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THE STRONG SOLUTION OF A CLASS OF GENERALIZED NAVIER-STOKES EQUATIONS*

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Abstract We study initial boundary value (IBV) problem for a class of generalized Navier-Stokes equations in $L^q([0,T); L^p(\Omega))$. Our main tools are regularity of analytic semigroup by Stokes operator and space-time estimates. As an application we can obtain some classical results of the Navier-Stokes equations such as global classical solution of 2-dimensional Navier-Stokes equation etc.

Key Words Admissible triple; generalized Navier-Stokes equations; initial boundary value problem; space-time estimates.
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1. Introduction and Main Results

In this paper we consider the following IBV problem for a class of generalized Navier-Stokes equations

$$\begin{cases} \dfrac{\partial u}{\partial t} - \Delta u + \nabla P = f(u, \nabla u), & (x,t) \in \Omega \times [0,T) \\ \operatorname{div} u(\cdot,t) = 0, & (x,t) \in \Omega \times [0,T) \\ u|_{\partial\Omega} = 0 & \\ u(x,0) = \varphi(x), & x \in \Omega \end{cases} \tag{1.1}$$

where $u = (u_1, \cdots, u_n)$ is a vector value function, $P(x,t)$ is a scalar value function, $\varphi = (\varphi_1, \cdots, \varphi_n)$ is an initial data. Let $f : \mathbb{R}^n \times \mathbb{R}^{n^2} \to \mathbb{R}^n$ be nonlinear vector functions, $\Omega \subset \mathbb{R}^n$ is a bounded smooth domain. For $1 \le p \le \infty$, $L^p = L^p(\Omega)$ denote the standard Lebesgue space with norm $\|\cdot\|_p$, $E^p(\Omega) = \{u = (u_1, \cdots, u_n) |\ u_i \in L^p(\Omega)$ and $\operatorname{div} u = 0$ in the sense of distribution $\}$. $L^q([0,T); L^p(\Omega))$ denotes space-time Lebesgue space, $L_T^{p,q} = L^q([0,T); E^p(\Omega))$ is the subspace of $L^q([0,T); (L^p(\Omega))^n)$ with norm

$$\|\cdot\|_{p,q,T} = \Big(\int_0^T \|\cdot\|_p^q dt \Big)^{1/q}$$

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Let $r \geq 1$ and σ, p satisfy $0 \leq \sigma = \left(\frac{1}{r} - \frac{1}{p}\right)\frac{n}{2} < 1$, we define

$$X_{p,T}^r = \left\{u | u \in C_b([0,T); E^p),\ \|u\|_{X_{p,T}^r} = \sup_{0\leq t<T} t^\sigma \|u(t)\|_p < \infty\right\} \tag{1.2}$$

where $\|u\|_p = \sum_{i=1}^n \|u_i\|_p$. When $T = \infty$, we usually denote $X_p^r = X_{p,\infty}^r$.

According to Helmholtz decomposition (See [1]) we have

$$(L^p)^n = E^p \oplus G^p \text{ (direct sum)} \tag{1.3}$$

where $G^p = \{\nabla g; g \in W^{1,p}\}$. Let $\mathcal{P}_p$ be the continuous projection from $(L^p)^n$ to E^p associated with this decomposition, and let B_p be the Laplace operator with zero boundary condition. Now we define $A_p = -\mathcal{P}_p\Delta$ with domain $D(A_p) = E^p \cap D(B_p)$. It is easy to verify that when $1 < p < \infty$, A_p generates an analytic semigroup $e^{-A_p t}$ in E_p, and A_p has a bounded inverse, where $D(A_p) = \{u | u \in W_0^{2,p} \cap E_p\}$. Hence we can define the fractional power $A_p^\alpha (\alpha \in \mathbb{R})$ and

$$\|A_p^\alpha e^{-A_p t}\| \leq C_\alpha t^{-\alpha}, \quad \text{for } \alpha \geq 0,\ t > 0 \tag{1.4}$$

(For detail see [1–4]). Usually we drop the subscript p attached to A and $\mathcal{P}$.

This paper is devoted to establish well-posedness theory of (1.1) in $L_T^{p,q}$ and $X_{p,T}^r$. As an application we can obtain well known classical results of the classical Navier-Stokes equations. In this problem the function P is automatically determined (up to a function of t) if u is a known vector function, indeed, $\partial P = (I - \mathcal{P})f(u, \nabla u)$, where $\mathcal{P}$ is the orthogonal projection of $(L^p)^n$ into E^p. For this reason it suffices to consider u only when we talk about the solution of (1.1).

For the sake of convenience we first introduce some notations. $\nabla = \left(\frac{\partial}{\partial x_1}, \cdots, \frac{\partial}{\partial x_n}\right)$, $\nabla_j = \frac{\partial}{\partial x_j}$, $(\cdot,\cdot)$ denotes usual L^2 inner product with respective to space variable.

Definition 1.1 *Let $q > r > 1$, $p \geq r$, we call (p, q, r) as admissible triple if*

$$\frac{1}{q} = \frac{n}{2}\left(\frac{1}{r} - \frac{1}{p}\right)$$

As is a standard practice, applying $\mathcal{P}$ to (1.1), we have

$$\frac{du}{dt} + A_p u = F(u, \nabla u), \quad t > 0; \quad u(0) = \varphi(x) \tag{1.5}$$

where $F(u, \nabla u) = \mathcal{P}f(u, \nabla u)$. Hence we study (1.1) *via* the corresponding integral equation

$$u(t) = e^{-At}\varphi(x) + \int_0^t e^{-A(t-s)} F(u, \nabla u) ds \tag{1.6}$$

in $L_T^{p,q}$ and $X_{p,T}^r$.

Definition 1.2 *A vector function $u(t) = u(t,x) = (u_1(t,x), \cdots, u_n(t,x))$ is said to be $L_T^{p,q}$-solution and $X_{p,T}^r$-solution of* (1.1) *if it is the solution of* (1.6) *in $L_T^{p,q}$ and $X_{p,T}^r$ respectively.*

Our main results can be stated as follows

Theorem A *Let $f(u, \nabla u)$ satisfy*

$$\|(-\Delta)^{-\frac{1}{2}}(f(u,\nabla u) - f(v,\nabla v)\|_p \leq \lambda\|(|u|^\alpha + |v|^\alpha)(u-v)\|_p, \quad 1 < p < \infty \tag{1.7}$$

where $\alpha > 0$, we have

(i) *Let $r \geq n\alpha > 1$, $\varphi(x) \in E^r(\Omega)$, $0 \leq \sigma = \frac{n}{2}\left(\frac{1}{r} - \frac{1}{p}\right) < \frac{1}{\alpha+1}$, then there exists at least a solution u of* (1.1) *such $u(t) \in X_{p,T}^r$.*

(ii) *If $p > \alpha + 1$, then the solution $u(t)$ which is constructed in* (i) *is unique.*

(iii) *If $r = n\alpha > 1$, then the solution $u(t)$ which is constructed in* (i) *can be extended to the infinity ($u(t) \in X_{p,\infty}^r$) provided that $\|\varphi(x)\|_r$ is sufficiently small.*

(iv) *Let $r \geq n\alpha > 1$. $[0, T^*)$ can be the maximal interval such that $u(t)$ solves* (1.1) *in X_{p,T^*}^r, $p > \max\left(\frac{n\alpha}{2}, \alpha + 1\right)$, then*

$$\|u(t)\|_2 \geq \frac{C}{(T^* - t)^{\frac{1}{2\alpha} - \frac{n}{2r}}} \tag{1.8}$$

(v) *If $n = 2, p > r = 2$. Further assume that*

$$(f(u, \nabla u), u) = 0, \quad for\ u \in E^p \tag{1.9}$$

then the solution $u(t)$ which is constructed in (i) *can be extended to infinity ($u(t) \in X_{p,\infty}^r$).*

Theorem B (i) *Let $f(u, \nabla u)$ satisfy* (1.7), *$r \geq n\alpha > 1$, (p, q, r) is any admissible triple with $q > 1 + \alpha$, $\varphi(x) \in E^r(\Omega)$. Then there exists $T > 0$ and at least a solution $u(t) \in L_T^{p,q} \cap C_b([0,T), E^r)$ of* (1.1).

(ii) *If $p, q > \alpha + 1$, the solution $u(t) \in L_T^{p,q}$ which was obtained in* (i) *is unique.*

(iii) *If $r = n\alpha > 1$, then the solution $u(t)$ which is obtained in* (i) *can be extended to the infinity provided that $\|\varphi(x)\|_r$ is sufficiently small.*

(iv) *If $n = 2, \alpha = 1, p > r = 2$. Further assume that $f(u, \nabla u)$ satisfies* (1.9), *then the solution $u(t)$ which is constructed in* (i) *can be extended to the infinity ($u(t) \in X_{p,\infty}^r$).*

Remarks (i) The main idea comes from [3], but I give a simple way to deal with (1.1) which is different with [3] in some sense. Similar to the wave and dispersive wave equations [5, 6], we introduce the concept of the admissible triple and obtain the space-time estimates of linear problem with respect to (1.1), see (2.2), (2.3) and Lamma 2.3 or [7]. By devoting to space-time estimates we easily establish the nonlinear estimates and obtain the proof of Theorem A and Theorem B by some classical iterative technology.

(ii) There are many vector functions $f(u,\nabla u)=(f_1(u,\nabla u),\cdots,f_n(u,\nabla u))$ satisfying (1.7) and (1.9). For example

$$f_i(u,\nabla u)=\sum_{j=1}^{n}\nabla_j(u_i^{\alpha}u_j),\quad i=1,2,\cdots,n \tag{1.10}$$

or

$$f_i(u,\nabla u)=\nabla_i g(u),\quad i=1,2,\cdots,n \tag{1.11}$$

where scalar function $g(u)$ satisfies

$$|g(u)-g(v)|\le\lambda_2(|u|^{\alpha}+|v|^{\alpha})|u-v| \tag{1.12}$$

It is easy to verify that $f(u,\nabla u)$ which is defined by (1.10) or (1.11) satisfies (1.7) and (1.9). So this paper generalizes the results in [3], [8].

This paper consists of four sections. In Section 2 we give some preliminary lemmas and some basic nonlinear estimates. Section 3 is devoted to the proof of our main results. In Section 4 we shall discuss some applications to the classical Navier-Stokes equations.

2. Nonlinear Estimates

It is well known [3, 9] that for $\varphi(x)\in E^r$ we have $e^{-At}\varphi\in E^p$ and

$$\|E^{-At}\varphi\|_p\le C|t|^{-\frac{n}{2}(\frac{1}{r}-\frac{1}{p})}\|\varphi\|_r \tag{2.1}$$

by Sobolev embedding theorem and the regularity of e^{-At}, where $0\le\frac{n}{2}\Big(\frac{1}{r}-\frac{1}{p}\Big)<1$ $(1\le r\le p)$. As the direct result of (2.1) we have

$$\|e^{-At}\varphi\|_{X^r_{p,T}}\le C\|\varphi\|_r \tag{2.2}$$

On the other hand, generalized Marcinkiewicz interpolation theorem implies

$$\|e^{-At}\varphi\|_{L_T^{p,q}}\le C\|\varphi\|_r,\quad 0<T\le\infty \tag{2.3}$$

where (p,q,r) is any admissible triple. Now we give some nonlinear estimates.

Lemma 2.1 *Let $u,v\in X^r_{p,T}$, $p>\alpha+1$, $r\ge n\alpha>1$, $0\le\frac{n}{2}\Big(\frac{1}{r}-\frac{1}{p}\Big)<1$, $f(u,\nabla u)$ satisfy (1.7). Then $Ju=\int_0^t e^{-A(t-s)}F(u,\nabla u)ds\in X^r_{p,T}$, $Jv\in X^r_{p,T}$ and*

$$\Big\|\int_0^t e^{-A(t-s)}F(u,\nabla u)ds\Big\|_{X^r_{p,T}}\le CT^{\frac{1}{2}-\frac{n\alpha}{2r}}\|u\|^{\alpha+1}_{X^r_{p,T}} \tag{2.4}$$

$$\Big\|\int_0^t e^{-A(t-s)}(F(u,\nabla u)-F(v,\nabla v))ds\Big\|_{X^r_{p,T}}$$

$$\leq CT^{\frac{1}{2}-\frac{n\alpha}{2r}}(\|u\|_{X^{\alpha}_{p,T}}+\|v\|^{\alpha}_{X^r_{p,T}})\|u-v\|_{X^r_{p,T}} \tag{2.5}$$

where C is independent of T.

Proof We only prove (2.4) and (2.5). In view of (1.7) and (2.1) we have

$$\begin{aligned}\|Ju\|_{X^r_{p,T}} \leq&\lambda \sup_{0\leq t<T}|t|^{\sigma}\Big\|\int_0^t(-\Delta)^{\frac{1}{2}}e^{-\frac{A}{2}(t-s)}\cdot e^{-\frac{A}{2}(t-s)}(-\Delta)^{-\frac{1}{2}}F(u,\nabla u)ds\Big\|_p\\ \leq&\lambda\sup_{0\leq t<T}|t|^{\sigma}\int_0^t|t-s|^{-\frac{1}{2}-\frac{n\alpha}{2p}}\|u\|_p^{\alpha+1}ds\\ \leq&\lambda\sup_{0\leq t<T}|t|^{\sigma}\int_0^t|t-s|^{-\frac{1}{2}-\frac{n\alpha}{2p}}s^{-(\alpha+1)\sigma}ds\|u\|^{\alpha+1}_{X^r_{p,T}}\\ \leq&\lambda\sup_{0\leq t<T}|t|^{\frac{1}{2}-\frac{n\alpha}{2p}-\alpha\sigma}\int_0^1|1-s|^{-\frac{1}{2}-\frac{n\alpha}{2p}}s^{-(\alpha+1)\sigma}ds\|u\|^{\alpha+1}_{X^r_{p,T}}\\ \leq&CT^{\frac{1}{2}-\frac{n\alpha}{2r}}\|u\|^{\alpha+1}_{X^r_{p,T}}\end{aligned} \tag{2.6}$$

by Hölder inequality. So we obtain the estimate (2.4).

Noting that

$$\|e^{-\frac{A}{2}(t-s)}(\Delta)^{-\frac{1}{2}}(F(u,\nabla u)-F(v,\nabla v))ds\|_p\leq\lambda|t-s|^{-\frac{n\alpha}{2p}}(\|u\|_p^{\alpha}+\|v\|_p^{\alpha})\|u-v\|_p \tag{2.7}$$

similar to the proof of (2.6) we easily obtain the estimate (2.5).

Lemma 2.2 *Let $p>r\geq n\alpha>1$, $0\leq\sigma=\left(\frac{1}{r}-\frac{1}{p}\right)\frac{n}{2}<\frac{1}{\alpha+1}<1$. If $p\leq\alpha+1$ then there exists $p_1>\alpha+1$ such that when $u,v\in X^r_{p,T}\cap X^r_{p_1,T}$, Ju and Jv belong to $X^r_{p,T}$ such that*

$$\|Ju\|_{X^r_{p,T}}\leq CT^{1-\frac{n\alpha}{2r}}\|u\|^{\alpha}_{X^r_{p_1,T}}(\|u\|_{X^r_{p,T}}+\|u\|_{X^r_{p_1,T}}) \tag{2.8}$$

$$\|Ju-Jv\|_{X^r_{p,T}}\leq CT^{1-\frac{n\alpha}{2r}}(\|u\|^{\alpha}_{X^r_{p_1,T}}+\|v\|^{\alpha}_{X^r_{p_1,T}})(\|u-v\|_{X^r_{p_1,T}}+\|u-v\|_{X^r_{p,T}}) \tag{2.9}$$

where $0\leq\sigma_1=\left(\frac{1}{r}-\frac{1}{p_1}\right)\frac{n}{2}<\frac{1}{\alpha+1}$ and C is constantly independent of T.

Proof In view of $1<n\alpha\leq r<p\leq\alpha+1$, we solve

$$\frac{1}{2\alpha}-\frac{n}{2p_1}=\frac{1}{\alpha+1} \tag{2.10}$$

when $\alpha<1$, that is

$$p_1=\frac{n\alpha(\alpha+1)}{1-\alpha}>\alpha+1 \tag{2.11}$$

which implies that there is $p_1>\alpha+1$ such that $0\leq\sigma_1=\left(\frac{1}{r}-\frac{1}{p_1}\right)\frac{n}{2}<\frac{1}{\alpha+1}<1$. When $\alpha\geq1$, the above fact is also valid by (2.10). Now we divide that into two cases to prove this Lemma.

(i) $p_1 < p(\alpha+1)$. Noting that (2.7)

$$\|Ju\|_{X^r_{p,T}} \le \lambda \sup_{0\le t<T} |t|^\sigma \int_0^t |t-s|^{-\frac{1}{2}-(\frac{1+\alpha}{p_1}-\frac{1}{p})\frac{n}{2}} \|u\|_{p_1}^{\alpha=1} ds$$

$$\le \lambda \sup_{0\le t<T} |t|^{\frac{1}{2}-(\frac{\alpha+1}{p_1}-\frac{1}{p})\frac{n}{2}-(1+\alpha)\sigma_1+\sigma} \int_0^1 |1-s|^{-\frac{1}{2}-(\frac{\alpha+1}{p_1}-\frac{1}{p})\frac{n}{2}} s^{-(\alpha+1)\sigma_1} ds \|u\|_{X^{\alpha+1}_{p_1,T}}$$

$$\le CT^{\frac{1}{2}-\frac{n\alpha}{2r}} \|u\|^{\alpha+1}_{X^r_{p_1,T}} \tag{2.12}$$

by Hölder inequality.

(ii) $p_1 \ge p(\alpha+1)$. In view of (1.5), similar to the conduction of (2.12) we obtain

$$\|Ju\|_{X^r_{p,T}} \le \lambda \sup_{0\le t<T} |t|^\sigma \int_0^t |t-s|^{-\frac{1}{2}-\frac{n\alpha}{2p_1}} \|u\|^\alpha_{p_1} \|u\|_p ds$$

$$\le \sup_{0\le t<T} |t|^{\frac{1}{2}-\frac{n\alpha}{2p_1}-\sigma_1\alpha} \int_0^1 |1-s|^{-\frac{1}{2}-\frac{n\alpha}{2p_1}} s^{-\alpha\sigma_1-\sigma} ds \|u\|^\alpha_{X^r_{p_1,T}} \|u\|_{X^r_{p,T}}$$

$$\le CT^{\frac{1}{2}-\frac{n\alpha}{2r}} \|u\|^\alpha_{X^r_{p_1,T}} \|u\|_{X^r_{p,T}} \tag{2.13}$$

by Hölder inequality. Collecting (2.11)–(2.13) we obtain (2.8). Using (2.7) and Hölder inequality we can conclude (2.9) by the exactly same way as leading to (2.8). So we complete the proof of Lemma 2.2.

Lemma 2.3 (i) *Let $r \ge n(l-1) > 1$, (p,q,r) be any admissible triple with $p,q > l > 1$, $f \in L^{q/l}([0,T]; L^{p/l})$. Then $GA^{1/2}f(x,t) \in L^q([0,T]; L^p)$ and*

$$\|GA^{1/2}f(x,t)\|_{p,q,T} \le CT^{\frac{1}{2}-\frac{n(l-1)}{2r}} \|f\|_{\frac{p}{l},\frac{q}{l},T} \tag{2.14}$$

where C is independent of $f(x,t)$ and T, and

$$GA^{1/2}f(x,t) = \int_0^t e^{-(t-\tau)A} A^{1/2}(x,\tau) d\tau$$

(ii) *Let (p',q',r) be any admissible triple with $q' > l$, $p' \le l$. Then there is an admissible triple (p,q,r) with $p,q \ge l$ such that $GA^{1/2}f(x,t) \in L^{q'}([0,T]; L^{p'})$ and*

$$\|GA^{1/2}f(x,t)\|_{p',q',T} \le CT^{\frac{1}{2}-\frac{n(l-1)}{2r}} \|f\|_{\frac{p}{l},\frac{q}{l},T}, \quad p < p'l,\ f \in L^{q/l}([0,T]; L^{p/l}) \tag{2.15}$$

or

$$\|GA^{1/2}f(x,t)\|_{p',q',T} \le CT^{\frac{1}{2}-\frac{n(l-1)}{2r}} (\|f\|_{\frac{p}{l},\frac{q}{l},T})^{\frac{l-1}{l}} \||f|^{\frac{1}{l}}\|_{p',q',T},$$

$$p \ge p'l,\ f \in L^{\frac{q}{l}}([0,T]; L^{\frac{p}{l}}),\quad |f|^{\frac{1}{l}} \in L^{q'}([0,T]; L^{p'}) \tag{2.16}$$

where C is independent of $f(x,t)$ and T in (2.15) and (2.16).

Proof

$$\begin{aligned}\|GA^{\frac{1}{2}}f(x,t)\|_{p',q'} \le& C\Big\|\int_0^t |t-s|^{-\frac{1}{2}-\frac{n}{2}(\frac{l}{p}-\frac{1}{p})}\|f(x,s)\|_{\frac{p}{l}}ds\Big\|_{q'}\\ \le& C|t|^{\frac{1}{2}-\frac{n(l-1)}{2r}}\|f(x,t)\|_{\frac{p}{l},\frac{q}{l}}\end{aligned}\tag{2.17}$$

by generalized Young inequality or Hardy-Littlewood-Sobolev inequality (when $r = (l-1)n$) [10]. This implies (i).

We now come to the proof of (ii). In view of $1 < n(l-1) \le r < p \le 1$ and Lemma 2.2 we know that there is a $p > l$ such that $0 \le \sigma = \Big(\frac{1}{r} - \frac{1}{p}\Big)\frac{n}{2} < \frac{1}{l} < 1$. Let $q = \frac{1}{\sigma}$, we easily obtain the estimate when $p < lp'$

$$\begin{aligned}\|GA^{\frac{1}{2}}f(x,t)\|_{p',q',T} \le& C\Big\|\int_0^t |t-s|^{-\frac{1}{2}-\frac{n}{2}(\frac{l}{p}-\frac{1}{p'})}\|f(x,s)\|_{\frac{p}{l}}ds\Big\|_{q'}\\ \le& C|t|^{\frac{1}{2}-\frac{n(l-1)}{2r}}\|f(x,t)\|_{\frac{p}{l},\frac{q}{l},T}\end{aligned}\tag{2.18}$$

by the generalized Young inequality or the Hardy-Littlewood-Sobolev inequality [10]. Therefore we obtain (ii).

When $p \ge lp'$, it is easy to verify that

$$\frac{l-1}{q} + \frac{1}{q'} = \frac{n(l-1)}{2}\Big(\frac{1}{r} - \frac{1}{p}\Big) + \frac{n}{2}\Big(\frac{1}{r} - \frac{1}{p'}\Big) < \frac{l-1}{l} + \frac{1}{l} = 1 \tag{2.19}$$

$$\frac{1}{q'} = \frac{l-1}{q} + \frac{1}{q'} + \frac{q-(l-1)}{q} - 1 \tag{2.20}$$

So we get

$$\begin{aligned}\|GA^{\frac{1}{2}}f(x,t)\|_{p',q',T} \le& C\Big\|\int_0^t |t-s|^{-\frac{1}{2}-\frac{n(l-1)}{2p}}\|f(x,s)\|_{\frac{p}{l}}^{\frac{l-1}{l}}\||f(x,s)|^{\frac{1}{l}}\|_{p'}ds\Big\|_{q'}\\ \le& C|t|^{\frac{1}{2}-\frac{n(l-1)}{2r}}\|f(x,s)\|_{\frac{p}{l},\frac{q}{l},T}^{\frac{l-1}{l}}\||f(x,s)|^{\frac{1}{l}}\|_{p',q',T}\end{aligned}\tag{2.21}$$

by generalized Young inequality or Hardy-Littlewood-Sobolev inequality. This completes the proof of Lemma 2.3.

As a direct consequence of Lemma 2.3, we have the following nonlinear estimates.

Lemma 2.4 (i) *Let $r \ge n\alpha > 1$, and let (p,q,r) be any admissible triple such that $p,q > \max(\alpha+1, r)$, $u, v \in L_T^{p,q}$, then Ju, $Jv \in L_T^{p,q}$ and*

$$\|Ju\|_{p,q,T} \le CT^{\frac{1}{2}-\frac{n\alpha}{2r}}\|u\|_{p,q,T}^{\alpha+1} \tag{2.22}$$

$$\|Ju - Jv\|_{p,q,T} \le CT^{\frac{1}{2}-\frac{n\alpha}{2r}}(\|u\|_{p,q,T}^{\alpha} + \|v\|_{p,q,T}^{\alpha})\|u-v\|_{p,q,T} \tag{2.23}$$

where C is constantly independent of T.

(ii) *Let* $q > \alpha + 1$, $p \le \alpha + 1$ *satisfy* $\frac{1}{q} = \left(\frac{1}{r} - \frac{1}{p}\right)\frac{n}{2} < \frac{1}{\alpha+1} < 1$. *Then there exist* $p_1 > \alpha + 1$ *such that when* $u, v \in L_T^{p,q} \cap L_T^{p_1,q_1}$, *Ju and Jv belong to* $L_T^{p,q}$, *such that*

$$\|Ju\|_{p,q,T} \le CT^{\frac{1}{2}-\frac{n\alpha}{2r}} \|u\|^{\alpha}_{p_1,q_1T}(\|u\|_{p,q,T} + \|u\|_{p_1,q_1,T}) \tag{2.24}$$

$$\begin{aligned}\|Ju - Jv\|_{p,q,T} \le CT^{\frac{1}{2}-\frac{n\alpha}{2r}} &(\|u\|^{\alpha}_{p_1,q_1,T} + \|v\|^{\alpha}_{p_1,q_1,T}) \\ &\times (\|u - v\|_{p_1,q_1,T} + \|u - v\|_{p,q,T})\end{aligned} \tag{2.25}$$

where $\frac{1}{q} = \left(\frac{1}{r} - \frac{1}{p}\right)\frac{n}{2}$ *and C is constantly independent of T.*

3. The Proof of Main Results

Before proving our main results we first prove the following iterative result.

Lemma 3.1 *Let* $\alpha > 0$ *and* $\{b_m\}$ *be a nonnegative sequence such that*

$$b_m \le b_0 + \lambda b_{m-1}^{\alpha+1}, \quad m = 1, 2, \cdots \tag{3.1}$$

then

$$b_m \le \frac{b_0}{1 - \lambda(2b_0)^{\alpha}}, \quad m = 1, 2, \cdots \tag{3.2}$$

provided that

$$2\lambda(2b_0)^{\alpha} < 1 \tag{3.3}$$

Proof We use an induction method to prove (3.2). When $m = 1$, by direct computation we easily find

$$b_0 + \lambda b_0^{\alpha+1} < \frac{b_0}{1 - \lambda(2b_0)^{\alpha}} \tag{3.4}$$

Collecting (3.1) and (3.4) yields (3.2).

Now we assume that (3.2) is valid when $m = k$. We come to the proof of (3.2) when $m = k + 1$. In fact, from (3.3) we only need to prove

$$\lambda\Big(\frac{b_0}{1 - \lambda(2b_0)^{\alpha}}\Big)^{\alpha+1} \le \frac{b_0}{1 - \lambda(2b_0)^{\alpha}} - b_0 \tag{3.5}$$

that is

$$\Big(\frac{b_0}{1 - \lambda(2b_0)^{\alpha}}\Big)^{\alpha} \le (2b_0)^{\alpha} \tag{3.6}$$

which is equivalent to (3.3). So we conclude that (4.2) is valid when $m = k + 1$. Therefore we complete the proof of Lemma 3.1 by the induction method.

The Proof of Theorem A (a) We first consider the following iterative function sequences

$$u_0(x, t) = e^{At}\varphi(x) \tag{3.7}$$

$$u_m = u_0(x,t) + \int_0^t e^{(t-s)A} F(u_{m-1}, \nabla u_{m-1}) ds \tag{3.8}$$

When $p > 1+\alpha$, it is easy to see

$$\|u_m\|_{X^r_{p,T}} \leq C_0\|\varphi\|_r + CT^{\frac{1}{2}-\frac{n\alpha}{2r}} |u_{m-1}\|^{\alpha+1}_{X^r_{p,T}} \tag{3.9}$$

by Lemma 2.1. From Lemma 3.1 it follows

$$\|u_m\|_{X^r_{p,T}} \leq \frac{C_0\|\varphi\|_r}{1 - CT^{\frac{1}{2}-\frac{n\alpha}{2r}} (2C_0\|\varphi\|_r)^\alpha} \leq C(T), \quad m = 1, 2, \cdots \tag{3.10}$$

provided that

$$2CT^{\frac{1}{2}-\frac{n\alpha}{2r}} (2C_0\|\varphi\|_r)^\alpha < 1 \tag{3.11}$$

On the other hand, it is easy to see

$$\|u_{m+1} - u_m\|_{X^r_{p,T}} \leq 2CT^{\frac{1}{2}-\frac{n\alpha}{2r}} \|u_{m+1} - u_m\|_{X^r_{p,T}} \tag{3.12}$$

Hence, there exist $T > 0$ and function $u(x,t)$ such that

$$\lim_{m\to 0} u_m = u, \quad \text{in} \quad X^r_{p,T} \tag{3.13}$$

and

$$u = e^{-At}\varphi + \int_0^t e^{-A(t-s)} F(u, \nabla u) ds \tag{3.14}$$

Now we prove that the solution u is unique. Let $u, v \in X^r_{p,T}$ be solutions of (1.1) with same initial data $\varphi(x) \in E^r$, then

$$u - v = \int_0^t e^{A(t-s)} (F(u, \nabla u) - F(v, \nabla v)) ds \tag{3.15}$$

For any $0 < t_0 \leq T$, we have

$$\|u - v\|_{X^r_{p,t_0}} \leq Ct_0^{\frac{1}{2}-\frac{n\alpha}{2r}} (\|u\|^\alpha_{X^r_{p,t_0}} + \|v\|^\alpha_{X^r_{p,t_0}}) \|u - v\|_{X^r_{p,t_0}} \tag{3.16}$$

Let t_0 be so small such that

$$Ct_0^{\frac{1}{2}-\frac{n\alpha}{2r}} (\|u\|^\alpha_{X^r_{p,t_0}} + \|v\|^\alpha_{X^r_{p,t_0}}) < 1 \tag{3.17}$$

We obtain

$$u \equiv v \quad \text{in} \quad X^r_{p,t_0} \tag{3.18}$$

by using (3.16) and (3.17). By a repeat of the above argument we also have

$$u \equiv v \quad \text{in} \quad [t_0, 2t_0] \times \Omega \tag{3.19}$$

By continuing the same way we easily complete the proof of (ii).

(b) Now we consider the case $p \le \alpha+1$. From Lemma 2.2 it follows that there exists a $p_1 > \alpha+1$ such that $\sigma_1 = \frac{n}{2}\Big(\frac{1}{r} - \frac{1}{p_1}\Big) < \frac{1}{\alpha+1}$. From the above discussion we know that there exists a T_1 and the solutions $u(t,x)$ of (1.1) such that $u \in X^r_{p,T_1}$. In view of Lemma 3.1 we have for iterative sequence in (3.7) and (3.8)

$$\|u_{m+1}\|_{X^r_{p,T_1}} \le CT_1^{\frac{1}{2}-\frac{n\alpha}{2r}} \|u_m\|^\alpha_{X^r_{p_1,T_1}} (\|u_m\|_{X^r_{p_1,T_1}} + \|u_m\|_{X^r_{p,T_1}}), \quad m = 1, 2, \cdots \tag{3.20}$$

$$\begin{aligned}\|u_{m+1} - u_m\|_{X^r_{p,T_1}} \le & CT_1^{\frac{1}{2}-\frac{n\alpha}{2r}} (\|u_{m+1}\|^\alpha_{X^r_{p_1,T_1}} + \|u_m\|^\alpha_{X^r_{p_1,T_1}}) \\ & \times (\|u_{m+1} - u_m\|_{X^r_{p,T_1}} + \|u_{m+1} - u_m\|_{X^r_{p_1,T_1}})\end{aligned} \tag{3.21}$$

Note that $\sum\limits_{m=0}^{\infty} \|u_{m+1} - u_m\|_{X^r_{p_1,T_1}}$ is a convergence series and $\|u_m\|_{X^r_{p_1,T_1}} \le C(T_1)$ $(m \doteq 1, 2, \cdots,)$, therefore there exists $0 < T \le T_1$ such that $u \sum\limits_{m=0}^{\infty} \|u_{m+1} - u_m\|_{X^r_{p,T}}$ is a convergence series. This implies that there exists at least $u \in X^r_{p,T}$ such that u satisfies (1.6). This completes the proof of (i).

(c) When $r = n\alpha$, (3.9) implies

$$\|u_m\|_{X^r_{p,T}} \le C_0\|\varphi\|_r + C\|u_{m-1}\|^{\alpha+1}_{X^r_{p,T}}, \quad p > \alpha+1, \quad m = 1, 2, \cdots \tag{3.22}$$

In view of Lemma 3.1, if $\|\varphi\|_r$ is sufficiently small, then the solution u which was obtained in (a) can be extended to the infinity, more explicitly $u \in X^r_{p,\infty}$. By the same reason we know that when $\|\varphi\|_r$ is sufficiently small, the solution u which was obtained in (b) can be extended to the infinity by (3.20) and (3.21). So (iii) is valid.

(d) Now we come to prove (iv) of Theorem A. Let $[0, T^*)$ be the maximal interval such that u solves (1.6) in space $C([0, T^*);\ L^p)$, then $\|u(T^*)\|_r = \infty$, otherwise, if $\|u(T^*)\|_r < \infty$, similar to the above discussion there is $T > T^*$, such that u solves (1.1) in $X^r_{p,T^*,T}$

$$u = e^{A(t-T^*)}u(T^*) + \int_{T^*}^t e^{A(t-s)}F(u, \nabla u)ds \tag{3.23}$$

by iterative method, where

$$\begin{aligned}X^r_{p,T^*,T} = & \{u | u \in C_b([T^*, T); E^p),\ \|u\|_{X^r_{p,T^*,T}} \\ = & \sup_{T^* \le t < T} (t - T^*)^\sigma \|u\|_p < \infty\}\end{aligned} \tag{3.24}$$

This is contradiction with that $[0, T^*)$ is a maximal interval. On the other hand, $\forall 0 < s < T^*$, we have $\|u(s)\|_r < \infty$. Hence u always solves in $X^r_{p,s,T}$ $(T \le T^*)$,

$$u = e^{A(t-s)}u(s) + \int_s^t e^{A(t-\tau)}F(u, \nabla u)d\tau \tag{3.25}$$

by iterative method. So if we take s sufficient close to T^* such that

$$4C_0C(T^*-s)^{\frac{1}{2}-\frac{n\alpha}{2r}}\|u(s)\|_r^\alpha < 1 \tag{3.26}$$

then $u(t)$ solves (3.25) in $[s,T^*]$. Naturally there exists constant C such that

$$(T^*-s)^{\frac{1}{2}-\frac{n\alpha}{2r}}\|u(s)\|_r^\alpha \geq C \tag{3.27}$$

which implies

$$\|u(s)\|_r \geq \frac{C}{(T^*-s)^{\frac{1}{2\alpha}-\frac{n}{2r}}}$$

Therefore we complete the proof of (iv) of Theorem A.

(e) In view of (1.9) and (1.1) we have

$$\|u\|_2^2 + \int_0^t \|\nabla u\|_2^2 ds = \|\varphi(x)\|_2^2 \tag{3.28}$$

which implies $\|u(t)\|_2 \leq \|\varphi\|_2$. Noting that $p > r = n = 2$, we have

$$\|u_m\|_{X^r_{p,T}} \leq \sup_{0<t\leq T} t^\sigma \|e^{At}\varphi\|_p + C\|u_{m-1}\|^{\alpha+1}_{X^r_{p,T}} \tag{3.29}$$

Because (For detail see [3])

$$\lim_{t\to 0} t^\sigma \|e^{At}\varphi\|_p = 0 \tag{3.30}$$

there exist $t_0 > 0$ and ε_0 such that

$$\sup_{0<t\leq t_0} t^\sigma \|e^{At}\varphi\|_p < \varepsilon_0 \tag{3.31}$$

$$4C_0\varepsilon_0 < 1 \tag{3.32}$$

So Lemma 4.1 implies that there exists a mild solution of (1.1) $u(t) \in X^r_{p,t_0}$ such that

$$u(t) = e^{-At}\varphi(x) + \int_0^t e^{-A(t-s)}F(u,\nabla u)ds, \quad 0 \leq t < t_0 \tag{3.33}$$

Noting (3.28) we have

$$\|u(t_0)\|_2 = \|\varphi\|_2, \quad \operatorname{div} u(t_0) = 0 \tag{3.34}$$

So we can continue to solve

$$u(t) = e^{-A(t-t_0)}u(t_0) + \int_{t_0}^t e^{-A(t-s)}F(u,\nabla u)ds \tag{3.35}$$

in space $X^r_{p,t_0,T}$. Similar to the proof of (3.9) we have

$$\|u_m\|_{X^r_{p,t_0,T}} \leq \sup_{0<t\leq T} (t-t_0)^\sigma \|e^{A(t-t_0)}\varphi\|_p + C\|u_{m-1}\|^{\alpha+1}_{X^r_{p,t_0,T}} \tag{3.36}$$

Note that

$$\lim_{t\to t_0}(t-t_0)^\sigma\|e^{A(t-t_0)}u(t_0)\|_p=0 \tag{3.37}$$

there exist $t_1>0$ and ε_1 such that

$$\sup_{t_0<t\le t_1}(t-t_0)^\sigma\|e^{A(t-t_0)}u(t_0)\|_p<\varepsilon_1 \tag{3.38}$$

$$4C_0\varepsilon_0<1 \tag{3.39}$$

So Lemma 4.1 implies that there exists a mild solution of (1.1) $u(t)\in X^r_{p,t_0,t_1}$. According to the same way we also can solve (1.6) in $X^r_{p,t_1,t_2},\cdots$. To obtain global solution, it is sufficient to prove $t_2-t_1=t_1-t_0$. For this purpose we only need to prove the following lemma.

Lemma 3.2 *Let* $\varphi^s\in\dot E^2(\Omega)$ *and* $\|\varphi^s\|_2\le\|\varphi\|_2$, $\sigma=\left(\frac{1}{2}-\frac{1}{p}\right)$, *then*

$$t^\sigma\|e^{At}\varphi^s\|_p\to 0\quad \textit{uniformly for } s,t\to 0$$

where $\dot E^2(\Omega)=\{\varphi|\varphi\in E^p \text{ and } \varphi|_{\partial\Omega}=0\}$.

Proof Let

$$\tilde\varphi^s=\begin{cases}\varphi^s, & x\in\Omega\\ 0, & x\in\mathbb{R}^n\backslash\overline{\Omega}\end{cases} \tag{3.40}$$

then $\tilde\varphi^s\in E^2(\mathbb{R}^n)$ such that

$$\|\tilde\varphi^s\|_2\le\|\varphi\|_2 \tag{3.41}$$

We now take

$$j(x)=\begin{cases}Ce^{\frac{-1}{1-|x|^2}}, & |x|<1\\ 0, & x\ge 1\end{cases} \tag{3.42}$$

such that $\int_{\mathbb{R}^n}j(x)dx=1$. Let $j_\delta(x)=\delta^{-2}j\left(\frac{x}{\delta}\right)$, $\tilde\varphi^s_\delta=j_\delta*\tilde\varphi^s$, we consider

$$\begin{aligned}t^\sigma\|e^{tA}\varphi^s\|_p=&t^\sigma\|e^{tA}\tilde\varphi^s\|_p\le t^\sigma\|e^{tA}(\tilde\varphi^s-\tilde\varphi^s_\delta\|_p+t^\sigma\|e^{tA}\tilde\varphi^s_\delta\|_p\\ \le&\|\tilde\varphi^s-\tilde\varphi^s_\delta\|_2+t^\sigma\|e^{tA}j_\delta)*\tilde\varphi^s\|_{H^1}\\ \le&C\|j_\delta*\tilde\varphi^s-\tilde\varphi^s\|_2+t^\sigma\left(\left\|\frac{\partial j_\delta}{\partial x}\right\|_1+\|j_\delta\|_1\right)\cdot\|\tilde\varphi^s\|_2\\ \le&C\|j_\delta*\tilde\varphi^s-\tilde\varphi^s\|_2+Ct^\sigma\|\varphi^s\|_2\end{aligned} \tag{3.43}$$

So $\forall\varepsilon>0$, noting that

$$\lim_{\delta\to 0}j_\delta*\tilde\varphi^s=\tilde\varphi^s,\quad \text{uniformly for } s \text{ in } L^2 \tag{3.44}$$

we take $\delta>0$ sufficient small such that

$$\|j_\delta*\tilde\varphi^s-\tilde\varphi^s\|_2\le\frac{\varepsilon}{2} \tag{3.45}$$

On the other hand, in view of $\sigma > 0$ there is a $\bar{t} > 0$ such that

$$Ct^{\sigma}\|\varphi\|_2 < \frac{\varepsilon}{2} \tag{3.46}$$

when $t \leq \bar{t}$. Collecting (3.43)–(3.46) yields Lemma 3.2. By Lemma 3.2 and continuing the above process in the same way we conclude $u \in C_b([0,\infty);L^p) \cap C_b((0,\infty);E^p)$. Therefore we complete the proof of Theorem A.

The Proof of Theorem B

In the exactly same way as leading to the proof of Theorem A we easily prove Theorem B by Lemma 2.4.

4. An Application to the Classical Navier-Stokes Equations

In this section we discuss and give an application of Theorem A and Theorem B to the classical Navier-Stokes equation. Now we consider the IBV problem for classical Navier-Stokes equations

$$\begin{cases} \dfrac{\partial u}{\partial t} - \Delta u + (u,\nabla)u + \nabla P = 0, & (x,t) \in \Omega \times [0,T) \\ \operatorname{div} u(\cdot,t) = 0, & (x,t) \in \Omega \times [0,T) \\ u|_{\partial\Omega} = 0 & \\ u(x,0) = \varphi(x), & x \in \Omega \end{cases} \tag{4.1}$$

Similar to Definition 2.2 we can introduce the concept of the weak solution for the IBV problem (4.1).

Definition 4.1 *A vector function $u(t) = u(t,x) = (u_1(t,x),\cdots,u_n(t,x))$ is said to be a weak solution of (4.1) if for any $C_0^\infty(\Omega)$ vector function $v(t,x) = (v_1(t,x),\cdots,v_n(t,x))$ defined on $\mathbb{R}\times\Omega$, such that $\operatorname{div}u = 0$, $v(t,x) = 0$, $t \geq T$, we have*

(a) $u(t) \in L_T^{p,q}$ *with* $p, q \geq 2$.

(b) $\int_0^T \int_\Omega < u,\, v_t + \Delta v + (\nabla v)u > dxdt = -\int_\Omega < \varphi,\, v(0,x) > dx.$

(c) $\operatorname{div}u(t,\cdot) = 0$ *in the distributions sense for almost every* $t \in [0,T)$. *Where* (∇v) *denotes* $n \times n$ *matrix* $\left(\frac{\partial v_i}{\partial x_j}\right)_{n\times n}$.

According to the discussion in Section 2, we can study the following integral equation

$$\frac{du}{dt} + A_p u = -\mathcal{P}(u,\nabla)u, \quad t > 0;\ u(0) = \varphi(x) \tag{4.2}$$

to replace the study of the problem (4.1). It is well known that $X_{p,T}^r$-solution and $L_T^{p,q}$-solution of (4.1) must be the weak solution of (4.1) [3, 8]. Let $F(u,\nabla u) = \mathcal{P}(u,\nabla)u$, then the corresponding integral equation of (4.1) or (4.2) is

$$u(t) = e^{-At}\varphi(x) - \int_0^t e^{-A(t-s)}F(u,\nabla u)ds \tag{4.3}$$

Since $\operatorname{div} u = 0$, we easily get $(u, \nabla)u = \sum_{j=1}^{n} \nabla_j(u^j u)$, therefore

$$F_i(u, \nabla u) = \mathcal{P} \sum_{i=j}^{n} \nabla_j(u^j u^i) \tag{4.4}$$

It is easy to see that F satisfies the conditions of Theorem A and Theorem B, Hence we obtain the same results for Naiver-Stokes initial boundary value problem (4.1). In particular according to regularity results of Naiver-Stokes equation [11, 12] any weak solution u is regular if $u \in L^q\ ([0,T); L^p(\Omega))$ with $\frac{2}{q} = n\Big(\frac{1}{n} - \frac{1}{p}\Big)$, $p > n$. Hence we have

Theorem C *Let $n = 2$, $\varphi \in E^2$, then for $p > 2$ there is a unique global smooth solution u of* (1.1) *such that*

$$u \in C^0([0,\infty), (W^{2,p}(\Omega))^2) \cap C^1([0,\infty), (L^p(\Omega))^2) \tag{4.5}$$

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