J.Partial Diff. Eqs. 13(2000), 59–74
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Vol.13, No.1

STABILITY AND HOPF BIFURCATION OF STATIONARY SOLUTION OF A DELAY EQUATION*

Zhou Li
(Department of Mathematics, Huazhong University of Science and Technology, Wuhan 430074, China)
Fu Yiping
(Department of Mathematics, South China University of Technology, Guangzhou 510641, China)
(Received Apr. 6, 1998)

Abstract In this paper we investigate a Logistic equation with delay and it is shown that if $b(x) > c(x)$, the stationary solution is globally asymptotically stable; if τ is small, $U(x)$ is locally stable; if $b(x) < c(x)$, there is Hopf bifurcation from $U(x)$.
Key Words Logistic equation; delay; stability; Hopf bifurcation.
1991 MR Subject Classification 35K57.
Chinese Library Classification O175.26, O175.29, O175.21.

1. Introduction

In this paper we study the following Logistic equation with instantaneous and delay effects

$$\begin{aligned} &\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + u(x,t)[a(x) - b(x)u(x,t) - c(x)u(x,t-\tau)], && \Omega \times [0,\infty) \\ &B[u](x,t) = 0, && \partial\Omega \times [0,\infty) \\ &u(x,t) = \eta(x,t), && \Omega \times [-\tau, 0] \end{aligned} \quad (1.1)$$

where $\eta \in C([-\tau,0], H_0^1[0,\pi])$, $\tau \geq 0$ is constant. The functions $a(x), b(x), c(x)$ are positive and Hölder continuous on $\overline{\Omega}$. The boundary condition is given by $Bu = u$ or $Bu = \dfrac{\partial u}{\partial n} + \gamma(x)u$ where $\gamma \in C^{1+\alpha}(\partial\Omega)$, $\gamma(x) \geq 0$ on $\partial\Omega$ and $\dfrac{\partial}{\partial n}$ denotes the outward normal derivative on $\partial\Omega$.

The problem (1.1) describes the evolution of population u subject to diffusion, having delay effects in the growth rate. The related problems when a, b, c are constants

* The project supported by National Natural Science Foundation of China.

or related ordinary differential equation have been treated extensively [1–5, and their references].

It is well known that the steady-state problem

$$\frac{\partial^2 u}{\partial x^2} + u(a(x) - b(x)u - c(x)u) = 0, \quad x \in \Omega$$
$$Bu = 0, \quad x \in \partial\Omega$$

(1) if $\lambda_1 \geq 1$, it has only the trivial solution 0 which is globally asymptotically stable with respect to every nonnegative initial function,

(2) if $\lambda_1 < 1$, it has a unique positive solution $U(x)$ which is globally asymptotically stable with respect to every nonnegative, nontrivial initial function,

where λ_1 is the smallest eigenvalue of the eigenvalue problem

$$\frac{\partial^2 \phi}{\partial x^2} + \lambda a(x)\phi = 0, \quad x \in \Omega, \quad B\phi = 0, \quad x \in \partial\Omega \tag{1.2}$$

Obviously, $U(x)$ is also a stationary solution of (1.1). However, as a solution of the delay equation (1.1), the stability of $U(x)$ is different.

The content of this paper is organized as follows. In Section 2 we show that when $b(x) > c(x)$, for any $\tau \geq 0$, the stationary solution $U(x)$ is globally asymptotically stable. In Section 3, for small τ and for any $b(x), c(x)$, it is given that $U(x)$ is linearized stable. Section 4 is devoted to the study of Hopf bifurcation from $U(x)$ as τ varies when $b(x) < c(x)$.

2. Globally Asymptotic Stability of $U(x)$ when $b(x) > c(x)$

Theorem 2.1 *Let* $L \equiv \max_{x\in\overline{\Omega}} \frac{c(x)}{b(x)}$ *and* $\tau > 0$. *If* $\lambda_1 < 1$ *and* $L < 1$, *then* $U(x)$ *is globally asymptotically stable in* (1.1) *with respect to every nonnegative initial function* $\eta(x,t)$ *with* $\eta(x,0) \equiv 0$.

Proof It is obvious that $c(x) \leq Lb(x)$ on $\overline{\Omega}$. Hence

$$c(x) \leq \frac{L}{L+1}(b(x) + c(x)), \ b(x) \geq \frac{1}{L+1}(b(x) + c(x)), \text{ on } \overline{\Omega} \tag{2.1}$$

Let U^* be the nonnegative solution of the following parabolic problem

$$\frac{\partial U^*}{\partial t} - \frac{\partial^2 U^*}{\partial x^2} = U^*(a(x) - b(x)U^*), \quad \Omega \times [0, \infty)$$
$$BU^* = 0, \quad \partial\Omega \times [0, \infty)$$
$$U^*(x, 0) = \eta(x, 0), \quad \Omega$$

Define the function $\tilde{U}$ as $\tilde{U}(x,t) = \eta(x,t)$ on $\overline{\Omega} \times [-\tau, 0]$ and $\tilde{U} = U^*$ on $\overline{\Omega} \times (0, \infty)$. Then $(\tilde{U}, 0)$ is a pair of upper and lower solutions of (1.1). Therefore, by the existence-comparison theorem [6, 7] there exists a unique solution u of (1.1) with $0 \le u \le U^*$ on $\overline{\Omega} \times [-\tau, \infty)$.

From the nonnegativity of u, we have

$$\frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} \le u(a(x) - b(x)u) \le u\Big(a(x) - \frac{1}{L+1}(b(x) + c(x))u\Big), \quad \Omega \times [0, \infty)$$

By a basic comparison argument for parabolic equations

$$u(x,t) \le (L+1)U(x,t), \quad \overline{\Omega} \times [0, \infty)$$

where $U(x,t)$ is the solution of the parabolic problem

$$\begin{aligned} &\frac{\partial U}{\partial t} - \frac{\partial^2 U}{\partial x^2} = U(a(x) - (b(x) + c(x))U), && \Omega \times [0, \infty) \\ &BU = 0, && \partial\Omega \times [0, \infty) \\ &U(x,0) = \eta(x,0), && \Omega \end{aligned}$$

Therefore

$$\overline{\lim_{t \to \infty}}[u(\cdot,t) - (L+1)U(\cdot)] \le \lim_{t \to \infty}(L+1)[U(\cdot,t) - U(\cdot)] = 0, \quad \text{in } C(\overline{\Omega}) \tag{2.2}$$

On the other hand, for each $\varepsilon > 0$, there exists a $T_\varepsilon > 0$ such that when $(x,t) \in \Omega \times (T_\varepsilon, \infty)$, $u(x,t) \le (L+1+\varepsilon)U(x)$. Hence

$$\begin{aligned} \frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} &\ge u[a(x) - b(x)u - (L+1+\varepsilon)c(x)U(x)] \\ &= u[a(x) - b(x)u - \alpha c(x)U(x) - (L+1+\varepsilon-\alpha)c(x)U(x)] \\ &\ge u\Big[a(x) - b(x)u - \alpha c(x)U(x) - (L+1+\varepsilon-\alpha)\frac{L(b(x)+c(x))}{L+1}U(x)\Big] \end{aligned}$$

where α is determined later. Then, by the same comparison argument, we have $u(x,t) \ge U_1(x,t)$ on $\overline{\Omega} \times [T_\varepsilon, \infty)$, where U_1 is the solution of the parabolic problem

$$\begin{aligned} &\frac{\partial U_1}{\partial t} - \frac{\partial^2 U_1}{\partial x_2} = U_1[a(x) - b(x)U_1 - \alpha c(x)U(x) \\ &\qquad - (L+1+\varepsilon-\alpha)\frac{L(b(x)+c(x))}{L+1}U(x)], \quad \Omega \times (T_\varepsilon, \infty) \\ &BU_1 = 0, \qquad \partial\Omega \times (T_\varepsilon, \infty) \end{aligned} \tag{2.3}$$

with

$$U_1(x, T_\varepsilon) = u(x, T_\varepsilon), \quad x \in \Omega$$

If there exists $0 < \alpha < 1$ such that

$$\alpha + \frac{L}{L+1}(L+1+\varepsilon-\alpha) = 1$$

which is equivalent to $\alpha = 1 - L^2 - L\varepsilon > 0$, then $\alpha U(x)$ is the positive stationary solution of (2.3) which is globally asymptotically stable. The arbitrariness of ε implies that for $L < 1$

$$\underline{\lim_{t\to 0}}[u(\cdot,t) - (1-L^2)U(\cdot)] \geq \underline{\lim_{t\to 0}}[U_1(\cdot,t) - (1-L^2)U(\cdot)] = 0, \quad \text{in } C(\overline{\Omega}) \tag{2.4}$$

Hence, it is known from (2.2) and (2.4) that

$$\overline{\lim_{t\to 0}}[u(\cdot,t) - (1+L)U(\cdot)] \leq 0 \leq \underline{\lim_{t\to 0}}[u(\cdot,t) - (1-L^2)U(\cdot)], \quad \text{in } C(\overline{\Omega}) \tag{2.5}$$

Assume by induction that for some integer k

$$\overline{\lim_{t\to 0}}[u(\cdot,t) - (1+L^{k-1})U(\cdot)] \leq 0 \leq \underline{\lim_{t\to 0}}[u(\cdot,t) - (1-L^k)U(\cdot)], \quad \text{in } C(\overline{\Omega}). \tag{2.6}$$

Then for any $\varepsilon > 0$, there exists a $T_\varepsilon > 0$ such that

$$\frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} \leq u[a(x) - b(x)u - (1-L^k-\varepsilon)c(x)U(x)], \quad \Omega \times (T_\varepsilon, \infty)$$

Hence for any $\beta > 1$

$$\begin{aligned}\frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} \leq & u\Big[a(x) - b(x)\Big(1 - \frac{1-L^k-\varepsilon}{\beta}\Big)u - b(x)\frac{1-L^k-\varepsilon}{\beta}u - (1-L^k-\varepsilon)c(x)U(x)\Big] \\ \leq & u\Big[a(x) - \frac{b(x)+c(x)}{L+1}\Big(1 - \frac{1-L^k-\varepsilon}{\beta}\Big)u \\ & - b(x)\frac{1-L^k-\varepsilon}{\beta}u - (1-L^k-\varepsilon)c(x)U(x)\Big]\end{aligned}$$

Then by the comparison argument we have $u(x,t) \leq U_2(x,t)$ on $\overline{\Omega} \times [T_\varepsilon, \infty)$ where U_2 is the solution of the parabolic problem

$$\begin{aligned}\frac{\partial U_2}{\partial t} - \frac{\partial^2 U_2}{\partial x_2} \leq & U_2\Big[a(x) - \frac{b(x)+c(x)}{L+1}\Big(1 - \frac{1-L^k-\varepsilon}{\beta}\Big)U_2 \\ & - b(x)\frac{1-L^k-\varepsilon}{\beta}U_2 - (1-L^k-\varepsilon)c(x)U(x)\Big], \quad \Omega \times (T_\varepsilon, \infty) \\ BU_2 = 0, & \quad \partial\Omega \times (T_\varepsilon, \infty)\end{aligned} \tag{2.7}$$

with

$$U_2(x, T_\varepsilon) = u(x, T_\varepsilon), \quad x \in \Omega$$

If there exists $\beta > 1$ such that

$$\frac{\beta}{L+1}\Big(1 - \frac{1-L^k-\varepsilon}{\beta}\Big) + (1 - L^k - \varepsilon) = 1$$

which is equivalent to $\beta = 1 + L^{k+1} + L\varepsilon$, then $\beta U(x)$ is the positive stationary solution of (2.7) which is globally asymptotically stable. Therefore, from the arbitrariness of ε, we have

$$\begin{aligned}\overline{\lim_{t\to 0}}[u(\cdot,t) - (1+L^{k+1})U(\cdot)] \le& \lim_{t\to 0}[U_2(\cdot,t) - (1+L^{k+1})U(\cdot)]\\ =&0 \quad \text{in} \qquad C(\overline{\Omega})\end{aligned} \tag{2.8}$$

Again for any $\varepsilon > 0$, there exists a $T_\varepsilon > 0$ such that

$$\begin{aligned}\frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} \ge& u\Big[a(x) - b(x)u - c(x)(1+L^{k+1}+\varepsilon)U(x)\Big]\\ \ge& u[a(x) - b(x)u - c(x)\delta U(x) - (1+L^{k+1}+\varepsilon-\delta)c(x)U(x)]\\ \ge& \Big[a(x) - b(x)u - c(x)\delta U(x) - \frac{L(1+L^{k+1}+\varepsilon-\delta}{L+1} \times (b(x)+c(x))U(x)\Big],\\ & \Omega\times(T_\varepsilon,\infty)\end{aligned} \tag{2.9}$$

Then by the comparison argument we have $u(x,t) > U_3(x,t)$ on $\overline{\Omega}\times[T_\varepsilon,\infty)$ where U_3 is the solution of the parabolic problem

$$\begin{aligned}&\frac{\partial U_3}{\partial t} - \frac{\partial^2 U_3}{\partial x^2} \le U_3\Big[a(x) - b(x)U_3 - c(x)\delta U(x)\\ &\qquad - \frac{L(1+L^{k+1}+\varepsilon-\delta)}{L+1}\Big)(b(x)+c(x))U(x)], \quad \Omega\times(T_\varepsilon,\infty)\\ &BU_3 = 0, \qquad \partial\Omega\times[T_\varepsilon,\infty)\end{aligned} \tag{2.10}$$

with $U_3(x,T_\varepsilon) = u(x,T_\varepsilon)$ in Ω. If there exists $0 < \delta < 1$ such that

$$\delta + \frac{L(1+L^{k+1}+\varepsilon-\delta)}{L+1} = 1$$

which is equivalent to $\delta = 1 - L^{k+2} - L\varepsilon$, then $\delta U(x)$ is the positive stationary solution of (2.10) which is globally asymptotically stable. Since ε is arbitrarily small, we have

$$\begin{aligned}\underline{\lim_{t\to 0}}[u(\cdot,t) - (1-L^{k+2})U(\cdot)] \ge& \lim_{t\to 0}[U_3(\cdot,t) - (1-L^{k+2})U(\cdot)]\\ =&0, \quad \text{in } C(\overline{\Omega})\end{aligned}$$

The induction argument as above shows that the relation(2.6) holds for any positive even integer k. Letting $k \to \infty$ in (2.6) yields

$$\lim_{t\to\infty}[u(\cdot,t) - U(\cdot)] = 0, \quad \text{uniformly on } \overline{\Omega}$$

Then Theorem 2.1 is proven.

3. The Linearized Stability of $U(x)$ when τ Is Small

In this section, we assume that the boundary condition is Dirichlet boundary condition. That is, we consider

$$\begin{aligned} &\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + u[a(x) - b(x)u - c(x)u_\tau], && \Omega\times[0,\infty) \\ &u = 0, && \partial\Omega\times[0,\infty) \\ &u(x,t) = \eta(x,t), && \Omega\times[-\tau,0] \end{aligned} \tag{3.1}$$

It is easily seen that the linearized equation of (3.1) at $U(x)$ is

$$\begin{aligned} &\frac{\partial V}{\partial t} = \frac{\partial^2 V}{\partial x^2} + [a(x) - (2b(x)+c(x))U(x)]V - c(x)U(x)V_\tau, && \Omega\times[0,\infty) \\ &V = 0, && \partial\Omega\times[0,\infty) \\ &V(x,t) = \psi(x,t), && \Omega\times[-\tau,0] \end{aligned} \tag{3.2}$$

where $\psi \in C([-\tau,0], L^2(\Omega)) \equiv C$.

If we introduce the operator $A : D(A) \to X$ defined by

$$A = \frac{\partial^2}{\partial x^2} + [a(x) - (2b(x)+c(x))U(x)]$$

with domain $D(A) = H_0^2$, and set $V(t) = V(\cdot,t)$, $\psi(t) = \psi(\cdot,t)$, then (3.2) can be rewritten as

$$\begin{aligned} &\frac{dV(t)}{dt} = AV(t) - cUV(t-\tau), \quad t > 0 \\ &V(t) = \psi(t), \quad t \in [-\tau,0], \ \psi \in C \end{aligned} \tag{3.3}$$

with A an infinitesimal generator of a compact C_0-semigroup [4]. The study of the stability of U therefore leads to the study of the eigenvalue problem

$$\Big(\frac{\partial^2}{\partial x^2} + [a(x) - (2b(x)+c(x)(1+e^{-\lambda\tau}))]U(x)\Big)y = \lambda y, \quad 0 \neq y \in H_0^2 \tag{3.4}$$

or the study of the point spectrum $\sigma(A_\tau)$, where A_τ is the infinitesimal generator of the semigroup induced by the solutions of (3.3) with

$$A_\tau \psi = \dot{\psi}$$

$$D(A_\tau) = \{\psi \in C \cap C^1 : \psi(0) \in H_0^2, \quad \dot{\psi}(0) = A\psi(0) - cU\psi(-\tau)\}$$

On the other hand, we consider the eigenvalue problem

$$\Big(\frac{\partial^2}{\partial x^2} + a(x) - 2(b(x) + c(x))U(x)\Big)y = \lambda y, \quad 0 \neq y \in H_0^2 \tag{3.5}$$

which corresponds with the equation (1.1) without delay.

We can get the following result through the comparison of the (3.4) with (3.5).

Theorem 3.1 *Let $\lambda_1 < 1$. Then for all $\tau \in [0, \tau^*)$, $U(x)$ is locally asymptotically stable, where τ^* satisfies $\tau^* \max_{x\in\overline{\Omega}}(c(x)U(x)) = 1$.*

Proof First, note that all the eigenvalues of (3.5) are negative by Theorem 2.1. Moreover, zero is eigenvalue of (3.4) if and only if it is an eigenvalue of (3.5). Since the eigenvalue of A_τ depends continuously on τ, it follows that the only way in which stability can change is having some complex eigenvalues crossing the imaginary axis on the complex plane as τ is increased. It is thus sufficient to prove that, under the hypothesis of the theorem, all nonreal eigenvalues have negative real parts.

Assume that λ is a complex eigenvalue of (3.4) with eigenfunction y. Then $\overline{\lambda}$ is also an eigenvalue with eigenfunction $\bar{y}$. Combining the two equations for (λ, y) and $(\overline{\lambda}, \bar{y})$ gives

$$(e^{-\lambda\tau} - e^{-\bar{\lambda}\tau}) \int_\Omega c(x)U(x)|y|^2 dx = (\lambda - \bar{\lambda}) \int_\Omega |y|^2 dx$$

Normalizing the eigenfunction y such that $\int_\Omega |y|^2 dx = 1$ and separating λ into real and imaginary parts, $\lambda = \alpha + \beta i$, give

$$-e^{-\alpha\tau} \sin(\beta\tau) \int_\Omega c(x)U(x)|y|^2 dx = \beta$$

From this relation it follows that

$$e^{\alpha\tau} \leq \tau\Big|\frac{\sin\beta\tau}{\beta\tau}\Big| \int_\Omega c(x)U(x)|y|^2 dx < \tau^* \max_{x\in\overline{\Omega}} |c(x)U(x)| = 1 \tag{3.6}$$

as long as $\beta \neq 0$. Therefore (3.6) yields $\alpha < 0$ and the result follows.

4. Hopf Bifurcation from $U(x)$ when $b(x) < c(x)$

In the rest of this work we set

$$k \equiv \int_\Omega a(x)dx, \quad \text{and } a_1(x) = \frac{a(x)}{k}$$

then $a(x) = ka_1(x)$, $\int_\Omega a_1(x)dx = 1$. Under this notation, (1.1) is equivalent to

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + u(ka_1(x) - b(x)u - c(x)u_\tau), \quad \Omega \times [0, \infty)$$

$$u = 0, \qquad [0, \infty) \tag{4.1}$$
$$u(x,t) = \eta(x,t), \qquad \Omega \times [-\tau, 0]$$

where we still denote $a_1(x)$ by $a(x)$ for convenience.

For the form of the equation (4.1), the eigenvalue problem corresponding to (1.2) becomes

$$\frac{\partial^2 \phi}{\partial x^2} + \lambda(k) k a(x) \phi = 0, \quad \text{in } \Omega, \ B\phi = 0, \text{ on } \partial\Omega$$

We assume $k_1 > 0$ is such that $\lambda_1(k_1) = 1$ and ϕ_1 is the corresponding positive eigenfunction with $\|\phi_1\|_{L^2} = 1$. Then it follows that the equation (4.1) has a unique positive stationary solution U_k if $k > k_1$ and has only zero solution if $k \le k_1$.

4.1. Eigenvalue Problems

Since the eigenvalues of A_τ depend continuously on τ, those values of τ for which $\sigma(A_\tau)$ contain a pure imaginary eigenvalue will play a key role in the analysis of the stability and bifurcation of periodic solutions. Furthermore, it is obvious that A_τ has an imaginary eigenvalue $\lambda = i\gamma$ $(\gamma \ne 0)$ for some $\tau > 0$ if and only if

$$\Big\{\frac{\partial^2}{\partial x^2} + k a(x) - 2b(x)U_k(x) - c(x)U_k(x)(1 + e^{-i\theta}) - i\gamma\Big\} y = 0 \tag{4.2}$$

is solvable for some value of $\gamma > 0$, $\theta \in [0, 2\pi)$. If we find a pair of (γ, θ) such that (4.2) has a solution $y \ne 0$, then

$$\tau_n = \frac{\theta + 2n\pi}{\gamma}, \quad n = 0, 1, 2, \cdots$$

will possibly be the candidate at which the stability changes and the Hopf bifurcation occurs. So an interesting question is: how many pairs of $(\gamma, \theta) \in R^+ \times [0, 2\pi)$ are there such that (4.2) is solvable?

We will show that for each k such that $0 < k - k_1 \ll 1$, there is a unique pair (γ, θ) which solves (4.2). For this purpose, we first prove two lemmas which will be used to conclude our assertion.

Set a linearized operator $D : H_0^2(\Omega) \to L^2(\Omega)$ by

$$Dy = \Big(\frac{\partial^2}{\partial x^2} + k_1 a(x)\Big) y$$

Then it is clear that, denoting by $N(B)$ and $R(B)$ the null space and the range of B

$$L^2(\Omega) = N(D) \oplus R(D)$$

where $N(D) = \text{span}\,\{\phi_1\}$ and

$$R(D) = \Big\{ y \in L^2(\Omega) : \langle \phi_1, y \rangle \equiv \int_\Omega \phi_1(x) \bar{y}(x) dx = 0 \Big\}$$

By projecting the positive equilibrium U_k into $N(D)$ and $R(D)$ we can give an expression for U_k as follows.

Lemma 4.1 *There are $k^* > k_1$ and a continuously differentiable mapping $k \to (\xi_k, \alpha_k)$ from $[k_1, k^*]$ to $H_0^2 \cap R(D) \times R^+$ such that*

$$U_k = \alpha_k(k - k_1)[\phi_1 + (k - k_1)\xi_k], \quad k \in [k_1, k^*]$$
$$\alpha_1 = \frac{\langle a\phi_1, \phi_1\rangle}{\langle (b+c)\phi_1^2, \phi_1\rangle}$$

and $\xi_1 \in H_0^2$ is the unique solution of the equation

$$D\xi + \phi_1[a - \alpha_1(b+c)\phi_1] = 0, \quad \langle \phi_1, \xi\rangle = 0$$

Proof Define a mapping $f : H_0^2 \times R^2 \to L^2 \times R$ by

$$f(\xi, \alpha, k) = (D\xi + [\phi_1 + (k - k_1)\xi][a - (b+c)\alpha(\phi_1 + (k - k_1)\xi)], \langle \phi_1, \xi\rangle)$$

By the definition of ξ_1, we have that

$$f(\xi_1, \alpha_1, k_1) = (D\xi_1 + \phi_1[a - \alpha_1(b+c)\phi_1], \langle \phi_1, \xi\rangle) = 0$$

and

$$D_{(\xi,\alpha)} f(\xi_1, \alpha_1, k_1)(\xi, \alpha) = (D\xi - (b+c)\alpha\phi_1^2, \langle \phi_1, \xi\rangle)$$

From the fact that $\dim N(D) = 1$ and $(b+c)\phi_1^2 \bar{\in} R(D)$, it clearly follows that $D_{(\xi,\alpha)} f(\xi_1, \alpha_1, k_1)$ is bijective from $H_0^2 \times R^2 \to L^2 \times R$. Therefore, it follows from the implicit function theorem that there exists $k^* > k_1$ and continuously differentiable mapping $k \to (\xi_k, \alpha_k) \in H_0^2 \times R^+$ for $k \in [k_1, k^*]$ such that

$$f(\xi_k, \alpha_k, k) \equiv 0, \quad k \in [k_1, k^*]$$

An easy calculation shows that $W_k = \alpha_k(k - k_1)[\phi_1 + (k - k_1)\xi_k]$ solves the equation

$$\frac{\partial^2 U}{\partial x^2} + U(a - bU - cU) = 0$$

The uniqueness of the solution of this equation ensures that

$$U_k = W_k = \alpha_k(k - k_1)[\phi_1 + (k - k_1)\xi_k], \quad k \in [k_1, k^*] \tag{4.3}$$

Lemma 4.2 *If (γ, θ, y) solves the equation (4.2) with $0 \neq y \in H_0^2$, then $\dfrac{\gamma}{k - k_1}$ is uniformly bounded for $k \in (k_1, k^*]$, and $\gamma\langle y, y\rangle = \sin\theta\langle cU_k y, y\rangle$.*

Proof From the hypothesis we have

$$\left\langle \frac{\partial^2 y}{\partial x^2} + [ka - 2bU_k - cU_k(1 + e^{-i\theta})]y - i\gamma y, y\right\rangle = 0$$

The identification of imaginary part of the above relation and Lemma 4.1 yield that

$$\gamma\langle y,y\rangle=\sin\theta\langle cU_k y,y\rangle=\sin\theta\alpha_k(k-k_1)\langle c(\phi_1+(k-k_1)\xi_k)y,y\rangle$$

Therefore

$$\begin{aligned}\frac{|\gamma|}{k-k_1}=&\frac{|\alpha_k\sin\theta|\langle c(\phi_1+(k-k_1)\xi_k)y,y\rangle}{\|y\|^2_{L^2}}\\ \le&|\alpha_k|\max_{x\in\overline{\Omega}}|c(x)|[1+(k^*-k_1)\|\xi\|_{H_0^2}]\end{aligned}$$

The boundedness of $\dfrac{\gamma}{k-k_1}$ follows from the continuity of $k\to(\|\xi_k\|_{H_0^2},\alpha_k)$.

Now for $k\in(k_1,k^*]$, suppose (γ,θ,y) is a solution of (4.2) with $0\neq y\in H_0^2$. If we ignore a scalar factor, y can be represented as

$$\begin{aligned}&y=\beta\phi_1+(k-k_1)z,\quad\langle\phi_1,z\rangle=0,\ \beta\ge 0\\ &\|y\|^2_{L^2}=\beta^2+(k-k_1)^2\|z\|^2_{L^2}=1\end{aligned}\tag{4.4}$$

Substituting (4.3), (4.4) and $\gamma=(k-k_1)h$ into (4.2), noting that $D\phi_1=0$, we obtain the equivalent system to (4.2),

$$\begin{aligned}&g_1(z,\beta,h,\theta,k)\equiv Dz+[a-ih-(2b+c(1+e^{-i\theta}))\\ &\qquad\qquad\times\alpha_k(\phi_1+(k-k_1)\xi_k)][\beta\phi_1+(k-k_1)z]=0\\ &g_2(z)\equiv\operatorname{Re}\langle\phi_1,z\rangle=0\\ &g_3(z)\equiv\operatorname{Im}\langle\phi_1,z\rangle=0\\ &g_4(z,\beta,k)\equiv(\beta^2-1)+(k-k_1)^2\|z\|^2_{L^2}=0\end{aligned}\tag{4.5}$$

Theorem 4.1 *Let $c(x)>b(x)$, $x\in\overline{\Omega}$). If $0<k^*-k_1\ll 1$, then there is a unique continuously differentiable mapping $k\to(z_k,\beta_k,h_k,\theta_k)$ from $[k_1,k^*]$ to $H_0^2\times R^3$ such that*

$$\begin{aligned}&z_1=\xi_1+\eta_1+il_1,\quad\beta_1=1,\quad h_1=\alpha_1\sqrt{\langle c\phi_1^2,\phi_1\rangle^2-\langle b\phi_1^2,\phi_1\rangle^2}\\ &\theta_1=\cos^{-1}\Big(-\frac{\langle b\phi_1^2,\phi_1\rangle}{\langle c\phi_1^2,\phi_1\rangle}\Big)\in\Big[\frac{\pi}{2},\pi\Big)\end{aligned}$$

and $(z_k,\beta_k,h_k,\theta_k)$ solves the system (4.5) for $k\in[k_1,k^*]$, where ξ_1 is as in Lemma 4.1 and η_1,l_1 is a unique solution of the following equations respectively

$$\begin{aligned}&D\eta-(b+c\cos\theta_1)\alpha_1\phi_1^2=0,\quad\langle\phi_1,\eta\rangle=0\\ &Dl-(h-c\sin\theta_1\alpha_1\phi_1)\phi_1=0,\quad\langle\phi_1,l\rangle=0\end{aligned}$$

Moreover, if $k\in(k_1,k^*)$, and $(z^k,\beta^k,h^k,\theta^k)$ solves the system (4.5) with $h^k>0$, and $\theta^k\in[0,2\pi)$, then

$$(z^k,\beta^k,h^k,\theta^k)=(z_k,\beta_k,h_k,\theta_k)$$

Proof Define $G : H_0^2 \times R^3 \times R \to L^2 \times R^3$ by $G = (g_1, g_2, g_3, g_4)$. Then it follows from the definition of z_1, ξ_1, η_1 and l_1 that

$$g_1(z_1, \beta_1, h_1, \theta_1, k_1) = Dz_1 + [a - ih_1 - (b+c)\alpha_1\phi_1 - (b + ce^{-i\theta_1})\alpha_1\phi_1]\phi_1 = 0$$

It is also trivial that

$$g_i(z_1) = 0, \quad i = 2, 3 \quad \text{and} \quad g_4(z_1, \beta_1, k_1) = 0$$

That is

$$G(z_1, \beta_1, h_1, \theta_1, k_1) = 0$$

Next, let $T = (T_1, T_2, T_3, T_4) : H_0^2 \times R^3 \to L^2 \times R^3$ be defined by

$$T = D_{(z,\beta,h,\theta)}G(z_1, \beta_1, h_1, \theta_1, k_1)$$

with this definition we can verify that

$$T_1(z, \beta, h, \theta) = Dz + [a - ih - (2b + c(1 + e^{-i\theta_1}))\alpha_1\phi_1]\phi_1\beta - i\phi_1 h + ie^{-i\theta_1}\alpha_1 c\phi_1^2\theta$$
$$T_2(z) = \mathrm{Re}\,\langle \Phi_1, z\rangle, \quad T_3(z) = \mathrm{Im}\,\langle \phi_1, z\rangle, \ T_4(\beta) = 2\beta$$

Noting that both ϕ_1 and $c\phi_1^2$ do not belong to $R(D)$, we are able to show that T is one-to-one and $H_0^2 \times R^3 \to L_2 \times R^3$. Hence our first conclusion follows from the implicit function theorem. To obtain the second conclusion, by virtue of the uniqueness of the implicit function theorem, it is sufficient to show that

$$(z^k, \beta^k, h^k, \theta^k) \to (z_1, \beta_1, h_1, \theta_1)$$

as $k \to k_1$ in the norm of $H_0^2 \times R^3$.

First, Lemma 4.1, Lemma 4.2 and the last equation of (4.5) imply that $\{\|\xi_k\|_{H_0^2}\}$, $\{h^k\}$ are bounded and

$$\|\beta^k\phi_1 + (k - k_1)z^k\|_{L^2} = 1, \quad \text{for } k \in [k_1, k^*] \tag{4.6}$$

Therefore, there is $M > 0$ such that

$$\sup_{k\in[k_1,k^*]} \|a - ih^k - (2b + c(1 + e^{-i\theta^k}))\alpha_k(\phi_1 + (k - k_1)\xi_k)\| < M \tag{4.7}$$

Since the operator $D : H_0^2 \times R(D) \to R(D)$ has a bounded inverse, it follows that

$$z^k + D^{-1}[a - ih^k - (2b + c(1 + e^{-i\theta^k}))\alpha_k(\phi_1 + (k - k_1)\xi_k][\beta^k\phi_1 + (k - k_1)z^k] = 0$$

from (4.6) and (4.7), we have

$$\|z^k\|_{H_0^2} \le M\|D^{-1}\|$$

Hence, $\{(z^k,\beta^k,h^k,\theta^k):k\in[k_1,k^*]\}$ is precompact in $L^2\times R^3$. Let $\{z^{k_n},\beta^{k_n},h^{k_n},\theta^{k_n}\}$ be any convergent subsequence such that

$$(z^{k_n},\beta^{k_n},h^{k_n},\theta^{k_n})\to(z^1,\beta^1,h^1,\theta^1),\ k_n\to k_1 \text{ as } n\to\infty$$

We claim that $(z^1,\beta^1,h^1,\theta^1)=(z_1,\beta_1,h_1,\theta_1)$. To see this, we take the limit in $G(z^{k_n},\beta^{k_n},h^{k_n},\theta^{k_n},k_n)=0$ as $n\to\infty$ to obtain

$$\begin{aligned}&Dz^1+[a-ih^1-(2b+c(1+e^{-i\theta^1}))\alpha_1\phi_1]\beta^1\phi_1=0\\&\langle\phi_1,z^1\rangle=0,\quad ,\beta^1-1=0\end{aligned}\tag{4.8}$$

Obviously, $\beta^1=\beta_1=1$. Taking the L^2-inner product of first equation of (4.8) with ϕ_1 and noting that

$$\alpha_1=\frac{\langle a\phi_1,\phi_1\rangle}{\langle(b+c)\phi_1^2,\phi_1\rangle}$$

we arrive at

$$-ih^1+i\alpha_1\sin\theta^1\langle c\phi_1^2,\phi_1\rangle-\alpha_1\langle(b+c\cos\theta^1)\phi_1^2,\phi_1\rangle=0$$

which leads to $\theta^1=\theta_1$, $h^1=h_1$. Therefore, from (4.8) and the uniqueness of the solution in H_0^2, we have $z^1=z_1$.

We have shown that $(z^k,\beta^k,h^k,\theta^k)\to(z_1,\beta_1,h_1,\theta_1)$, as $k\to k_1$, with the convergence being in $L^2\times R^3$. However, recalling the definition of g_1 in (4.5) and the fact that $D^{-1}:R(D)\to H_0^2\cap R(D)$ is a continuous linear operator, we get the convergence in $H_0^2\times R^3$. The proof of Theorem 4.1 is completed.

From Theorem 4.1, we immediately have the following conclusion

Corollary 4.1 *If $0<k^*-k_1\ll1$, then for each $k\in(k_1,k^*)$, the eigenvalue problem (4.2) with $\gamma>0$, $\tau>0$, $\theta\in[0,2\pi)$ and $0\neq y\in H_0^2$ has a solution (γ,θ,y), or equivalently, $i\gamma\in\sigma(A_\tau)$ if and only if*

$$\gamma=\gamma_k=(k-k_1)h_k,\ \tau=\tau_{k_n}=\frac{\theta_k+2n\pi}{\gamma_k},\quad n=0,1,2,\cdots$$

and $y=cy_k$, $y_k=\beta_k\phi_1+(k-k_1)z_k$. Here c is any nonzero constant, and z_k,β_k,h_k,θ_k are defined as in Theorem 4.1.

Let D^* denote the adjoint operator of D, since D is a Fredholm operator, we have

$$D^*\phi_1^*=0,\ N(D^*)=\text{span}\{\phi_1^*\},\ \lim N(D^*)=1,\ \text{with } \langle\phi_1,\phi_1^*\rangle=1$$

Moreover, the adjoint eigenvalue problem corresponding to the eigenvalue problem (4.2) is

$$\Big\{\frac{\partial^2}{\partial x^2}+[ka(x)-2b(x)U_k(x)-c(x)U_k(x)(1+e^{i\theta})+i\gamma\Big\}y^*=0\tag{4.9}$$

By using the same method as above we can get

Theorem 4.2 *There is a k^* with $0 < k^* - k_1 \ll 1$ such that, for each $k \in (k_1, k^*)$, $(\gamma_k, \theta_k; y_k^*)$ is a solution of the eigenvalue problem (4.9), where γ_k, θ_k are defined as in Theorem 4.1, and y_k^* can also be represented as*

$$y_k^* = \beta_k^* \phi_1^* + (k - k_1) z_k^*, \quad \text{with } \beta_k^* \to 1, \text{ as } k \to k_1$$

and $\langle \phi_1^*, z_k^* \rangle = 0$, $z_k^* \in R(D^*)$.

4.2. Stability and Hopf Bifurcation from U_k

Now, we turn to the study of the stability of U_k with $k \in (k_1, k^*]$ fixed, and the delay τ treated as a bifurcation parameter. To describe the stability of U_k, it is enough to investigate how the eigenvalue $\lambda = i\gamma_k$ varies as the delay τ passes through τ_{k_n}, $n = 0, 1, 2, \cdots$. For this purpose, we first show that

Lemma 4.3 *If $0 < k^* - k_1 \ll 1$, then for each $k \in (k_1, k^*]$*

$$S_{k_n} = \langle (1 - \tau_{k_n} e^{-i\theta_k} cU_k) y_k, y_k^* \rangle \neq 0, \quad n = 0, 1, 2, \cdots \tag{4.10}$$

Proof From Lemma 4.1, Theorem 4.1, Corollary 4.1 and Theorem 4.2, we have

$$\begin{aligned} y_k &= \beta_k \phi_1 + O(k - k_1), \quad y_k^* = \beta_k^* \phi_1^* + O(k - k_1) \\ U_k &= \alpha_k (k - k_1) \phi_1 + O((k - k_1)^2), \quad \tau_{k_n} = \frac{\theta_k + 2n\pi}{(k - k_1) h_k} \end{aligned}$$

and $\theta_k \to \theta_1 \in \left[\frac{\pi}{2}, \pi\right)$, $h_k \to h_1 > 0$, $\beta_k \to 1$, $\beta_k^* \to 1$, $\alpha_k \to \alpha_1$ as $k \to k_1$. Let

$$t_k = \operatorname{Arg} \langle y_k, y_k^* \rangle, \quad -\pi < t_k \le \pi$$

then

$$|\langle y_k, y_k^* \rangle| e^{it_k} = \langle y_k, y_k^* \rangle \to \langle \phi_1, \phi_1^* \rangle = 1, \quad \text{as } k \to k_1$$

Hence $t_k \to 0$ as $k \to k_1$. Furthermore, we have

$$\begin{aligned} \frac{\langle e^{-i(t_k + \theta_k)} cU_k y_k, y_k^* \rangle}{k - k_1} &= \langle e^{-i(t_k + \theta_k)} [c\alpha_k \beta_k \phi_1^2 + O(k - k_1)], \beta_k^* \phi_1^* + O(k - k_1) \rangle \\ &\to \langle e^{-i\theta_1} \alpha_1 c \phi_1^2, \phi_1^* \rangle \\ &= (\cos\theta_1 - i \sin\theta_1) \alpha_1 \langle c\phi_1^2, \phi_1^* \rangle, \quad \text{as } k \to k_1 \end{aligned} \tag{4.11}$$

Therefore, it follows from (4.11) that for $n = 0, 1, 2, \cdots$, as $k \to k_1$

$$\begin{aligned} \operatorname{Im} (e^{-it_k} \langle (1 - \tau_{k_n} e^{-i\theta_k} cU_k) y_k, y_k^* \rangle) &= -\frac{\theta_k + 2n\pi}{(k - k_1) h_k} \operatorname{Im} \langle e^{-i(t_k + \theta_k)} cU_k y_k, y_k^* \rangle \\ &\to \frac{\theta_1 + 2n\pi}{h_1} \alpha_1 \sin\theta_1 \langle c\phi_1^2, \phi_1^* \rangle \neq 0 \end{aligned}$$

and so for $0 \le k - k_1 \ll 1$ and all τ_{k_n}, $n = 0, 1, 2, \cdots$, we have $S_{k_n} \neq 0$.

Lemma 4.4 *For each* $k \in (k_1, k^*]$ $(0 < k^* - k_1 \ll 1)$ *and* $n = 0, 1, 2, \cdots, \lambda = i\gamma_k$ *is a simple eigenvalue.*

Proof First, it follows from Corollary 4.1 that

$$\dim N(A_{\tau_{k_n}} - i\gamma_k) = 1, \quad N(A_{\tau_{k_n}} - i\gamma_k) = \operatorname{span}(e^{i\gamma_k\theta} y_k)$$

Now suppose

$$\psi \in D(A_{\tau_k}) \cap D(A^2_{\tau_{k_n}}), \quad (A_{\tau_{k_n}} - i\gamma_k)^2 \psi = 0$$

it follows that

$$(A_{\tau_{k_n}} - i\gamma_k)\psi \in N(A_{\tau_{k_n}} - i\gamma_k) = \operatorname{span}(e^{i\gamma_k\theta} y_k)$$

Hence there is a constant c_1 such that

$$(A_{\tau_{k_n}} - i\gamma_k)\psi = c_1 e^{i\gamma_k\theta} y_k$$

or

$$\begin{aligned} &\dot\psi(\theta) = i\gamma_k\psi(\theta) + c_1 e^{i\gamma_k\theta} y_k, \quad \theta \in [-\tau_{k_n}, 0) \\ &\dot\psi(0) = \frac{\partial^2}{\partial x^2}\psi(0) + [ka(x) - (2b(x) + c(x))U_k(x)]\psi(0) - c(x)U_k(x)\psi(-\tau_{k_n}) \end{aligned} \tag{4.12}$$

The first equation of (4.12) gives

$$\dot\psi(0) = i\gamma_k\psi(0) + c_1 y_k, \quad \psi(\theta) = (\psi(0) + c_1\theta y_k)e^{i\gamma_k\theta} \tag{4.13}$$

Substituting (4.13) into (4.12) we have

$$\begin{aligned} &\Big\{\frac{\partial^2}{\partial x^2} + [ka(x) - (2b(x) + c(x))U_k(x)]\Big\}\psi(0) - i\gamma_k\psi(0) - c(x)U_k(x)e^{-i\theta_k}\psi(0) \\ &= c_1[1 - \tau_{k_n} e^{-i\theta_k} c(x)U_k(x)]y_k \end{aligned}$$

and therefore

$$\begin{aligned} 0 &= \Big\langle \psi(0), \Big\{\frac{\partial^2}{\partial x^2} + ka(x) - (2b(x) + c(x))U_k(x) - c(x)U_k(x)e^{i\theta_k} + i\gamma_k\Big\} y_k^* \Big\rangle \\ &= \Big\langle \Big\{\frac{\partial^2}{\partial x^2} + ka(x) - (2b(x) + c(x))U_k(x) - c(x)U_k(x)e^{-i\theta_k} - i\gamma_k\Big\}\psi(0), y_k^* \Big\rangle \\ &= c_1\langle [1 - \tau_{k_n} e^{-i\theta_k} U_k(x)c(x)]y_k, y_k^* \rangle \end{aligned}$$

As a consequence of Lemma 4.1, we see that $c_1 = 0$. So $(A_{\tau_{k_n}} - i\gamma_k)\psi = 0$ and $\psi \in N((A_{\tau k_n} - i\gamma_k)$. By induction we have

$$N((A_{\tau_{k_n}} - i\gamma_k)^j) = N(A_{\tau_{k_n}} - i\gamma_k), \quad j = 1, 2, \cdots, n = 0, 1, 2, \cdots$$

Therefore, $\lambda = i\gamma_k$ is a simple eigenvalue of $A_{\tau_{k_n}}$ for $n = 0, 1, 2, \cdots$.

Since $\lambda = i\gamma_k$ is a simple eigenvalue of $A_{\tau_{k_n}}$, by using the implicit function theorem it is not difficult to show that there is a neighborhood $O_{k_n} \times C_{k_n} \times H_{k_n} \subset R \times C \times H_0^2$ of $(\tau_{k_n}, i\gamma_k, y_k)$ and a continuously differential function $(\lambda, y) : O_{k_n} \to C_{k_n} \times H_{k_n}$ such that for each $\tau \in O_{k_n}$, the only eigenvalue of A_τ in C_{k_n} is $\lambda(\tau)$, and

$$\lambda(\tau_{k_n}) = i\gamma_k, \quad y(\tau_{k_n}) = y_k$$
$$\Big\{\frac{\partial^2}{\partial x^2} + [ka - (2b + c)U - ce^{-\lambda(\tau)\tau}U - \lambda(\tau)]\Big\}y(\tau) = 0, \quad \tau \in O_{k_n}$$

Differentiating the above equality with respect to τ at $\tau = \tau_{k_n}$, we have

$$\Big\{\frac{\partial^2}{\partial x^2} + [ka - 2bU_k - cU_k(1 + e^{-i\theta_k}) - i\gamma_k]\Big\}\frac{dy(\tau_{k_n})}{d\tau} + \lambda(\tau_{k_n})e^{-i\theta_k}cU_k y_k$$
$$+ (-1 + \tau_{k_n}e^{-i\theta_k}cU_k)y_{k_n}\frac{d\lambda(\tau_{k_n})}{d\tau} = 0$$

Taking the L^2-inner product of the above equality with y^*, we obtain

$$\begin{aligned}\frac{d\lambda(\tau_{k_n})}{d\tau} &= \frac{\langle i\gamma_k e^{-i\theta_k}cU_k y_k, y_k^*\rangle}{\langle(1 - \tau_{k_n}e^{-i\theta_k}cU_k)y_k, y_k^*\rangle}\\ &= \frac{1}{|S_{k_n}|^2}[i\gamma_k e^{-i(\theta_k+t_k)}\langle cU_k y_k, y_k^*\rangle|\langle y_k, y_k^*\rangle| - i\gamma_k\tau_{k_n}|\langle cU_k y_k, y_k^*\rangle|^2]\end{aligned} \tag{4.14}$$

Lemma 4.5 *For each $k \in (k_1, k^*)$ $(0 < k^* - k_1 \ll 1)$*

$$\mathrm{Re}\,\frac{d\lambda(\tau_{k_n})}{d\tau} > 0, \quad n = 0, 1, 2, \cdots$$

Proof Let t_k be defined as in Lemma 4.3, then

$$\langle y_k, y_k^*\rangle = |\langle y_k, y_k^*\rangle|e^{-it_k}$$

So from (4.14)

$$\mathrm{Re}\,\frac{d\lambda(\tau_{k_n})}{d\tau} = \frac{1}{|S_{k_n}|^2}\mathrm{Re}\,(i\gamma_k e^{-i(\theta_k+t_k)}\langle cU_k y_k, y_k^*\rangle|\langle y_k, y_k^*\rangle|) \tag{4.15}$$

Moreover

$$\begin{aligned}\frac{ie^{-i(\theta_k+t_k)}\langle cU_k y_k, y_k^*\rangle}{k - k_1} &= ie^{-i(\theta_k+t_k)}\langle c\alpha_k\beta_k\phi_1^2 + O(k - k_1), \beta_k^*\phi_1^* + O(k - k_1)\rangle\\ &\to ie^{-i\theta_1}\langle c\alpha_1\phi_1^2, \phi_1^*\rangle\\ &= (\sin\theta_1 + i\cos\theta_1)\alpha_1\langle c\phi_1^2, \phi_1^*\rangle\end{aligned} \tag{4.16}$$

It follows from (4.15) and (4.16) that if $0 < k - k_1 \ll 1$, then $\mathrm{Re}\frac{d\lambda(\tau_{k_n})}{d\tau} > 0$.

From Corollary 4.1 and Lemma 4.5 we immediately have

Theorem 4.3 *Let $b(x) < c(x)$ in $\overline{\Omega}$. For each fixed k, $0 < k - k_1 \ll 1$, the infinitesimal generator A_τ has exactly $2(n+1)$ eigenvalues with positive real part if $\tau \in (\tau_{k_n}, \tau_{k_{n+1}}]$, $n = 0, 1, 2, \cdots$, and all eigenvalues of A_τ have negative real part if $0 \leq \tau < \tau_{k_0}$.*

Theorem 4.4 *Let $b(x) < c(x)$ in $\overline{\Omega}$. The positive equilibrium U_k is locally asymptotically stable if $0 \leq \tau < \tau_{k_0}$ and unstable if $\tau > \tau_{k_0}$, and $\tau_{k_0} \max_{x \in \overline{\Omega}} |c(x)U_k(x)| \geq 1$.*

As application of the general Hopf bifurcation theorem [8], we have the final conclusion.

Theorem 4.5 *Let $b(x) < c(x)$ in $\overline{\Omega}$. For each fixed $k \in (k_1, k^*)$, a Hopf bifurcation will occur as the delay τ increasingly passes through each points τ_{k_n}, $n = 0, 1, 2, \cdots$. Specifically, for each τ_{k_n}, there is a $\delta_{k_n} > 0$ such that for each $\tau \in (\tau_{k_n}, \tau_{k_n} + \delta_{k_n})$ the equation (4.1) has a periodic solution $U_{k_n,\tau}$ near U_k with period $\approx \frac{2\pi}{\gamma_k}$. Furthermore, $U_{k_0,\tau}$ is locally asymptotically stable, and $U_{k_n,\tau}$ is unstable for $n \geq 1$.*

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