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INITIAL-BOUNDARY VALUE PROBLEM FOR THE LANDAU-LIFSHITZ SYSTEM WITH APPLIED FIELD

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Abstract In this paper, the existence and partial regularity of weak solution to the initial-boundary value problem of Landau-Lifshitz equations with applied fields in a 2D bounded domain are obtained by the penalty method.
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1. Introduction

The Landau-Lifshitz (LL) system which describes the evolution of spin fields in continuum ferromegnets bears a fundamental role in the understanding of nonequilibrium magnetism, just as the Navier-Stokes equation does in that of fluid dynamics. The LL system for a spin chain with an easy plane

$$u_t = u \times u_{xx} + u \times Ju$$

has been studied by the inverse scattering method in [1–3] where $u = (u^1, u^2, u^3)$ is the spin vector, $J = \operatorname{diag}\{J_1, J_2, J_3\}$ with $J_1 \le J_2 \le J_3$ and "×" denotes the vector cross product in R^3. More general LL system of the following form

$$u_t = u \times F_{\text{eff}} - \lambda u \times (u \times F_{\text{eff}})$$

was also studied in [4] where $F_{\text{eff}} = \nabla^2 u - 2A(u \cdot n)n + \mu B$, $n = (0, 0, 1)$, A is the anisotropy paramenter ($A > 0$, easy plane; $A < 0$, easy axis), μ is the gyromagnetic ration in Bohr magnetons, λ is the Gilbert damping constant and $\boldsymbol{B}$ is the external magnetic field.

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A lot of works contributed to the study of solutions to the Landau-Lifshitz systems

$$u_t = -\alpha_1 u \times (u \times \Delta u) + \alpha_2 u \times \Delta u$$

of 1- or 2-dimensional spin chain motion have been made by mathematicians.

In 1993, Guo and Hong [5] established the global existence and partial regularity theorems concerning the weak solutions from a 2-dimensional Riemannian manifold (without boundary) into the unit sphere S^2 with standard metric and revealed the links between the solutions and the harmonic maps. They found that the solutions have the same partial regularity as that of the harmonic map heat flow [6]. Also for $m = 2$, the uniqueness of weak solution to the initial problem satisfying energy inequality can be found in [7]. The conclusions of [5] were extended to a class of generalized Landau-lifshitz system in [8].

The existence and partial regularity results for the weak solution of the nonhomogeneous initial-boundary value problem with $m = 2$ (without applied fields) were established in [9] in which the authors introduced a method much different from before which originates from the study of Ginzburg-Landau functional [10].

In this paper, we let $\Omega \subset R^n$ $(n = 1, 2)$ be a bounded smooth domain and consider the following nonhomogeneous initial-boundary value problem

$$u_t = -u \times (u \times \Delta u) + u \times \Delta u + u \times \boldsymbol{H}(u, x, t), \quad \text{in } \Omega \times R_+ \tag{1.1}$$

$$u|_{\partial\Omega \times R_+} = \psi(x), \quad u|_{\Omega \times \{t=0\}} = \varphi(x), \quad |\varphi(x)| = 1 \tag{1.2}$$

which is a natural general form including both "easy plane" and the external field.

Because of the action of the external field $\boldsymbol{H}$, one can not expect to get the smoothness away from a set consisting of only finitely many points as for the case $\boldsymbol{H} \equiv 0$. We can only get the smoothness away from at most countably many lines in $\overline{\Omega} \times [0, \infty)$.

Our main results are Theorem 4.1 (existence and partial regularity) and Theorem 5.2 (smooth solution of 1-D problem). In this paper we denote $\Omega(t) = \Omega \times \{t\}$, $\Omega_t = \Omega \times (0, t)$, $B_r(x)$ the disk centered at x with radius r.

2. A Penalty Problem and Weak Solution to (2.1)–(1.2)

Since $|\varphi(x)| = 1$ on $\overline{\Omega}$, it is easy to verify that $|u(x, t)| \equiv 1$ and it follows from [5] that u is a solution of (1.1)–(1.2) if and only if u is a solution in the classical sense of the following system

$$\frac{1}{2}u_t - \frac{1}{2}u \times u_t = \Delta u + u|\nabla u|^2 + \frac{1}{2}u \times \boldsymbol{H} - \frac{1}{2}u \times (u \times \boldsymbol{H})$$

Therefore, it is natural to consider the following equation

$$\frac{1}{2}u_t - \frac{1}{2}u \times u_t = \Delta u + u|\nabla u|^2 + u \times \boldsymbol{f}(u, x, t) \tag{2.1}$$

and its corresponding penalty equation

$$\frac{1}{2}u_t - \frac{1}{2}u \times u_t = \Delta u + \frac{1}{\varepsilon^2}u(1-|u|^2) + u \times f(u, x, t) \tag{2.2}$$

subject to the initial-boundary condition (1.2).

For the nonhomogeneous term $f(p, x, t)$ in (2.1), we assume

(F_1) There exists a nonnegative function $g(x, t) \in L^\infty(\Omega \times R_+)$ such that

$$|f(p, x, t)| \le g(x, t), \quad \text{for } (p, x, t) \in \{R^3 : |p| \le 1\} \times \overline{\Omega} \times R_+$$

(F_2) (i) $f(p, \cdot, \cdot)$ is Hölder continuous of index α uniformly in $p \in \{R^3 : |p| \le 1\}$.

(ii) $\left|\frac{\partial f}{\partial p}(p, x, t)\right| \le h(x, t)$, for $(p, x, t) \in \{R^3 : |p| \le 1\} \times \Omega \times R_+$ where $h(x, t)$ is a nonnegative function in $L^\infty(\Omega \times (0, T))$ for any given $0 < T < +\infty$.

The aim of this section is to prove the following theorem. In the sequel, we denote by ν and τ the unit outer normal vector and unit tangential vector to $\partial\Omega$. We also put $C_f = \max\limits_{\overline{\Omega} \times R_+} g(x, t)$.

Theorem 2.1 *For any given $\varepsilon > 0$, the problem (2.2)–(1.2) admits a global smooth solution $u \in C^{2+\alpha, 1+\alpha/2}(\overline{\Omega}_T)$ $(\forall T < \infty)$ satisfying*

$$|u| \le 1 \tag{2.3}$$

and there are constants $C_1 > 0$ independent of ε and $C_2 > 0$ such that

$$\int_0^T \int_{\partial\Omega} \left|\frac{\partial u}{\partial \nu}\right|^2 \le C_1 \tag{2.4}$$

$$\|u\|_{W_2^{2,1}(\Omega_T)} \le C_2 \tag{2.5}$$

Proof It follows from standard Galerkin's method that there exists a global weak solution u to (2.2)–(1.2). Therefore the proof left over is to verify its smoothness. Note the following version of (2.2). This can be finished by proving (2.5) and applying Schauder estimate as well as the boot-strap method since we have the condition (F_2).

$$u_t = G(u)\Delta u + G(u)\frac{1}{\varepsilon^2}u(1-|u|^2) + G(u)u \times f(u, x, t) \tag{2.2$'$}$$

where

$$G(u) = \frac{1}{1+|u|^2}\begin{pmatrix} 1+(u^1)^2 & u^1u^2 - u^3 & u^1u^3 + u^2 \\ u^1u^2 + u^3 & 1+(u^2)^2 & u^2u^3 - u^1 \\ u^1u^3 - u^2 & u^2u^3 + u^1 & 1+(u^3)^2 \end{pmatrix}$$

After proving (2.3), we see that the equation (2.2)$'$ is strongly parabolic since

$$\xi G(u)\xi^T = \frac{1}{1+|u|^2}[|\xi|^2 + (u \cdot \xi)^2], \quad \forall \xi \in R^3$$

Test (2.2) by $\phi = u - \min\{1,|u|\}\frac{u}{|u|}$ and note $|u(x,0)| = 1$, $|u(x,t)| = 1$ on $\partial\Omega$ to give

$$\frac{1}{4}\partial_t \int_{|u|>1} |u|^2\Big(1-\frac{1}{|u|}\Big) + \int_{|u|>1} |\nabla u|^2(1-|u|) \le 0$$

This implies (2.3).

It remains to prove (2.4) and (2.5). First of all, testing (2.2) by u_t, we get

$$\frac{1}{2}\int_\Omega |u_t|^2 + \frac{d}{dt}\int_\Omega \Big[\frac{1}{2}|\nabla u|^2 + \frac{1}{4\varepsilon^2}(1-|u|^2)^2\Big] = \int_\Omega u_t(u\times \boldsymbol{f}(u,x,t))$$

Hence

$$\begin{aligned}\frac{1}{2}\int_0^T\int_\Omega |u_t|^2 + \int_{\Omega(T)} \Big[\frac{1}{2}|\nabla u|^2 + \frac{1}{4\varepsilon^2}(1-|u|^2)^2\Big] \\ =\frac{1}{2}\int_\Omega |\nabla\varphi|^2 + \int_0^T\int_\Omega u_t(u\times \boldsymbol{f}(u,x,t))\end{aligned} \tag{2.6}$$

It then follows from Hölder inequality and (F_1) that

$$\begin{aligned}\frac{1}{4}\int_0^T\int_\Omega |u_t|^2 + \int_{\Omega(T)} \Big[\frac{1}{2}|\nabla u|^2 + \frac{1}{4\varepsilon^2}(1-|u|^2)^2\Big] \\ \le\frac{1}{2}\int_\Omega |\nabla\varphi|^2 + \int_0^T\int_\Omega |\boldsymbol{f}(u,x,t)|^2 \le \frac{1}{2}\int_\Omega |\nabla\varphi|^2 + \int_0^T\int_\Omega g^2(x,t) \le C_T\end{aligned} \tag{2.7}$$

where C_T is independent of ε.

Let $\boldsymbol{v} = (v_1, v_2)$ be a smooth vector field defined on $\overline{\Omega}$ such that $\boldsymbol{v} = \nu$ on $\partial\Omega$. Multiply (2.2) by $\boldsymbol{v}\cdot\nabla u$ and note that

$$\int_\Omega \Delta u(\boldsymbol{v}\cdot\nabla u) = \int_{\partial\Omega}\Big|\frac{\partial u}{\partial\nu}\Big|^2 - \int_\Omega \sum_{i=1}^2 u_{x_i}(\boldsymbol{v}\cdot\nabla u)_{x_i}$$

It follows from (2.7) that

$$\begin{aligned}\int_\Omega \sum_{i=1}^2 u_{x_i}(\boldsymbol{v}\cdot\nabla u)_{x_i} &= \int_\Omega \sum_{i=1}^2 u_{x_i}(v_1 u_{x_1x_i} + v_2 u_{x_2x_i}) + \int_\Omega u_{x_i}\boldsymbol{v}_{x_i}\cdot\nabla u \\ &= \int_\Omega \sum_{i=1}^2 u_{x_i}(v_1 u_{x_1x_i} + v_2 u_{x_2x_i}) + O(1) \\ &= \frac{1}{2}\int_\Omega v_1\sum_{i=1}^2(|u_{x_i}|^2)_{x_1} + v_2\sum_{i=1}^2 v_2(|u_{x_i}|^2)_{x_2} + O(1) \\ &= \frac{1}{2}\int_{\partial\Omega}|\nabla u|^2 + O(1)\end{aligned}$$

and then

$$\int_\Omega \Delta u(\boldsymbol{v}\cdot\nabla u) = \int_{\partial\Omega}\Big|\frac{\partial u}{\partial\nu}\Big|^2 - \frac{1}{2}\int_{\partial\Omega}|\nabla u|^2 + O(1)$$

On the other hand, we have

$$\frac{1}{\varepsilon^2}\int_\Omega u(1-|u|^2)(v\cdot\nabla u)=\frac{1}{4\varepsilon^2}\int_\Omega(1-|u|^2)^2\mathrm{div}\,(v)=O(1)$$

We finally obtain

$$\begin{aligned}\int_0^T\int_{\partial\Omega}\Big[\Big|\frac{\partial u}{\partial\nu}\Big|^2-\frac{1}{2}|\nabla u|^2\Big]=&\frac{1}{2}\int_0^T\int_{\partial\Omega}\Big(\Big|\frac{\partial u}{\partial\nu}\Big|^2-\Big|\frac{\partial u}{\partial\tau}\Big|^2\Big)\\ =&\int_0^T\int_\Omega\Big(\frac{1}{2}u_t-\frac{1}{2}u\times u_t-u\times f(u,x,t)\Big)(v\cdot\nabla u)+O(1)\\ =&O(1)\end{aligned}$$

(2.4) follows since $\dfrac{\partial u}{\partial\tau}=\dfrac{\partial\psi}{\partial\tau}$ on $\partial\Omega$.

In order to prove (2.5), we derive a Bochner-type inequality. Let $A=\frac{1}{2}|\nabla u|^2$, it follows from (2.2) that

$$\begin{aligned}&\frac{1}{2}A_t-\Delta A+|D^2u|^2+\frac{2}{\varepsilon^2}|u\nabla u|^2\\ &=2A\frac{1-|u|^2}{\varepsilon^2}+\frac{1}{2}\nabla u\cdot\nabla(u\times u_t)+\nabla u\cdot\nabla(u\times f(u,x,t))\end{aligned}\tag{2.8}$$

and then

$$\frac{1}{2}A_t-\Delta A+|D^2u|^2\le 2\varepsilon^{-2}A+\frac{1}{2}\nabla u\cdot\nabla(u\times u_t)+\nabla u\cdot\nabla(u\times f(u,x,t))\tag{2.9}$$

Let $P_r=B_r(x_0)\times[t_0,t_0+r^2]\subset\Omega_T$. Take a standard cut-off function $\xi=\xi(x)\in C_0^\infty(B_{2r}(x_0))$ such that $0\le\xi\le 1$ and $\xi\equiv 1$ in $B_r(x_0)$. Multiply (2.9) by ξ^2 and integrate it over P_r by part to give

$$\begin{aligned}\int_{P_r}|D^2u|^2\xi^2(x)\le&\frac{1}{2}\int_{B_{2r}(x_0)}\xi^2(x)[A(t_0+r^2)-A(t_0)]+\int_{P_r}\Big[A\Delta\xi^2\\ &+2\varepsilon^{-2}A\xi^2-\frac{1}{2}\nabla\cdot(\xi^2\nabla u)(u\times u_t)-\nabla\cdot(\xi^2\nabla u)(u\times f(u,x,t)\Big]\\ \le&C\int_\Omega A+C\int_{P_r}A+\frac{1}{2}\int_{P_r}|\Delta u|^2\xi^2+C\int_{P_r}|u_t|^2+C\int_{P_r}|f(u,x,t)|^2\end{aligned}$$

Hence, it follows from this inequality and (2.7) that

$$\int_{P_r}|D^2u|^2\xi^2\le C$$

This implies that (2.5) holds on P_r.

Next we give estimate near the boundary. Let $x_0\in\partial\Omega$. Without loss of generality we assume that $\partial\Omega$ near x_0 is flat, i.e., $\Omega\cap B_{2r}(x_0)=\{(x_1,x_2),x_2>0\}\cap B_{2r}(x_0)$. Choose the cut-off function $\xi(x)$ as above. We have

$$\int_{P_r}|D^2u|^2\xi^2(x)\le\int_{P_r}\Big[-\frac{1}{2}\frac{\partial A}{\partial t}\xi^2+\Delta A\cdot\xi^2+2\varepsilon^{-2}A\xi^2$$

$$+\frac{1}{2}\xi^2\nabla u\cdot\nabla(u\times u_t)+\xi^2\nabla u\cdot\nabla(u\times f(u,x,t))\Big]$$

The fourth term on the right-hand side of the above inequality can be treated as before since $u_t|_{\partial\Omega}=0$, the last term can be rewritten as

$$\int_{t_0}^{t_0+r^2}\int_{\partial\Omega}\xi^2(\psi\times f(\psi,x,t))\frac{\partial u}{\partial\nu}-\int_{P_r}\nabla\cdot(\xi^2\nabla u)(u\times f(u,x,t))$$

and then can be also treated as before in view of (2.4).

The second term equals to

$$\int_{P_r}A\Delta\xi^2+\int_{B_{2r}(x_0)\cap\{x_2=0\}\times[t_0,t_0+r^2]}\Big[A\frac{\partial\xi^2}{\partial x_2}-\xi^2\frac{\partial A}{\partial x_2}\Big]$$

however

$$\int_{B_{2r}(x_0)\cap\{x_2=0\}\times[t_0,t_0+r^2]}\xi^2\frac{\partial A}{\partial x_2}=-2\int_{B_{2r}(x_0)\cap\{x_2=0\}\times[t_0,t_0+r^2]}u_{x_2}(\xi^2\psi_{x_1x_1}+\xi\xi_{x_1}\psi_{x_1})$$

Hence, from (2.4) we obtain $\int_{P_r}\xi^2|D^2u|^2\le C$, and (2.5) follows.

Note that C_2 in (2.5) may depend on ε at this moment.

Lemma 2.2 *For any given $T>0$, there exists a weak solution $u\in V$ of (2.1)-(1.2) where*

$$V=\{u:\Omega\times[0,T]\to S^2|u\ \textit{is measurable and}\ \int_0^T\int_\Omega|u_t|^2dxdt+\operatorname*{esssup}_{0\le t\le T}\int_\Omega|\nabla u(\cdot,t)|^2dx<\infty\}$$

and the following identity holds

$$\frac{1}{2}\int_0^T\int_\Omega|u_t|^2+\frac{1}{2}\int_{\Omega(T)}|\nabla u|^2=\frac{1}{2}\int_\Omega|\nabla\varphi|^2+\int_0^T\int_\Omega u_t(u\times f(u,x,t))\tag{2.10}$$

Proof The existence of weak solution in V follows from (2.7), (F_1) and (F_2). Moreover, testing (2.1) by u_t and integrating by parts, we can get (2.10).

Lemma 2.3 *Let u_ε be the solutions of (2.2)-(1.2). Then we have a subsequence denoted by u_{ε_n} such that*

$$u_{\varepsilon_nt}\to u_t\ \textit{strongly in}\ L^2(0,T;L^2(\Omega))\tag{2.11}$$

$$\nabla u_{\varepsilon_n}(\cdot,t)\to\nabla u(\cdot,t)\ \textit{strongly in}\ L^2(\Omega),\forall t\ge0\tag{2.12}$$

where u is a weak solution of (2.1)-(1.2).

Proof Since we have from (2.7) that there is some sequence of u_ε, denoted by u_{ε_n}, such that

$$u_{\varepsilon_n}\to u\ \text{strongly in}\ L^2(0,T;L^2\Omega))$$

$$u_{\varepsilon_n t} \to u_t \text{ weakly in } L^2(0,T;L^2(\Omega))$$
$$\nabla u_{\varepsilon_n} \to \nabla u \text{ weakly * in } L^\infty(0,T;L^2(\Omega))$$

let $w_{\varepsilon_n} = u_{\varepsilon_n} - u$ we obtain from these relations and (2.6), (2.10) that as $\varepsilon_n \to 0$

$$\begin{aligned}
\int_0^T\int_\Omega |w_{\varepsilon_n t}|^2 + \int_\Omega |\nabla w_{\varepsilon_n}|^2 =& \int_0^T\int_\Omega [|u_t|^2 + |u_{\varepsilon_n t}|^2] \\
&+ \int_\Omega [|\nabla u|^2 + |\nabla u_{\varepsilon_n}|^2] - 2\int_0^T\int_\Omega u_{\varepsilon_n t} u_t - 2\int_\Omega \nabla u_{\varepsilon_n}\cdot \nabla u \\
\le& 2\int_0^T\int_\Omega |u_t|^2 + 2\int_\Omega |\nabla u|^2 - 2\int_0^T\int_\Omega u_{\varepsilon_n t}u_t - 2\int_\Omega \nabla u_{\varepsilon_n}\cdot\nabla u \\
&+ 2\int_0^T\int_\Omega [u_{\varepsilon_n t}(u_{\varepsilon_n}\times f(u_{\varepsilon_n},x,t)) - u_t(u\times f(u,x,t))] \\
=& 2\int_0^T\int_\Omega (u_t - u_{\varepsilon_n t})u_t + 2\int_\Omega \nabla u\cdot(\nabla u - \nabla u_{\varepsilon_n}) \\
&+ 2\int_0^T\int_\Omega [u_{\varepsilon_n t}(u_{\varepsilon_n}\times f(u_{\varepsilon_n},x,t)) - u_t(u\times f(u,x,t))] \\
=& o(1) + 2\int_0^T\int_\Omega [u_{\varepsilon_n t}(u_{\varepsilon_n}\times f(u_{\varepsilon_n},x,t)) - u_t(u\times f(u,x,t))]
\end{aligned}$$

Therefore, (2.11) and (2.12) can be proved if we have

$$\int_0^T\int_\Omega [u_{\varepsilon_n t}(u_{\varepsilon_n}\times f(u_{\varepsilon_n},x,t)) - u_t(u\times f(u,x,t))] = o(1) \tag{2.13}$$

Now we prove (2.13). In fact, the left-hand side of (2.13) equals to

$$\begin{aligned}
&\int_0^T\int_\Omega (u_{\varepsilon_n t} - u_t)(u_{\varepsilon_n}\times f(u_{\varepsilon_n},x,t)) + \int_0^T\int_\Omega u_t[u_{\varepsilon_n}\times f(u_{\varepsilon_n},x,t) - u\times f(u,x,t)] \\
&= \int_0^T\int_\Omega (u_{\varepsilon_n t} - u_t)[u_{\varepsilon_n}\times (f(u_{\varepsilon_n},x,t) - f(u,x,t))] \\
&\quad + \int_0^T\int_\Omega (u_{\varepsilon_n t} - u_t)[(u_{\varepsilon_n} - u)\times f(u,x,t)] \\
&\quad + \int_0^T\int_\Omega (u_{\varepsilon_n t} - u_t)[u\times f(u,x,t)] \\
&\quad + \int_0^T\int_\Omega u_t(u_{\varepsilon_n} - u)\times f(u_{\varepsilon_n},x,t) \\
&\quad + \int_0^T\int_\Omega u_t[u\times (f(u_{\varepsilon_n},x,t) - f(u,x,t))] \\
&=: I_1 + I_2 + I_3 + I_4 + I_5
\end{aligned}$$

For I_1, we have from (2.7) and (F_2) that

$$|I_1| \le \Big\|\frac{\partial f}{\partial p}(u_{\varepsilon_n},x,t)\Big\|_{L^\infty(\Omega_T)}\Big\{\int_0^T\int_\Omega (|u_{\varepsilon_n t}|^2 + |u_t|^2)\Big\}^{1/2}\Big\{\int_0^T\int_\Omega |u_{\varepsilon_n} - u|^2\Big\}^{1/2}$$

$$=o(1)$$

since we have $u_{\varepsilon_n} \to u$ strongly in $L^2(\Omega_T)$. The estimates for the other terms can be done in the similar manner.

3. Estimates Uniformly in ε

Lemma 3.1 *For any given $T > 0$, there is a constant $C > 0$ independent of ε and T such that for the solution, u_ε, of (2.2)-(1.2) obtained in Theorem 2.1, we have*

$$\sup_{t\in[0,T]} |\nabla u_\varepsilon(\cdot,t)\|_{L^\infty(\Omega)} \le C\varepsilon^{-1} \tag{3.1}$$

Proof In view of (F_1), this Lemma is a consequence of the following claim.

Claim Let $G(s)$ be a smooth matrix in $s \in R^3$ and $u \in C^{2+\alpha,1+\alpha/2}(\Omega_T)$, for any $T < \infty$, solves the following strongly parabolic system

$$u_t = G(u)\Delta u + f(u) \qquad \text{on } \Omega \times (0,\infty)$$
$$u(x,0) = 0 \text{ on } \Omega, \quad u = 0 \quad \text{on } \partial\Omega$$

Then we have for some $C > 0$ independent of T that

$$\sup_{0\le t\le T} \|\nabla u(\cdot,t)\|^2_{L^\infty(\Omega)} \le C(\|f\|_{L^\infty(\Omega\times[0,\infty))}\|u\|_{L^\infty(\Omega\times[0,\infty))} + \|u\|^2_{L^\infty(\Omega\times[0,\infty))}) \tag{$*$}$$

In fact, this claim is a parabolic version of Lemma A.1 and Lemma A.2 of [11]. Therefore, it can be proved similarly to [11]. Now we give a sketch for the interior estimate and omit the estimate near the boundary.

Assume for simplicity that $0 \in \Omega$ and set $d = \text{dist}(0,\partial\Omega)$. We shall prove $(*)$ at $x = 0$, $t = t_0$ where $t_0 \in [0,T]$.

Let $0 < \lambda \le d$ be a parameter to be determined later. It is clear that the function $v(y,\tau) = u(\lambda y, t_0 + \lambda^2\tau)$ defined on $B_1 \times [0,1]$ ($B_1 = B(0,1)$) solves

$$v_\tau = G(v)\Delta v + \lambda^2 f(\lambda y, t_0 + \lambda^2\tau) \quad \text{on } B_1 \times [0,1]$$

where $f(\cdot,\cdot) = f(u(\cdot,\cdot))$. Then we have from the standard parabolic estimates (See [12]) that

$$|\nabla v(0,\tau)| \le C(\lambda^2\|f(\lambda y, t_0+\lambda^2\tau)\|_{L^\infty(B_1\times[0,1])} + \|v\|_{L^\infty(B_1\times[0,1])}), \quad \forall 0 \le \tau \le 1$$

In particular we get

$$\lambda|\nabla u(0,t_0)| \le C(\lambda^2\|f\|_{L^\infty(\Omega\times[0,\infty))} + \|u\|_{L^\infty(\Omega\times[0,\infty))}) \tag{$*$}_1$$

If $d \ge (\|u\|_{L^\infty}/\|f\|_{L^\infty})^{1/2}$, we take $\lambda = (\|u\|_{L^\infty}/\|f\|_{L^\infty})^{1/2}$ in $(*)_1$ and obtain

$$|\nabla u(0,t_0)| \le 2C\|f\|^{1/2}_{L^\infty}\|u\|^{1/2}_{L^\infty} \tag{$*$}_2$$

If $d < (\|u\|_{L^\infty}/\|f\|_{L^\infty})^{1/2}$, we take $\lambda = d$ in $(*)_1$ and get

$$|\nabla u(0,t_0)| \le C((\|u\|_{L^\infty}\|f\|_{L^\infty})^{1/2} + \frac{1}{d}\|u\|_{L^\infty}) \qquad (*)_3$$

In $(*)_2$ and $(*)_3$, C is independent of T. Combining $(*)_2$ with $(*)_3$, we have for $0 \le t \le T$

$$|\nabla u(x,t)|^2 \le C\Big(\|u\|_{L^\infty}\|f\|_{L^\infty} + \frac{1}{\mathrm{dist}^2(x,\partial\Omega)}\|u\|^2_{L^\infty}\Big)$$

This implies that $(*)$ holds on $K \times [0,T]$ for any compact subset K of Ω.

Similarly, we can prove $(*)$ near the boundary. This finishes the proof of Lemma 3.1.

Lemma 3.2 *There exists constant $\lambda_0 > 0$, $\mu_0 > 0$ independent of ε and t such that if*

$$\frac{1}{\varepsilon^2}\int_{\Omega\cap B_{2l}} (1-|u_\varepsilon|^2)^2 \le \mu_0 \tag{3.2}$$

provided that $l/\varepsilon \ge \lambda_0$, $0 < \lambda \le 1$, then

$$|u_\varepsilon| \ge \frac{1}{2}, \quad \forall x \in \Omega \cap B_l \tag{3.3}$$

where B_l is any sphere in R^2 with radius l.

Proof In view of Lemma 3.1, the proof is just the same as that of Theorem III.3 in [10].

According to Lemma IV.1 of [10], we have, at this time, a family of disks $\{B(x_i,\lambda_0\varepsilon)\}_{i\in I}$ such that $x_i \in \Omega$, $B(x_i,\lambda_0\varepsilon/4) \cap B(x_j,\lambda_0\varepsilon/4) = \phi$ $(i \ne j)$ and $\Omega \subset \cup_{i\in I} B(x_i,\lambda_0\varepsilon)$. We call $B(x_i,\lambda_0\varepsilon)$ "good disk" if

$$\frac{1}{\varepsilon^2}\int_{\Omega\cap B(x_i,2\lambda_0\varepsilon)} (1-|u_\varepsilon|^2)^2 \le \mu_0$$

Otherwise, we call it "bad disk". Denote

$$J = \{j \in I, B(x_j,\lambda_0\varepsilon) \text{ is bad disk}\}$$

Then we can prove the following version of Lemma IV.2 of [10].

Lemma 3.3 *There exists a positive integer N independent of ε and t such that Card $J \le N$ and*

$$|u_\varepsilon| \ge \frac{1}{2} \text{ on } \Omega\backslash \cup_{j\in J} B(x_j,\lambda_0\varepsilon) \tag{3.4}$$

Moreover, we can choose (See [Section IV.2, 10]) $J' : J' \subset J$ and $\lambda \ge \lambda_0$ such that

$$\begin{cases} |x_i - x_j| \ge 8\lambda\varepsilon, \quad i \ne j,\ i,j \in J' \\ \cup_{j\in J} B(x_j,\lambda_0\varepsilon) \subset \cup_{j\in J'} B(x_j,\lambda_\varepsilon) \\ |u_\varepsilon| \ge \dfrac{1}{2} \quad \text{on } \Omega\backslash \cup_{j\in J'} B(x_j,\lambda\varepsilon) \end{cases} \tag{3.5}$$

In the following of this section, we want to derive some estimates uniformly in ε for the solution u_ε of (2.2)–(1.2).

Lemma 3.4 *Let $x_0 \in \overline{\Omega}$, $P_r = B_r(x_0) \times [t_0, t_0 + r^2]$. If $|u_\varepsilon| \geq \alpha_0 > 0$ on P_r, then there exists a constant $C > 0$ independent of ε such that*

$$\int_{P_r} |D^2 u_\varepsilon|^2 \leq C$$

Proof The proof can be done by modifying the proof of (2.5). We only give the interior estimate for simplicity.

In fact, since $|u_\varepsilon| \geq \alpha_0 > 0$ on P_r, for $A = A_\varepsilon = \frac{1}{2}|\nabla u_\varepsilon|^2$, $u = u_\varepsilon$, we have from (2.8) and (2.2) that on P_r, there holds

$$\begin{aligned}\frac{1}{2}A_t - \Delta A + |D^2 u|^2 \leq& \frac{A}{|u|}[u_t - u \times u_t - 2\Delta u - 2u \times f(u,x,t)]\\ &+ \frac{1}{2}\nabla u \cdot \nabla(u \times u_t) + \nabla u \cdot \nabla(u \times f(u,x,t))\end{aligned}$$

Therefore

$$\begin{aligned}\frac{1}{2}A_t - \Delta A + \frac{1}{2}|D^2 u|^2 \leq& c\Big(|u_t|^2 + \frac{A^2}{|u|^2} + |f(u,x,t)|^2\Big)\\ &+ \frac{1}{2}\nabla u \cdot \nabla(u \times u_t) + \nabla u \cdot \nabla(u \times f(u,x,t))\\ \leq& c(1 + A^2 + |u_t|^2) + \frac{1}{2}\nabla u \cdot \nabla(u \times u_t) + \nabla u \cdot \nabla(u \times f(u,x,t))\end{aligned}$$

with c independent of ε. Let ξ be as before. Since the last two terms can be handled as before, it suffices to estimate the term $\int_{P_r} |\nabla u|^4 \xi^2(x)$. Since $W^{1,1}(\Omega)$ can be embedded into $L^2(\Omega)$ and

$$\Big(\int \phi^2\Big)^{1/2} \leq C \int (|\nabla \phi| + |\phi|), \qquad \forall \phi \in W^{1,1}(\Omega)$$

we have by taking $\phi = \xi |\nabla u|^2$

$$\int_{P_r} |D^2 u|^2 \xi^2 \leq C + C\Big[\int_{t_0}^{t_0 + r^2} \int_\Omega \xi |\nabla u||D^2 u|\Big]^2 \leq C + C\int_{P_r} |\nabla u|^2 \int_{P_r} \xi^2 |D^2 u|^2$$

Note also that we have from (2.6), $cC\int_{P_r} |\nabla u|^2 \leq 1/8$ if r is small enough. The conclusion follows on P_r. Near the boundary we have the same estimate.

Lemma 3.5 *Let $|u_\varepsilon| \geq \alpha_0 > 0$ on $Q_{r,s} = B_r(x_0) \times [t_0 - s, t_0 + s]$. Then for any $q > 2$, there is a constant $C_q > 0$ independent of ε such that*

$$\|u_\varepsilon\|_{W_q^{2,1}(Q_{r/2,s/2})} \leq C_q \tag{3.6}$$

Proof First of all, we have from Lemma 3.4 that $\|\nabla u_\varepsilon\|_{L^q(Q_{r,s})} \le C_q$. Moreover we have for $\Psi = \frac{1}{\varepsilon^2}(1-|u_\varepsilon|^2)$ that

$$\frac{1}{2}\varepsilon^2\Psi_t - \varepsilon^2\Delta\Psi + 2\alpha_0^2\Psi \le 2|\nabla u_\varepsilon|^2 \quad \text{in } Q_{r,s} \tag{3.7}$$

Take the cut-off function $\xi(x) \in C_0^\infty(B_r(x_0))$, $\xi \equiv 1$ in $B_{r/2}(x_0)$, $\eta(t) \in C_0^\infty([t_0 - s, t_0 + s])$, $\eta \equiv 1$ in $[t_0 - s/2, t_0 + s/2]$, $|\nabla\xi| \le C/r$, $|\eta_t| \le C/s$, $0 \le \xi \le 1$, $0 \le \eta \le 1$. Multiply (3.7) by $\xi^2(x)\eta^2(t)\Psi^{q-1}$ and integrate it over $Q_{r,s}$ to give

$$\begin{aligned}\frac{\varepsilon^2}{2q}\int_{B_r}\xi^2(x)\eta^2(t)\Psi^q\Big|_{t_0-s}^{t_0+s} - \varepsilon^2\int_{Q_{r,s}}\xi^2\eta^2\Psi^{q-1}\Delta\Psi + 2\alpha_0^2\int_{Q_{r,s}}\xi^2\eta^2\Psi^q \\ \le 2\int_{Q_{r,s}}\xi^2\eta^2|\nabla u_\varepsilon|^2\Psi^{q-1} + \frac{\varepsilon^2}{q}\int_{Q_{r,s}}\xi^2\eta|\eta_t|\Psi^q\end{aligned}$$

i.e.

$$\begin{aligned}\frac{1}{2}\varepsilon^2(q-1)\int_{Q_{r,s}}\xi^2\eta^2\Psi^{q-2}|\nabla\Psi|^2 + 2\alpha_0^2\int_{Q_{r,s}}\xi^2\eta^2\Psi^q \\ \le \sigma\int_{Q_{r,s}}\xi^2\eta^2\Psi^q + C_\sigma\int_{Q_{r,s}}\xi^2\eta^2|\nabla u_\varepsilon|^{2q} \\ + \frac{\varepsilon^2}{q}\int_{Q_{r,s}}\xi^2\eta|\eta_t|\Psi^q + \frac{2\varepsilon^2}{q-1}\int_{Q_{r,s}}\eta^2|\nabla\xi|^2\Psi^q\end{aligned}$$

Setting $\sigma = \alpha_0^2$ in the above inequality, we have

$$\alpha_0^2\int_{Q_{r,s}}\xi^2\eta^2\Psi^q \le C\int_{Q_{r,s}}\xi^2\eta^2|\nabla u_\varepsilon|^{2q} + \frac{2\varepsilon^2}{q-1}\int_{Q_{r,s}}\eta^2|\nabla\xi|^2\Psi^q + \frac{\varepsilon^2}{q}\int_{Q_{r,s}}\xi^2\eta|\eta_t|\Psi^q$$

Hence

$$\alpha_0^2\int_{Q_{r,s}}\xi^2\eta^2\Psi^q \le C_q + C\varepsilon^2\int_{Q_{r,s}\setminus Q_{r/2,s/2}}\Big(\frac{1}{r^2}\Psi^q + \frac{1}{s}\Psi^q\Big)$$

Fixing r, s and taking ε small enough such that

$$\frac{C\varepsilon^2}{r^2} \le \frac{1}{4}\alpha_0^2, \quad \frac{C\varepsilon^2}{s} \le \frac{1}{4}\alpha_0^2$$

we obtain

$$\alpha_0^2\int_{Q_{r,s}}\xi^2\eta^2\Psi^q \le C_q + \frac{\alpha_0^2}{2}\int_{Q_{r,s}\setminus Q_{r/2,s/2}}\Psi^q$$

It follows from hole-filling method that

$$\int_{Q_{r/2,s/2}}\Psi^q \le C_q, \quad \forall q > 2 \tag{3.8}$$

It is concluded from (3.8) and L^q theory of parabolic systems that (3.6) holds.

Corollary 3.6 *Under the assumption of Lemma 3.5, we have for any $\gamma \in (0,1)$*

$$\|\nabla u_\varepsilon\|_{L^\infty(Q_{r,s})} \leq C \tag{3.9}$$

$$\|u_\varepsilon\|_{C^{1+\gamma,(1+\gamma)/2}(Q_{r,s})} \leq C \tag{3.10}$$

with C independent of ε.

Proof (3.9) can be proved by virtue of Lemma 3.5 and boot-strap method and (3.10) follows from Lemma 3.5 and Krylov estimates.

4. The Partial Regularity

In this section, let $\Omega \subset R^2$ be a bounded smooth domain, we shall prove one of our main results as follows.

Theorem 4.1 *There exist $0 < T_1 < T_2 < \cdots$ and $a_j^i \in \overline{\Omega}$, $j = 1, \cdots, N_i$, $i = 1, 2, \cdots, N_i \leq N$, such that, $\forall \gamma \in (0,1)$, we have for some sequence $\{u_{\varepsilon_n}\}$ that*

$$u_{\varepsilon_n} \to u \quad in\ C^{1+\gamma,(1+\gamma)/2}_{\text{loc}}(\overline{\Omega} \times [0,\infty) \backslash A)$$

where $A = \cup_i \cup_{j=1}^{N_i} (\{a_j^i\} \times [T_i, \infty))$, u is a solution of (2.1)-(1.2). Moreover, u is also in $C^{2+\alpha,1+\alpha/2}_{\text{loc}}(\overline{\Omega} \times [0,\infty) \backslash A)$.

According to Section 3, it suffices to give $C^{1+\gamma,(1+\gamma)/2}$-estimates ($\forall \gamma \in (0,1)$) uniformly in ε for u_ε on the compact subset of $(\overline{\Omega} \times [0,\infty) \backslash A)$.

Lemma 4.2 *There exists $\widetilde{T}_1 > 0$ independent of ε such that*

$$|u_\varepsilon| \geq \frac{1}{2} \quad on\ \overline{\Omega} \times [0, \widetilde{T}_1] \tag{4.1}$$

Proof $\forall x_0 \in \overline{\Omega}$, let ξ be the standard cut-off function on $B_{2R}(x_0)$ such that $0 \leq \xi \leq 1$, $\xi \equiv 1$ on $B_R(x_0)$, $|\nabla \xi| \leq \frac{1}{R}$. Test (2.2) by $\xi^2 u_{\varepsilon t}$ to give for any $\beta > 0$

$$\begin{aligned}
&\frac{1}{2}\int_0^t \int_\Omega \xi^2 |u_{\varepsilon t}|^2 + \sup_{0 \leq \tau \leq t} \left[\frac{1}{2}\int_{\Omega(\tau)} \xi^2 |\nabla u_\varepsilon|^2 + \frac{1}{4\varepsilon^2}\int_{\Omega(\tau)} \xi^2 (1 - |u_\varepsilon|^2)^2\right] \\
\leq &\frac{1}{2}\int_\Omega \xi^2 |\nabla \varphi|^2 + \beta \int_0^t \int_\Omega \xi^2 |u_{\varepsilon t}|^2 + C\beta \int_0^t \int_\Omega |\nabla \xi|^2 |\nabla u_\varepsilon|^2 \\
&+ C_\beta \int_0^t \int_\Omega \xi^2 |f(u_\varepsilon, x, t)|^2
\end{aligned}$$

Taking $\beta = 1/4$ in above inequality, we have from (F_1) that

$$\begin{aligned}
&\frac{1}{4}\int_0^t \int_\Omega \xi^2 |u_{\varepsilon t}|^2 + \sup_{0 \leq \tau \leq t} \left[\frac{1}{2}\int_{\Omega(\tau)} \xi^2 |\nabla u_\varepsilon|^2 + \frac{1}{4\varepsilon^2}\int_{\Omega(\tau)} \xi^2 (1 - |u_\varepsilon|^2)^2\right] \\
\leq &\frac{1}{2}\int_\Omega \xi^2 |\nabla \varphi|^2 + C\int_0^t \int_\Omega |\nabla \xi|^2 |\nabla u_\varepsilon|^2 + C\int_0^t \int_\Omega \xi^2 |f(u_\varepsilon, x, t)|^2
\end{aligned}$$

$$\leq \frac{1}{2}\int_{\Omega}\xi^2|\nabla\varphi|^2 + \frac{4Ct}{R^2}\int_{\Omega}|\nabla\varphi|^2 + CR^2t \tag{4.2}$$

where $C = CC_f^2$. Fixing $R = R_0 > 0$, $t = \widetilde{T}_1 > 0$ in (4.2) such that

$$\frac{1}{2}\int_{B_{2R_0}}|\nabla\varphi|^2 \leq \mu_0/8, \quad CR_0^2\widetilde{T}_1 + \frac{4C\widetilde{T}_1}{R_0}\int_{\Omega}|\nabla\varphi|^2 \leq \mu_0/8$$

we deduce

$$\sup_{0\leq t\leq \widetilde{T}_1}\frac{1}{\varepsilon^2}\int_{B_{R_0}(x_0)}(1-|u_\varepsilon|^2)^2 \leq \mu_0$$

It follows from this and Lemma 3.2 that

$$|u_\varepsilon| \geq \frac{1}{2} \quad \text{on} \quad B_{R_0}(x_0)\times[0,\widetilde{T}_1]$$

This implies the desired result.

Now we define $T_1 > \widetilde{T}_1$ by

$$T_1 = \inf\{T|T>0, \text{ there is } x_0\in\Omega \text{ such that } \liminf_{\varepsilon\to 0}|u_\varepsilon(x_0,T)| = 0\} \tag{4.3}$$

From the definition of T_1 we know that there is no bad disk on $\Omega(t)$ if $0 \leq t < T_1$ and for any $0 \leq T < T_1$ there holds $\|u_\varepsilon\|_{C^{1+\gamma,(1+\gamma)/2}(\overline{\Omega}\times[0,T])} \leq C$.

Denote the bad disks on $\Omega(T_1)$ by $\{B(x_i^\varepsilon, \lambda\varepsilon)\times\{T_1\}\}$, $i = 1,\cdots,\widetilde{N}_1$, where $\widetilde{N}_1 \leq N$ and N is determined by Lemma 3.3. Passing to a subsequence, we assume

$$x_i^{\varepsilon_n} \to a_j^1, \quad j = 1,\cdots,N_1, \ N_1 \leq \widetilde{N}_1; \ a_l^1 \neq a_k^1, \ l \neq k$$

At this time, on any compact subset of $\overline{\Omega}\times[0,T_1]\setminus\cup_{j=1}^{N_1}(\{a_j^1\}\times\{T_1\})$, we have $|u_{\varepsilon_n}| \geq 1/2$ if n is large enough. Therefore the conclusion of Corollary 3.6 holds on such compact subset.

Now we work starting from $t = T_1$. We first prove

Lemma 4.3 *For the function Ψ defined in (3.7) we have*

$$\Psi \in L^\infty_{\mathrm{loc}}(\overline{\Omega}\times[0,T_1]\setminus\cup_{j=1}^{N_1}(\{a_j^1\}\times\{T_1\})) \tag{4.4}$$

Proof The interior estimates and the estimates near the boundary are done in the following one step. Denote

$$K = B_{2r}(x_0)\times[0,T_1] \subset \Omega\times[0,T_1]\setminus\cup_{j=1}^{N_1}(\{a_j^1\}\times\{T_1\}), \quad x_0\in\Omega$$
$$\tilde{K} = (B_{2r}(x_0)\cap\Omega)\times[0,T_1]\setminus\cup_{j=1}^{N_1}(\{a_j^1\}\times\{T_1\}), \quad x_0\in\partial\Omega$$

Again, denoting by ξ the standard cut-off function of $B_{2r}(x_0)$, we get

$$\varepsilon_n^2\frac{\partial}{\partial t}(\xi\Psi) - 2\varepsilon_n^2\Delta(\xi\Psi) + \xi\Psi \leq 4\xi|\nabla u_{\varepsilon_n}|^2 - 4\varepsilon_n^2\nabla\xi\cdot\nabla\Psi - 2\varepsilon_n^2\Psi\Delta\xi \tag{4.5}$$

It follows from above that, on the compact subset K and $\tilde{K}$, the right-hand side of (4.5) is bounded uniformly in n. Then Lemma 4.3 follows from the maximum principle (See also the proof of Step A.5 of [11]).

Lemma 4.4 *There exists $\tilde{T}_2 > T_1$ independent of ε_n such that on any compact subset M of $\overline{\Omega} \times [T_1, \tilde{T}_2] \setminus \cup_{j=1}^{N_1} (\{a_j^1\} \times [T_1, \tilde{T}_2])$*

$$|u_{\varepsilon_n}| \geq \frac{1}{2} \quad on\ M \tag{4.6}$$

Proof For any $x_0 \in \overline{\Omega} \setminus \cup_{j=1}^{N_1} \{a_j^1\}$, take $R > 0$ so small that $B_{2R}(x_0)$ doesn't contain a_j^1 $(1 \leq j \leq N_1)$. Let $\xi(x)$ be the cut-off function of $B_{2R}(x_0)$ and define

$$E_\xi(u) = \frac{1}{2} \int_\Omega \xi^2 |\nabla u|^2 + \frac{1}{4\varepsilon^2} \int_\Omega \xi^2 (1 - |u|^2)^2$$

it follows from simple computations that, for $t > T_1$,

$$\begin{aligned} E_\xi(u_{\varepsilon_n}(x,t)) \leq & E_\xi(u_{\varepsilon_n}(x,T_1)) + C \int_{T_1}^t \int_\Omega |\nabla \xi|^2 |\nabla u_{\varepsilon_n}|^2 + CC_f^2 R^2 (t - T_1) \\ \leq & \frac{1}{2} \int_{B_{2R}(x_0) \times \{T_1\}} \xi^2 |\nabla u_{\varepsilon_n}|^2 + \frac{1}{4\varepsilon_n^2} \int_{B_{2R}(x_0) \times \{T_1\}} \xi^2 (1 - |u_{\varepsilon_n}|^2)^2 \\ & + \frac{C}{R^2} \int_{T_1}^t \int_{B_{2R}(x_0)} |\nabla u_{\varepsilon_n}|^2 + CR^2 (t - T_1) \end{aligned}$$

Hence we have from this inequality, (2.7), Lemma 2.3 and Lemma 4.3 that

$$E_\xi(u_{\varepsilon_n}(x,t)) \leq o(1) + CR^2 + \frac{C(t - T_1)}{R^2} + CR^2(t - T_1)$$

Now the desired conclusion follows from Lemma 3.2 if one fixed $R = R_0$, $t = \tilde{T}_2 > T_1$ such that

$$o(1) + CR_0^2 + \frac{C(\tilde{T}_2 - T_1)}{R_0^2} + CR_0^2(\tilde{T}_2 - T_1) \leq \frac{\mu_0}{4}$$

As before, we define $T_2 > T_1$ by

$$\begin{aligned} T_2 = \inf\{T | T > T_1, \text{ there is } x_0 \in \Omega \setminus \cup_{j=1}^{N_1} \{a_j^1\} \text{ such that} \\ \liminf_{\varepsilon \to 0} |u_\varepsilon(x_0, T)| = 0\} \end{aligned} \tag{4.7}$$

Denote the bad disks on $\Omega(T_2)$ by $B(x_k^\varepsilon, \lambda\varepsilon)$, $k = 1, \cdots, \tilde{N}_2$, $\tilde{N}_2 \leq N$. Passing to a further subsequence, still denoted by u_{ε_n}, we assume $x_k^{\varepsilon_n} \to a_l^2$, $l = 1, \cdots, N_2 \leq \tilde{N}_2$ with a_l^2 different from each other. On the compact subset of $\overline{\Omega} \times [T_1, T_2] \setminus (\cup_{j=1}^{N_1} \{a_j^1\} \times [T_1, T_2] \cup \cup_{l=1}^{N_2} \{a_l^2\} \times \{T_2\})$, repeating the above proof, we obtain

Lemma 4.5 *For any $\gamma \in (0,1)$ and any compact subset M of*

$$\overline{\Omega} \times [T_1, T_2] \setminus (\cup_{j=1}^{N_1} \{a_j^1\} \times [T_1, T_2] \cup \cup_{l=1}^{N_2} \{a_l^2\} \times \{T_2\})$$

we have for some constant $C > 0$ independent on n that

$$\|u_{\varepsilon_n}\|_{C^{1+\gamma,(1+\gamma)/2}(M)} \le C$$

Summing up, we have proved Theorem 4.1 by virtue of Lemma 4.5 and Schauder method.

Remark It is clear that the energy $E_\varepsilon(u_\varepsilon(x,\cdot))$ needn't be nonincreasing. Therefore, we can not get the smoothness away from finitely many points as in [13].

5. Smooth Solution of 1-Dimensional Problem

According to the above discussion, in order to get a global smooth solution, it suffices to prove that for any $T > 0$ there holds $|u_\varepsilon| > 1/2$ on $\overline{\Omega} \times [0,T]$ when ε is small enough.

Lemma 5.1 *Let $\Omega \subset R^1$ be a bounded open set. Then for any $0 < T < \infty$ we have*

$$|u_\varepsilon| \to 1 \quad as \quad \varepsilon \to 0 \tag{5.1}$$

uniformly in $(x,t) \in \overline{\Omega} \times [0,T]$.

Proof It suffices to prove (5.1) on $K \times [0,T]$, $\forall K \subset\subset \Omega$ since the proof near the boundary is similar to the above. Note that $\forall t > 0$

$$\frac{1}{\varepsilon^2}\int_\Omega (1-|u_\varepsilon|^2)^2 \le C = \int_\Omega |\nabla\varphi|^2$$

Let $x_0 \in K$ and denote $\alpha_\varepsilon(t) = |u_\varepsilon(x_0,t)|$. It follows from (3.1) that in $B(x_0,\rho)$ where $\rho < d =$disk $(K, \partial\Omega)$

$$|u_\varepsilon(x,t)| \le \alpha_\varepsilon(t) + \frac{C}{\varepsilon}\rho$$

Hence, on $B(x_0,\rho)$ there holds

$$(1-|u_\varepsilon|^2)^2 \ge (1-|u_\varepsilon|)^2 \ge \Big(1-\alpha_\varepsilon(t) - \frac{C}{\varepsilon}\rho\Big)^2$$

where we have assumed $C\rho/\varepsilon \le 1-\alpha_\varepsilon$. We conclude that

$$\rho\Big(1-\alpha_\varepsilon - \frac{C\rho}{\varepsilon}\Big)^2 \le \int_{B(x_0,\rho)} (1-|u_\varepsilon|^2)^2 \le C\varepsilon^2$$

Taking $\rho = \dfrac{\varepsilon(1-\alpha_\varepsilon)}{2C} < d$, we finally get

$$(1-\alpha_\varepsilon)^3 \le 8CC_0\varepsilon$$

with C independent of ε and t, i.e., $\alpha_\varepsilon(t) \to 1$ uniformly on $[0,T]$. Lemma 5.1 follows.

In one word, we have proved

Theorem 5.2 *Let $\Omega \subset R^1$ be a bounded open set. Then for any given $T > 0$, there exists at least one solution $u \in C^{2+\alpha,1+\alpha/2}(\overline{\Omega}\times[0,T])$ for the problem (2.1)-(1.2).*

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