J.Partial Diff. Eqs. **13**(2000), 89–96
©International Academic Publishers Vol.13, No.1

HÖLDER ZYGMUND SPACE TECHNIQUES TO THE NAVIER-STOKES EQUATIONS IN THE WHOLE SPACES

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Dedicated to Professor Chen Wenyuan on his 70th birthday

(Received Apr.19, 1999)

Abstract With the use of Hölder Zygmund space techniques, local regular solutions to the Navier-Stokes equations in R^n are shown to exist when the initial data are in the space

$$\{a|(-\Delta)^{-\beta/2}a \in C^0(R^n)^n\} \quad (0 < \beta < 1)$$

Key Words Navier-Stokes equations; local solutions; Hölder Zygmund spaces.
1991 MR Subject Classification 35Q30, 76D05.
Chinese Library Classification O175.29.

1. Introduction

Consider the incompressible viscous fluid motion governed by the Navier-Stokes equations in R^n, $n \geq 2$:

$$\begin{cases} \partial u/\partial t - \Delta u + \nabla \cdot (u \otimes u) + \nabla \pi = 0 \\ \nabla \cdot u = 0 \\ u(0) = a \end{cases} \tag{1}$$

with unknown velocity $u = (u_1(x,t), \cdots, u_n(x,t))$ and unknown pressure $\pi = \pi(x,t)$. Here $\nabla =$ the gradient $(\partial_1, \cdots, \partial_n)$ and $\Delta =$ the Laplacian $\nabla \cdot \nabla$.

Mathematical theory of the Navier-Stokes equations stems from the poineering work of Leray [1] in 1934, where the existence of a global weak solution was established when the initial velocity $a \in L_2(R^n)^n$. The regularity of this weak solution, however, still remains foundamentally unknown. To understand the regularity problem, Fabes, Jones and Riviere [2] obtained the local existence of regular solutions with initial data in $L_p(R^n)^n$ with $n < p < \infty$ and the global existence of regular solutions with small initial data in $L_r(R^n)^n \cap L_p(R^n)^n$ with $1 \leq r < n < p < \infty$. This result has been extensively studied by many authors. For example, [3–7] and [8–9] are concerned with regular solutions when the initial velocity is in the Lebesgue space $L_p(R^n)^n$ with $p < \infty$ and the Lorentz space $L_{n,\infty}(R^n)^n$, respectively. It has become clear than $L_n(R^n)^n$ is a critical space in obtaining regularity solutions in the following sense: regular solution exists locally when the initial velocity $a \in L_p(R^n)^n$ with $n < p < \infty$, small regular solution exists globally when $a \in L_n(R^n)^n$, and no regular solution is found to exist when $a \in L_p(R^n)^n$ with $p < n$ no matter how small the $\|a\|_{L^p}$ is. One can also refer to [10–14] for stability study on fluid motions and [15–18] for bifurcation analysis of Navier-Stokes flows.

The purpose of this paper is to present a new approach showing the local existence of regular solutions when the initial data are in a new function space containing $L_p(R^n)^n$ with $n < p < \infty$.

To state our result, we denote by F the Fourier transform in R^n and set the Riesz potential $(-\Delta)^{\lambda/2} = F^{-1}|\xi|^\lambda F$. Moreover, we introduce the Hölder Zygmund space $\mathcal{C}^\alpha(R^n)$:

$$[u]_{\mathcal{C}^\alpha} \equiv \sup_{y\neq 0} \frac{\|u(\cdot+y)-u(\cdot)\|_{L_\infty}}{|y|^\alpha} \quad \text{for } 0<\alpha<1$$

$$[u]_{\mathcal{C}^0} \equiv [(-\Delta)^{-1/4}a]_{\mathcal{C}^{1/2}},\ [u]_{\mathcal{C}^\alpha} \equiv [(-\Delta)^{\alpha/2-1/4}a]_{\mathcal{C}^{1/2}} \quad \text{for } \alpha \geq 1$$

$$\mathcal{C}^\alpha(R^n) \equiv \begin{cases} \{u \in L_\infty(R^n) | \|u\|_{\mathcal{C}^\alpha} \equiv \|u\|_{L_\infty} + [u]_{\mathcal{C}^\alpha} < \infty\} & \text{for } \alpha > 0 \\ \{u \in S'(R^n) | \|u\|_{\mathcal{C}^0} \equiv [u]_{\mathcal{C}^0} < \infty\} & \text{for } \alpha = 0 \end{cases}$$

where $S'(R^n)$ denotes the dual space of $S(R^n)$, the Schwartz space of repidly decreasing smooth scalar functions.

The main result of this paper reads as follows:

Theorem 1.1 *Let $n \geq 2$, $0 < \beta < 1$, $(-\Delta)^{-\beta/2}a \in \mathcal{C}^0(R^n)^n$ and $\nabla \cdot a = 0$ in the sense of distribution. Then there exists a constant $T > 0$ such that Eq. (1) admits a regular solution u satisfying*

$$(-\Delta)^{-\beta/2}u \in C_{w-*}([0,T]; \mathcal{C}^0(R^n)^n)$$

and

$$\|(-\Delta)^{-\beta/2}u(t)\|_{\mathcal{C}^0} + t^{\beta/2}\|u(t)\|_{L_\infty} + t\|u(t)\|_{\mathcal{C}^{2-\beta}} \in L_\infty(0,T)$$

where C_{w-*} denotes the continuity in the weak−∗ topology.

Theorem 1.1 is to be proved in Section 2 based on elementary properties of the Hölder Zygmund spaces described in Section 2.

Let us mention that Giga, Inui and Matsui [19] recently obtained the local existence of regular solutions with initial data in $L_\infty(R^n)^n$ together with its subspaces. However, our study ion is rather different from those of [19] due to the fact that Theorem 1.1 shows the sharp regularity estimate in Hölder Zygmund spaces and the initial data

$$a \in \{a \in S'(R^n)^n | (-\Delta)^{-\beta/2}a \in \mathcal{C}^0(R^n)^n\}$$

which contains $L_p(R^n)^n$ with $p = n/\beta$, by the homogeneity and the Sobolev imbedding theorem (See [20]).

2. Preliminary Results in Hölder Zygmund Spaces

This section is devoted to some base properties of the Hölder Zygmund space.

Lemma 2.1 *There hold*

$$[(-\Delta)^{\lambda/2}a]_{\mathcal{C}^\alpha} \cong [a]_{\mathcal{C}^{\alpha+\lambda}}, \quad \alpha+\lambda \geq 0, \alpha \geq 0$$

and

$$\|(-\Delta)^{\lambda/2}u\|_{L_\infty} \leq C\|u\|_{\mathcal{C}^0}^{1/2}\|(-\Delta)^{\lambda}u\|_{\mathcal{C}^0}^{1/2}, \quad \lambda > 0$$

where and in what follows C denotes a generic constant independent of the quantities s, t, T, a and u.

Proof Let us denote by $B^\beta_{p,q}(R^n)$ (See [20, Definition 2.3.1/2]) the Besov space, and by $\dot{B}^\beta_{p,q}(R^n)$ (See [20, Definition 5.1.3/2]) the homogeneous Besov space. Note that, for $\alpha > 0$,

$$\mathcal{C}^\alpha(R^n) = B^\alpha_{\infty,\infty}(R^n), \text{ and } [u]_{\mathcal{C}^\alpha} \cong \|u\|_{\dot{B}^\alpha_{\infty,\infty}} \text{ (See [20, Theorems 2.5.7, 5.2.3/2])}$$

By the lifting property of the operator $(-\Delta)^\alpha$ in the class of homogeneous Besov spaces (See [20, Theorem 5.2.3/1]), we have

$$[(-\Delta)^{\lambda/2}u]_{\mathcal{C}^\alpha} \cong [u]_{\mathcal{C}^{\alpha+\lambda}}, \ \alpha+\lambda \geq 0$$

and hence

$$\mathcal{C}^0(R^n) = \dot{B}^0_{\infty,\infty}(R^n), \quad [(-\Delta)^{\lambda/2}u]_{\mathcal{C}^0} \cong [u]_{\mathcal{C}^\lambda}, \quad \lambda \geq 0$$

Moreover, by the homogeneity, it is readily seen that $B^0_{\infty,1}(R^n) = \dot{B}^0_{\infty,1}(R^n)$ and so, by [20, Proposition 2.5.7],

$$B^0_{\infty,1}(R^n) \subset L_\infty(R^n)$$

On the other hand, it follows from [20, Subsection 5.2.5] that

$$\dot{B}^\beta_{\infty,1}(R^n) = (\dot{B}^0_{\infty,\infty}(R^n), \dot{B}^{2\beta}_{\infty,\infty}(R^n))_{1/2,1}$$

where $(\cdot,\cdot)_{\theta,p}$ represents the real interpolation functor defined by [20, Definition 2.4.1]. Consequently,

$$\begin{aligned}\|(-\Delta)^{\beta/2}u\|_{L_\infty} &\leq C\|(-\Delta)^{\beta/2}u\|_{\dot{B}^0_{\infty,1}} \leq C\|u\|^{1/2}_{\dot{B}^0_{\infty,\infty}}\|(-\Delta)^\beta u\|^{1/2}_{\dot{B}^0_{\infty,\infty}} \\ &\leq C\|u\|_{\mathcal{C}^0}^{1/2}\|(-\Delta)^\beta u\|_{\mathcal{C}^0}^{1/2}\end{aligned}$$

The proof is complete.

Lemma 2.2 *There hold true the following estimates:*

$$\|\nabla^k e^{-tA}u\|_{L_\infty} \leq Ct^{-k/2}\|u\|_{L_\infty}, \quad k \geq 0 \tag{2}$$

$$[e^{-tA}u]_{C^{\alpha+\lambda}} \le Ct^{-\lambda/2}[u]_{C^\alpha}, \quad \alpha, \lambda \ge 0 \tag{3}$$

$$\|(-\Delta)^{\lambda/2}e^{-tA}u\|_{L_\infty} \le Ct^{-\lambda/2}[u]_{C^0}, \quad \lambda > 0 \tag{4}$$

where the semigroup

$$e^{-tA}u(x) = (4\pi t)^{-n/2}\int_{R^n} e^{-\frac{|x-y|^2}{4t}}u(y)dy$$

Proof Eq. (2) is simple and follows immediately from the integral representation of e^{-tA}. Eq. (3) is due to the following observation

$$[e^{-tA}u]_{C^{\alpha+\lambda}} = t^{-\lambda/2}[F^{-1}(|\xi|^2t)^{\lambda/2}e^{-|\xi|^2t}Fu]_{C^\alpha} \le Ct^{-\lambda/2}[u]_{C^\alpha}$$

where the use is made of Mikhlin theorem on Fourier multiplers (See [20, Theorem 5.2.2)]. Finally, Lemma 2.1 and Eq. (3) yield

$$\|(-\Delta)^{\lambda/2}e^{-tA}u\|_{L_\infty} \le C\|e^{-tA}u\|_{C^0}^{1/2}\|(-\Delta)^\lambda e^{-tA}u\|_{C^0}^{1/2} \le Ct^{-\lambda/2}\|u\|_{C^0}$$

This shows Eq. (4) and completes the proof.

Lemma 2.3 *For $0 < \alpha < 4$, there holds true the equivalence*

$$[u]_{C^\alpha} \cong |[u]|_{C^\alpha}, \quad u \in C^\alpha(R^n)$$

where the seminorm (See [21])

$$|[u]|_{C^\alpha} \equiv \sup_{t>0} t^{k-\alpha/2}\|\Delta^k e^{-tA}u\|_{L_\infty} \quad \textit{for integer } k > \alpha/2$$

Proof From [20, Theorem 2.12.2] it follows that

$$\|u(\cdot)\|_{L_\infty} + [u(\cdot)]_{C^\alpha} \cong \|u(\cdot)\|_{L_\infty} + |[u(\cdot)]|_{C^\alpha}$$

and so

$$\|u(\lambda\ \cdot)\|_{L_\infty} + [u(\lambda\ \cdot)]_{C^\alpha} \cong \|u(\lambda\ \cdot)\|_{L_\infty} + |[u(\lambda\ \cdot)]|_{C^\alpha}, \quad \lambda > 0$$

and hence

$$\lambda^{-\alpha}\|u(\cdot)\|_{L_\infty} + [u(\cdot)]_{C\alpha} \cong \lambda^{-\alpha}\|u(\cdot)\|_{L_\infty} + |[u(\cdot)]|_{C^\alpha}$$

Passing to the limit as $\lambda \to \infty$ gives the desired equivalence. The proof is complete.

3. Proof of Theorem 1.1

Let us begin with the definition of the notations, $T > 0$:

$P =$ the projection operator such that $(Pu)_j = \sum_{i=1}^n F^{-1}(\delta_{ij} - \xi_i\xi_j|\xi|^{-2})Fu_i$

$U_T = \{u|(-\Delta)^{-\beta/2}u \in L^\infty(0,T;C^0(R^n)^n), \nabla\cdot u = 0, \|u\|_{U_T} < \infty\}$ with

$$\|u\|_{U_T} = \sup_{0<t<T}(\|(-\Delta)^{-\beta/2}u(t)\|_{C^0} + t^{\beta/2}\|u(t)\|_{L_\infty} + t^{1/2}[u(t)]_{C^{1-\beta}} + t[u(t)]_{C^{2-\beta}}$$

$$Mu(t) = e^{-tA}a - \int_0^t e^{-(t-s)A}P\nabla\cdot(u(s)\otimes u(s))ds,\ u\in U_T$$

By Lemma 2.2, we have

$$\begin{aligned}\|(-\Delta)^{-\beta/2}e^{-tA}a\|_{C^0} + t^{\beta/2}\|e^{-tA}a\|_{L_\infty} + t^{1/2}[e^{-tA}a]_{C^{1-\beta}} + t[e^{-tA}a]_{C^{2-\beta}}\\ \le C\|(-\Delta)^{-\beta/2}a\|_{C^0}\end{aligned}$$

Additionally, using Lemmas 2.1, 2.2 and 2.3 and the potential estimates in Hölder Zygmund spaces

$$[\nabla(-\Delta)^{-1/2}u]_{C^\alpha} = [F^{-1}\xi|\xi|^{-1}Fu]_{C^\alpha} \le C[u]_{C^\alpha}$$

(See Mikhlin theorem [20, Theorem 5.2.2]), we have, for $f(s)=u(s)\otimes u(s)$ and $\alpha=1,2$,

$$\begin{aligned}
&[(-\Delta)^{-\beta/2}(Mu(t)-e^{-tA}Pa)]_{C^\alpha} \le C\|[(-\Delta)^{-\beta/2}\int_0^t e^{-(t-s)A}P\nabla\cdot f(s)ds]|_{C^\alpha}\\
&\le C\sup_{\tau>0}\tau^{2-\alpha/2}\int_0^t \|\Delta^2 e^{-(t-s+\tau)A}P\nabla\cdot(-\Delta)^{-\beta/2}f(s)\|_{L_\infty}ds\\
&\le C\sup_{\tau>0}\tau^{2-\alpha/2}\int_0^t (t+\tau-s)^{-1}\|\Delta^2 e^{-(t+\tau-s)A/2}P\nabla\cdot(-\Delta)^{-(2+\beta)/2}f(s)\|_{L_\infty}ds\\
&\le C\sup_{\tau>0}\tau^{2-\alpha/2}\int_0^t (t+\tau-s)^{-2}\|[P\nabla^2\cdot(-\Delta)^{-(2+\beta)/2}f(s)]|_{C^2}ds\\
&\le C\sup_{\tau>0}\tau^{2-\alpha/2}\int_0^t (t+\tau-s)^{-2}[P\nabla^2\cdot(-\Delta)^{-(2+\beta)/2}f(s)]_{C^2}ds\\
&\le C\sup_{\tau>0}\tau^{2-\alpha/2}\int_0^t (t+\tau-s)^{-2}[\Delta^{-(1+\beta)/2}f(s)]_{C^2}ds\\
&\le C\sup_{\tau>0}\tau^{2-\alpha/2}\int_0^t (t+\tau-s)^{-2}s^{-(1+\beta)/2}ds\sup_{0<s<T}s^{(1+\beta)/2}[f(s)]_{C^{1-\beta}}\\
&\le C\sup_{\tau>0}\tau^{2-\alpha/2}\Big(\int_0^t+\int_{t/2}^t\Big)(t+\tau-s)^{-2}s^{-(1+\beta)/2}ds\sup_{0<s<T}s^{(1+\beta)/2}\\
&\quad\cdot[u(s)]_{C^{1-\beta}}\|u(s)\|_{L_\infty}\\
&\le C\sup_{\tau>0}\tau^{2-\alpha/2}(t+\tau)^{-2}t^{(1-\beta)/2}\|u\|_{U_T}^2\\
&\quad+C\sup_{\tau>0}\tau^{2-\alpha/2}t^{-(1+\beta)/2}(\tau^{-1}-(t+\tau)^{-1})\|u\|_{U_T}^2\\
&\le Ct^{-(\alpha-1+\beta)/2}\|u\|_{U_T}^2
\end{aligned}$$

and, furthermore, for $\beta>\delta>0$,

$$\|Mu(t)-e^{-tA}a\|_{L_\infty} \le \int_0^t \|\Delta e^{-(t-s)A}P\nabla\cdot\Delta^{-1}f(s)\|_{L_\infty}ds$$

$$
\begin{aligned}
&\le C\int_0^t (t-s)^{-\delta/2}\|[e^{-(t-s)A/2}P\nabla\cdot\Delta^{-1}f(s)]\|_{C^{2-\delta}}ds\\
&\le C\int_0^t (t-s)^{-\beta/2}\|[P\nabla\cdot(-\Delta)^{-(2+\beta-\delta)/2}f(s)]\|_{C^{2-\delta}}ds\\
&\le C\int_0^t (t-s)^{-\beta/2}[P\nabla\cdot(-\Delta)^{-(2+\beta-\delta)/2}f(s)]_{2-\delta}ds\\
&\le C\int_0^t (t-s)^{-\beta/2}[(-\Delta)^{-(1+\beta-\delta)/2}f(s)]_{2-\delta}ds\\
&\le C\int_0^t (t-s)^{-\beta/2}[f(s)]_{1-\beta}ds\\
&\le C\int_0^t (t-s)^{-\beta/2}s^{-(1+\beta)/2}\sup_{0<s<T} s^{(1+\beta)/2}ds[f(s)]_{C^{1-\beta}}\\
&\le Ct^{-\beta+1/2}\|u\|_{U_T}^2
\end{aligned}
$$

and, finally,

$$
\begin{aligned}
\|(-\Delta)^{-\beta/2}(Mu(t)-e^{-tA}a)\|_{C^0}\le&\int_0^t \|e^{-(t-s)A}P\nabla(-\Delta)^{-\beta/2}\cdot f(s)\|_{C^0}ds\\
\le& C\int_0^t[\nabla\cdot(-\Delta)^{-\beta/2}f(s)]_{C^0}ds\\
\le& C\int_0^t s^{-(1+\beta)/2}ds\sup_{0<s<T} s^{(1+\beta)/2}[f(s)]_{C^{1-\beta}}\\
\le& Ct^{(1-\beta)/2}\|u\|_{U_T}^2
\end{aligned}
$$

Collecting terms, we arrive at the sharp estimate

$$
\|Mu\|_{U_T}\le C\|(-\Delta)^{-\beta/2}a\|_{C^0}+CT^{(1-\beta)/2}\|u\|_{U_T}^2
$$

Likewise, we have

$$
\|Mu-Mv\|_{U_T}\le CT^{(1-\beta)/2}(\|u\|_{U_T}+\|v\|_{U_T})\|u-v\|_{U_T},\ u,v\in U_T
$$

Noting $PMu(t)=Mu(t)$, we have $\nabla\cdot Mu=0$. We thus can choose a small constant $T>0$ and a large constant $r>0$ such that M is a contraction operator mapping the complete metric space $\{u\in U_T|\|u\|_{U_T}\le r\}$ into itself. By the contraction mapping principle, we obtain the local solution, which can be represented in the integral form

$$
u(t+\tau)=e^{-tA}u(\tau)-\int_0^t e^{-(t-s)A}P\nabla\cdot(u(s+\tau)\otimes u(s+\tau))ds,\ t+\tau\le T
$$

To verify the weak$-*$ continuity of $u(\tau)$ at $T>\tau\ge 0$, we note, for $\phi\in S(R^n)^n$,

$$
\begin{aligned}
&\int_{R^n}(-\Delta)^{-\beta/2}(u(t+\tau)-u(\tau))\cdot\phi dx\\
&\le\Big|\int_{R^n}(-\Delta)^{-\beta/2}(e^{-tA}u(\tau)-u(\tau))\cdot\phi dx\Big|
\end{aligned}
$$

$$+\int_0^t \left|\int_{R^n} (-\Delta)^{-\beta/2} e^{-(t-s)A} P\nabla\cdot(u(s+\tau)\otimes u(s+\tau))\cdot\phi dx\right| ds$$
$$\le \|(-\Delta)^{-\beta/2}u(\tau)\|_{C^0}\|e^{-tA}\phi-\phi\|_{\dot B^0_{1,1}}$$
$$+\int_0^t \|(-\Delta)^{-\beta/2}P\nabla\cdot(u(s+\tau)\otimes u(s+\tau))\|_{C^0}\|\phi\|_{\dot B^0_{1,1}} ds$$
$$\le \|u\|_{U_T}\|e^{-tA}\phi-\phi\|_{\dot B^0_{1,1}} + Ct^{(1-\beta)/2}\|u\|^2_{U_T}\|\phi\|_{\dot B^0_{1,1}}$$

where the use is made of the duality (See [20])

$$(\dot B^0_{1,1}(R^n))^* = \dot B^0_{\infty,\infty}(R^n) = C^0(R^n)$$

Obviously, e^{-tA} is strongly continuous in $\dot B^0_{1,1}(R^n)$. Hence the proof of Theorem 1.1 is complete.

References

[1] Leray J., Sur le mouvement d'un liquide visquex emplissant l'espace, *Acta. Math.*, **63** (1934), 193–248.

[2] Fabes E.B., Jones B.F. and Riviere N.M., The initial value problem for the Navier-Stokes equations with data in L^p, *Arch. Rational Mech. Anal.*, **45** (1972), 222–240.

[3] Beirao de Veiga H., Existence and asymptotic behavior for strong solutions of the Navier-Stokes equations in the whole space, *Indiana Univ. Math. J.*, **36** (1987), 149–166.

[4] Chen Z.M., A sharp decay result on strong solutions of the Navier-Stokes equations in the whole space, *Commun. Partial Diff. Eqs.*, **16** (1991), 801–820.

[5] Giga Y., Solutions for semilinear parabolic equations in L^p and regularity of weak solutions of the Navier-Stokes system, *J. Diff. Eqs.*, **62** (1986), 182–212.

[6] Kato T., Strong L^p-solutions of the Navier-Stokes equations in R^n, with applications to weak solutions, *Math. Z.*, **187** (1984), 471–480.

[7] Miyakawa T., Application of Hardy space techniques to the time-decay problem for incompressible Navier-Stokes flows in R^n, *Funkcialaj Ekvacioj*, **41** (1998), 383–434.

[8] Barraza O.A., Self-similar solutions in weak L^p-spaces of the Navier-Stokes equations, *Revista Matemática Iberoamericana*, **12** (1996), 411-439.

[9] Barraza O.A., Regularity and stability for the solutions of the Navier-Stokes equations in Lorentz spaces, *Nonl. Anal.*, **35** (1999), 747–764.

[10] Borchers W. and Miyakawa T., On stability of exterior stationary Navier-Stokes flows, *Acta Math.* **174** (1995), 311–382.

[11] Chen Z.M., L_n solutions of the stationary and nonstationary Navier-Stokes equations in R^n, *Pacific J. Math.*, **158** (1993), 293–303.

[12] Chen Z.M. and Miyakawa T., Decay properties of weak solutions to a perturbed Navier-Stokes system in R^n, *Adv. Math. Sci. Appl.*, **7** (1997), 741–770.

[13] He C., Weighted estimates for nonstationary Navier-Stokes equations, *J. Diff. Eqs.*, **148** (1998), 422–444.

[14] Hishida T., On a class of stable steady flows to the exterior convection problem, *J. Diff. Eqs.*, **141** (1997), 54–85.

[15] Chen Z.M., Bifurcations of a steady-state solution to the two-dimensional Navier-Stokes equations, *Commun. Math. Phys.*, **201** (1999), 117–138.

[16] Chen Z.M. and Price W.G., Remarks on time dependent periodic Navier-Stokes flows on a two-dimensional torus, *Commun. Math. Phys.*, (to appear).

[17] Chen Z.M., Price W.G., Time dependent periodic Navier-Stokes flows on a two-dimensional torus, *Commun. Math. Phys.*, **179** (1996), 577–597.

[18] Chen Z.M. and Price W.G., Long time behavior of Navier-Stokes flows on a two-dimensional torus excited by a sinusoidal force, *J. Statist. Phys.*, **86** (1997), 301–335.

[19] Giga Y., Inui K. and Matsui S., On the Cauchy problem for the Navier-Stokes equations with nondecaying initial data, Preprint.

[20] Triebel H., Theory of Function Spaces. Birkhäuser Verlag, Basel-Boston, 1983.

[21] Butzer P.L. and Berens H., Semi-groups of Operators and Approximation, Grundlehren Math., Wissensch. Vol.145, Springer-Verlag, Berlin 1967.