A)R^2}$$

Then $v \in S^*(1)$ and $|v| \le 1$ in Q^*. By Lemma 3, if $|B_r(0) \times (1-r^2, 1]| > 2h^{-\gamma}$, then there exists a point $(y_1, \tau_1) \in B_r(0) \times (1 - r^2, 1] \cap D_h(u) \cap \hat{D}_h(u)$. Scaling back, we can easily conclude the corollary.

In the following, we assume that u is a viscosity solution of (1) and F satisfies (3)-(6). Moreover, $|u| \le M$. For $Q_R(x_0,t_0) = B_R(x_0) \times (t_0 - R^2, t_0] \subset Q_T$, we denote by K the Lipschitz constant in $Q_R(x_0,t_0)$, and we always assume that $R < 1$, $R^{1/2} \le K \le M_1$, where M_1 is a positive constant.

Lemma 5 $u \in S^*(A)$, *where $A = \mu_2(M, M_1)$.*

Proof For any $\varphi \in C^{2,1}(Q_T)$, suppose $u - \varphi$ has local minimum at (x_0, t_0) and $\varphi_t(x_0, t_0) - \mu^-(D^2\varphi(x_0, t_0)) < -A$. Then by local minimum property

$$u(x, t_0) - u(x_0, t_0) \geq \langle D\varphi(x_0, t_0), x - x_0\rangle + o(|x - x_0|)$$

and the above inequality implies

$$|D\varphi(x_0, t_0)| \leq M_1$$

Hence, at (x_0, t_0) we have

$$\begin{aligned}\varphi_t + F(x, t, u, D\varphi, D^2\varphi) \leq& \varphi_t + F(x, t, u, D\varphi, 0) - \lambda \text{ Tr } (D^2\varphi)^+ \\ =& \varphi_t - \mu^-(D^2\varphi) + F(x, t, u, D\varphi, 0) < 0\end{aligned}$$

This contradicts u being a viscosity solution of (1). Therefore

$$u \in \bar{S}(-A)$$

Similarly, we can prove $u \in \underline{S}(A)$.

By Corollary 4 and Lemma 5, there exists a point (x_1, t_1) with $|x_0 - x_1| \leq h^{-\gamma/n+2}R^1$, $0 < t_0 - t_1 \leq h^{-2\gamma/n+2}R^2$ such that (10) holds.

Let

$$\begin{aligned}\tilde{B}_r(x_1) &= \{(x, y) : |x - x_1|^2 + |y - x_1|^2 < r^2\} \\ \tilde{Q}_r(x_1, t_1) &= \tilde{B}_r(x_1) \times (t_1 - r^2, t_1]\end{aligned}$$

Lemma 6 *There exists a constant θ and a unit vector $a \in \mathbf{R}^N$ such that for any $0 < \delta < 1$ and $l \in \mathbf{R}^N$ with $|l| = K$, $\langle a, l\rangle \geq -\frac{1}{2}K$, we have*

$$|u(x, t) - u(y, t) - \langle l, x - y\rangle| \leq \Big(2 - \frac{1}{4}\Big)K|x - y|$$
$$\text{on } \{(x, y, t) \in \tilde{Q}_{\delta\theta R}(x_1, t_1) : |x - y| = \delta^2\theta R\}$$

Proof For any $(x, y, t) \in \tilde{Q}_{\delta\theta R}(x_1, t_1)$ and $|x - y| = \delta^2\theta R$, if $R^{1/2} \leq \dfrac{A+1}{h}$ by Corollary 4,

$$\begin{aligned}|u(x, t) - u(y, t) - \langle \hat{a}, x - y\rangle| \leq& \frac{C}{R^2}\Big(\underset{Q_R(x_0,t_0)}{\text{OSC}} u + AR^2 + R^2\Big)(|x - x_1|^2 \\ &+ |y - x_1|^2 + 2(t_1 - t)) \\ \leq& \frac{2CK}{R}(|x - x_1|^2 + |y - x_1|^2 + 2(t_1 - t)) \\ \leq& 8CK\delta^2\theta^2 R = 4\theta CK|x - y|\end{aligned}$$

Let $a = \hat{a}/|\hat{a}|$, then

$$|\langle \hat{a} - l, x - y\rangle| \le |\hat{a} - l||x - y| = (|\hat{a}|^2 + |l|^2 - 2|\hat{a}|\langle a, l\rangle)^{1/2}|x - y| \le \sqrt{3}K|x - y|$$

By taking $\theta = \left(2 - \sqrt{3} - \frac{1}{4}\right)/4c$, the lemma is proved.

Lemma 7 *Let θ be the constant and l the vector in Lemma 6. Then there exist constants θ_1, δ such that*

$$|u(x,t) - u(y,t) - \langle l, x - y\rangle| \le \left(2 - \frac{1}{16}\right)K|x - y| \ in \ \tilde{Q}_{\theta_1\theta\delta R}(x_1, t_1)$$

Proof Without loss of generality, we can assume $(x_1, t_1) = (0, 0)$. In the domain $S = \{(x, y, t) \in \tilde{Q}_{\tilde{R}}(0, 0) : |x - y| < \delta\tilde{R}\}$, we consider the function

$$\Psi(x, y, t) = \frac{K}{4\tilde{R}^2}|x - y|(|x|^2 + |y|^2 - t) + \psi(|x - y|) + \langle l, x - y\rangle$$

$$\psi(|x - y|) = \left(2 - \frac{1}{8}\right)K|x - y| - \frac{\delta\tilde{R}K}{\delta l^1(1)} l\left(\frac{|x - y|}{\delta\tilde{R}}\right)$$

where $l(\sigma) = \int_0^\sigma ds \int_0^s \mu(\eta)/\eta d\eta$, $\tilde{R} = \theta\delta R$. With the help of Lemma 6, it is easy to see

$$u(x, t) - u(y, t) \le \bar{\Psi} \text{ on } \partial^* S, \text{ the parabolic boundary of } S.$$

We want to prove

$$u(x, t) - u(y, t) \le \Psi \quad \text{in} \quad S \tag{11}$$

If not, there exists $(\bar{x}, \bar{y}, \bar{t}) \in S$, with $\bar{x} \ne \bar{y}$, and $X, Y \in \mathbf{M}^N$ such that (7)–(9) hold. (9) implies

$$\begin{pmatrix} X - \frac{K}{2\tilde{R}^2}|\bar{x} - \bar{y}|I & O \\ O & Y - \frac{K}{2\tilde{R}^2}|\bar{x} - \bar{y}|I \end{pmatrix} + Q \le \begin{pmatrix} B & -B \\ -B & B \end{pmatrix} \tag{12}$$

where

$$Q = \frac{K}{2\tilde{R}^2}(|x - y|_z \otimes z + z \otimes |x - y|_z)|_{(x,y)=(\bar{x},\bar{y})}, \ z = (x, y),$$

$$B = \left\{D^2\psi(x - y) + \frac{K}{4\tilde{R}^2}(|x|^2 + |y|^2 - t)|x - y|_{xx}\right\}\Big|_{(x,y,t)=(\bar{x},\bar{y},\bar{t})}$$

A simple calculation shows

$$\|Q\| \le CK/\tilde{R}, \quad \|B\| \le CK/|\bar{x} - \bar{y}|$$

Multiplying (12) by $\begin{pmatrix} I & I \\ I & -I \end{pmatrix}$ from right and left, we get

$$\begin{pmatrix} X + Y - \frac{K}{\tilde{R}^2}|\bar{x} - \bar{y}|I & X - Y \\ X - Y & X + Y - \frac{K}{\tilde{R}^2}|\bar{x} - \bar{y}|I \end{pmatrix} + \bar{Q} \le \begin{pmatrix} 0 & 0 \\ 0 & 4B \end{pmatrix}$$

which implies

$$X+Y-\frac{K}{\tilde{R}^2}|\bar{x}-\bar{y}|I+Q_1\le 0,\quad X+Y-\frac{K}{\tilde{R}^2}|\bar{x}-\bar{y}|I+Q_2\le 4B$$

where $\|Q_1\|$, $\|Q_2\|\le CK/\tilde{R}$. Let

$$H=\frac{K}{\tilde{R}^2}|\bar{x}-\bar{y}|I-Q_1,\quad P=\frac{1}{|\bar{x}-\bar{y}|^2}(\bar{x}-\bar{y})\otimes(\bar{x}-\bar{y}).$$

Then

$$\begin{aligned}\text{Tr }(X+Y-H)\le&\text{Tr }P(X+Y-H)\le\text{Tr }P(4B-Q_2+Q_1)\\ &\le-\frac{K}{2l'(1)|\bar{x}-\bar{y}|}\mu\Big(\frac{|\bar{x}-\bar{y}|}{\delta\hat{R}}\Big)+CK/\tilde{R}\end{aligned}$$

Hence

$$|\text{Tr }(X+Y-H)|\ge\frac{K}{2l'(1)|\bar{x}-\bar{y}|}\mu\Big(\frac{|\bar{x}-\bar{y}|}{\delta\hat{R}}\Big)-CK/\tilde{R}\tag{13}$$

From (3), (4), (5), (8), we get

$$\begin{aligned}&\lambda|\text{Tr }(X+Y-H)|-\Lambda N\|H\|+\frac{\partial\Psi}{\partial t}(\bar{x},\bar{y},\bar{t})\\ &\le C\{1+[\mu(|\bar{x}-\bar{y}|)+|D_x\Psi+D_y\Psi|^{1/2}\mu(|D_x\Psi+D_y\Psi|)](\|X\|+\|Y\|)\}\end{aligned}\tag{14}$$

It is easy to calculate that

$$|D_x\Psi(\bar{x},\bar{y},\bar{t})+D_y\Psi(\bar{x},\bar{y},\bar{t})|=\frac{K}{4\tilde{R}}|\bar{x}-\bar{y}|\cdot|2\bar{x}+2\bar{y}|\le\frac{K}{\tilde{R}}|\bar{x}-\bar{y}|\tag{15}$$

As in [4], we have

$$\|X\|+\|Y\|\le C(N)\{1+\|B\|^{1/2}|\text{Tr }(X+Y-H)|^{1/2}+|\text{Tr }(X+Y-H)|\}\tag{16}$$

So from (14)–(16)

$$\begin{aligned}|\text{Tr }(X+Y-H)|\le&C+CK/\tilde{R}+C\mu\Big(\frac{M_1}{\tilde{R}}|\bar{x}-\bar{y}|\Big)(\|B\|^{1/2}|\text{Tr }(X+Y-H)|^{1/2}\\ &+|\text{Tr }(X+Y-H)|)\end{aligned}$$

Using Cauchy inequality and picking δ small, we can get a contradiction with (13). Thus we have proved (11). The lemma follows at once if we take θ_1 small.

In the following, we denote $\nu=\frac{1}{2}\theta_1\theta\delta$.

Lemma 8 *Let l be the vector in Lemma 6. Then there exists a constant η_1 such that if R is small enough, we have*

$$|u(x,t)-u(y,t)-\langle l,x-y\rangle\le(2-\eta_1)K|x-y|\quad in\quad\tilde{B}_{R/4}(x_0)\times(t_1-(\nu R)^2,t_1]$$

Proof Let $(x_1, t_1) = 0$. Consider the function

$$\Psi = \langle l, x-y\rangle + 2K|x-y| + \frac{1}{2}K\eta\Big(\frac{(2\nu)^m}{2} - W^{-\frac{m}{2}}\Big) \cdot \Big(|x-y| + \frac{R}{l'(1)}l\Big(\frac{|x-y|}{R}\Big)\Big) + \frac{1}{2}K\eta(2\nu)^m|x-y|$$

where $W = \frac{1}{(\nu R)^2}\Big(|x|^2 + |y|^2 - \frac{1}{16\nu^2}t\Big)$, and the domain

$$S = \Big\{(x,y,t) : |x|^2 + |y|^2 - \frac{1}{16\nu^2}t < \Big(\frac{R}{2}\Big)^2,\ (\nu R)^2 < |x|^2 + |y|^2 < \Big(\frac{R}{2}\Big)^2\Big\}$$

Obviously

$$u(x,t) - u(y,t) \le \Psi \quad \text{on} \quad \Big\{(x,y,t) : |x|^2 + |y|^2 - \frac{1}{16\nu^2}t = \Big(\frac{R}{2}\Big)^2\Big\}$$

For $(x,y,t) \in S \cap \Big\{\Big|\frac{x+y}{2}\Big| \le \nu R\Big\}$, if $x, y \in B_{\sqrt{2}\nu R}(0)$, by Lemma 7, we have

$$|u(x,t) - u(y,t) - \langle l, x-y\rangle| \le \Big(2 - \frac{1}{16}\Big)K|x-y|$$

if $x \notin B_{\sqrt{2}\nu R}(0)$ or $y \notin B_{\sqrt{2}\nu R}(0)$, let $\overline{xy} \cap B_{\nu R}(0) = x_2$ and $\overline{xy} \cap B_{\sqrt{2}\nu R}(0) = x_3$, then $|x_2 - x_3| \ge (\sqrt{2}-1)\nu R \ge (\sqrt{2}-1)\nu|x-y|$, hence

$$\begin{aligned} |u(x,t) - u(y,t) - \langle l, x-y\rangle| &\le 2K|x-x_2| + 2K|y-x_3| + \Big(2 - \frac{1}{16}\Big)K|x_2 - x_3| \\ &\le \Big(2 - \frac{\sqrt{2}-1}{16}\nu\Big)K|x-y| \end{aligned}$$

By taking $\eta = \frac{\sqrt{2}-1}{16}\nu$, on the parabolic boundary of S and in $S \cap \Big\{(x,y,t) : \Big|\frac{x+y}{2}\Big| \le \nu R\Big\}$ we have

$$u(x,t) - u(y,t) \le \Psi$$

We want to prove the above inequality holds in $\Delta = S \cap \Big\{(x,y,t) : \Big|\frac{x+y}{2}\Big| > \nu R\Big\}$. If not, there exists $(\bar{x}, \bar{y}, \bar{t}) \in \Delta$, with $\bar{x} \ne \bar{y}$, and $X, Y \in \mathrm{M}^N$ such that (7)–(9) hold. From (9) we get

$$\begin{pmatrix} X+Y & X-Y \\ X-Y & X+Y \end{pmatrix} \le \begin{pmatrix} Q_1 & Q \\ Q^T & Q_2 \end{pmatrix} \tag{17}$$

where

$$\begin{aligned} Q_2 &= D_{xx}\Psi + D_{xy}\Psi + D_{yx}\Psi + D_{yy}\Psi|_{(x,y)=(\bar{x},\bar{y})} \\ Q_2 &= D_{xx}\Psi - D_{xy}\Psi - D_{yx}\Psi + D_{yy}\Psi|_{(x,y)=(\bar{x},\bar{y})} \\ Q &= D_{xx}\Psi + D_{xy}\Psi - D_{yx}\Psi - D_{yy}\Psi|_{(x,y)=(\bar{x},\bar{y})} \end{aligned}$$

A calculation shows

$$Q_1 = K\eta\Big\{\frac{m}{(\nu R)^2}W^{-\frac{m}{2}-1}fI - \frac{m(m+2)}{2(\nu R)^4}W^{-\frac{m}{2}-2}f(\bar{x}+\bar{y})\otimes(\bar{x}+\bar{y})\Big\}$$

$$\begin{aligned} Q_2 =& K\eta\Big\{\frac{m}{(\nu R)^2}W^{-\frac{m}{2}-1}fI - \frac{m(m+2)}{2(\nu R)^4}W^{-\frac{m}{2}-2}f(\bar{x}-\bar{y})\otimes(\bar{x}-\bar{y}) \\ &+ \frac{2m}{(\nu R)^2}W^{-\frac{m}{2}-1}g|\bar{x}-\bar{y}|P + 2\Big(\frac{(2\nu)^m}{2} - W^{-\frac{m}{2}}\Big)g\frac{1}{|\bar{x}-\bar{y}|}(I-P) \\ &+ \frac{2}{l'(1)}\Big(\frac{(2\nu)^m}{2} - W^{-\frac{m}{2}}\Big)l''\Big(\frac{|\bar{x}-\bar{y}|}{R}\Big)\frac{1}{R}P + \frac{(2\nu)^m}{2}\cdot\frac{1}{|\bar{x}-\bar{y}|}(I-P)\Big\} \\ &+ \frac{8K}{|\bar{x}-\bar{y}|}(I-P) \end{aligned}$$

$$\begin{aligned} Q =& K\eta\Big\{\frac{2m}{(\nu R)^2}W^{-\frac{m}{2}-1}g\frac{1}{|\bar{x}-\bar{y}|}(\bar{x}+\bar{y})\otimes(\bar{x}-\bar{y}) \\ &- \frac{m(m+2)}{2(\nu R)^4}W^{-\frac{m}{2}-2}f(\bar{x}+\bar{y})\otimes(\bar{x}-\bar{y})\Big\} \end{aligned}$$

where $f = |\bar{x}-\bar{y}| + \frac{R}{l'(1)}l\Big(\frac{|\bar{x}-\bar{y}|}{R}\Big)$, $g = 1 + \frac{1}{l'(1)}l'\Big(\frac{|\bar{x}-\bar{y}|}{R}\Big)$, $P = \frac{1}{|\bar{x}-\bar{y}|^2}(\bar{x}-\bar{y})\otimes(\bar{x}-\bar{y})$. (17) implies

$$X + Y - Q_1 \le 0, \quad X + Y - Q_1 \le Q_2 - Q_1$$

So

$$\begin{aligned} \mathrm{Tr}\ (X+Y-Q_1) \le& \mathrm{Tr}\ \varepsilon P(Q_2 - Q_1) = \varepsilon K\eta\Big\{\frac{2}{Rl'(1)}\Big(\frac{(2\nu)^m}{2} - W^{-\frac{m}{2}}\Big)l''\Big(\frac{|\bar{x}-\bar{y}|}{R}\Big) \\ &+ \frac{2m}{(\nu R)^2}W^{-\frac{m}{2}-1}g|\bar{x}-\bar{y}| \\ &+ \frac{m(m+2)}{(\nu R)^4}W^{-\frac{m}{2}-2}f\Big\langle\bar{x}, \frac{\bar{x}-\bar{y}}{|\bar{x}-\bar{y}|}\Big\rangle\Big\langle\bar{y}, \frac{\bar{x}-\bar{y}}{|\bar{x}-\bar{y}|}\Big\rangle\Big\} \end{aligned}$$

Let

$$\tilde{Q}_1 = -K\eta\frac{m(m+2)}{2(\nu R)^4}W^{-\frac{m}{2}-2}f(\bar{x}+\bar{y})\otimes(\bar{x}+\bar{y})$$

Then

$$|\mathrm{Tr}\ \tilde{Q}_1| \ge K\eta\frac{m(m+2)}{(\nu R)^2}W^{-\frac{m}{2}-2}|\bar{x}-\bar{y}|$$

By taking ε sufficiently small, we get

$$|\mathrm{Tr}\ (X+Y-Q_1)| + |\mathrm{Tr}\ \tilde{Q}_1| \ge \frac{\varepsilon K\eta}{Rl'(1)}W^{-\frac{m}{2}}l''\Big(\frac{|\bar{x}-\bar{y}|}{R}\Big) + \frac{1}{2}K\eta\frac{m(m+2)}{(\nu R)^2}W^{-\frac{m}{2}-2}|\bar{x}-\bar{y}| \tag{18}$$

From (3), (4), (5), (8), we have

$$\lambda(|\mathrm{Tr}\ (X+Y-Q_1)| + |\mathrm{Tr}\ \tilde{Q}_1|) - \Lambda N\|Q_1 - \tilde{Q}_1\| + \frac{\partial\Psi}{\partial t}$$

$$\leq C\{1+[\mu(|\bar{x}-\bar{y}|) + |D_x\Psi + D_y\Psi|^{1/2}\mu(|D_x\Psi + D_y\Psi|)](\|X\|+\|Y\|)\} \tag{19}$$

Noting that

$$\left|\frac{\partial\Psi}{\partial t}(\bar{x},\bar{y},\bar{t})\right|,\ \|Q_1-\bar{Q}_1\| \leq CK\eta\frac{m}{R^2}W^{-\frac{m}{2}-1}|\bar{x}-\bar{y}|$$

By taking m large, we get from (18), (19)

$$\frac{\lambda}{2}(|\mathrm{Tr}\ (X+Y-Q_1)|+|\mathrm{Tr}\ \bar{Q}_1|) \leq C[\mu(|\bar{x}-\bar{y}|) + |D_x\Psi + D_y\Psi|^{1/2}\mu(D_x\Psi + D_y\Psi|)](\|X\|+\|Y\|)$$

Using the same argument as the proof of Lemma IV. 1 in [4], we have

$$\|X\|+\|Y\| \leq C(N)\{\|Q\|+|\mathrm{Tr}\ (X+Y-Q_1)|+\|Q_2-Q_1\|^{1/2}|\mathrm{Tr}(X+Y-Q_1)|^{1/2}\}$$

Direct calculation shows

$$|D_x\Psi + D_y\Psi| = \frac{1}{2}K\eta\frac{m}{(\nu R)^2}W^{-\frac{m}{2}-1}(|\bar{x}-\bar{y}|)+\frac{R}{l'(1)}l\Big(\frac{|\bar{x}-\bar{y}|}{R}\Big)\Big)|\bar{x}+\bar{y}| \leq Cm/2^{\frac{m}{2}}$$

$$\|Q\| \leq CK\eta\frac{m(m+2)}{W^{m/2}}/R,\ \|Q_2-Q_1\| \leq CK/|\bar{x}-\bar{y}|$$

Using Cauchy inequality and taking m large, R small, we can get a contradiction with (18). Thus

$$|u(x,t)-u(y,t)-\langle l,x-y\rangle| \leq (2-\eta_1)K|x-y| \quad \text{in} \quad \bar{B}_{R/2\sqrt{2}}(x_1)\times(t_1-(\nu R)^2,t_1]$$

where $\eta_1 = \dfrac{\sqrt{2}-1}{32}\nu\Big[\Big(\dfrac{4}{3}\Big)^{\frac{m}{2}}-\dfrac{3}{2}\Big](2\nu)^m$.

The lemma is proved.

Lemma 9 *Let l be the vector in Lemma 6. Then there exist constants $\tilde{\eta}$ and $\tilde{\theta}$, such that*

$$|u(x,t)-u(y,t)-\langle l,x-y\rangle| \leq (2-\tilde{\eta})K|x-y| \ \text{in}\ \bar{Q}_{\tilde{\theta}R}(x_0,t_0)$$

Proof Assume $x_0=0$. Consider the function

$$\Psi = \langle l,x-y\rangle + 2K|x-y| + \frac{1}{2}K\eta\Big(\frac{9}{4}-W^{-1}\Big)\Big(|x-y|+\frac{R}{l'(1)}l\Big(\frac{|x-y|}{R}\Big)\Big)+K\eta\frac{9}{4}|x-y|$$

where $W=\dfrac{1}{9}+\dfrac{1}{R^2}(|x|^2+|y|^2+\sigma(t-t_1))$, and the domain

$$S=\Big\{(x,y,t):|x|^2+|y|^2+\sigma(t-t_1)<\Big(\frac{R}{3}\Big)^2,\quad 0<\sigma(t-t_1)\leq\Big(\frac{R}{3}\Big)^2\Big\}$$

If we take $\eta \le \frac{2}{9}\eta_1$, then on the parabolic boundary of S

$$u(x,t) - u(y,t) \le \Psi$$

Now we want to prove the above inequality holds in S by the same way in previous lemma. At this time,

$$\lambda|\mathrm{Tr}\ (X+Y-Q_1)| - \Lambda N\|Q_1\| + \frac{\partial \Psi}{\partial t} \ge \frac{\lambda K\eta}{Rl'(1)} W^{-1} l''\Big(\frac{|\bar{x}-\bar{y}|}{R}\Big) \tag{20}$$

provided σ large, while

$$\begin{aligned}\lambda|\mathrm{Tr}\ (X+Y-Q_1)| - \Lambda N\|Q_1\| + \frac{\partial \Psi}{\partial t} &\le C\{1 + [\mu(|\bar{x}-\bar{y}|) \\ &\quad + |D_x\Psi + D_y\Psi|^{1/2}\mu(|D_x\Psi + D_y\Psi|)] \\ &\quad \cdot(\|Q\| + |\mathrm{Tr}\ (X+Y-Q_1)| + \|Q_2 - Q_1\|^{1/2}|\mathrm{Tr}\ (X+Y-Q_1)|^{1/2})\}\end{aligned} \tag{21}$$

Noting that

$$|D_x\Psi + D_y\Psi| \le CK\eta$$
$$\|Q\| \le CK\eta/R, \quad \|Q_2 - Q_1\| \le CK/|\bar{x}-\bar{y}|$$

we find (20) contradicts (21) by taking η small, R small.

Taking h large such that $\frac{\sigma(t_0-t_1)}{R^2}$ small, and η small again, we get the lemma immediately.

The proof of Theorem 1 is included in [1-2], [4-5]. So we have $u \in C^{1+\alpha,\frac{1+\alpha}{2}}(Q_T)$.

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