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GRADIENT CATASTROPHE IN THE CLASSICAL SOLUTIONS OF NONLINEAR HYPERBOLIC SYSTEMS

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Abstract Classical solutions of hyperbolic systems, generally, collapse in finite time, even for small and smooth initial data. Here, we consider a type of these systems and prove a blow up result.
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1. Introduction

In this work, we are concerned with the Cauchy problem for one-dimensional first order quasilinear hyperbolic systems. In fact this problem has been discussed by many authors and several results concerning existence and formation of singularities have been established. Here, we consider a strictly hyperbolic system of the form:

$$\begin{cases} u_t(x,t) = a(u(x,t), v(x,t))v_x(x,t) \\ v_t(x,t) = b(u(x,t), v(x,t))u_x(x,t) \end{cases} \tag{1.1}$$

where a subscript denotes a partial derivative to the relevant variable; $x \in \mathbb{R}$, and $t > 0$.

In is indeed well known that, generally, classical solutions for such systems break down in finite time, even for smooth and small initial data. Lax [1] and MacCamy and Mizel [2] studied the system, for a depending on v only and $b \equiv 1$, and showed that the solutions blow up in a finite time, even if the initial data are smooth and small. Note in this particular case, the system is reduced to the nonlinear wave equation. For a and b depending on v only (or u only), similar results were established, for systems with dissipation, by Slemrod [3], Kosinskii [4] and Messaoudi [5].

It is interesting to mention that global existence for the system considered in [6] has been established by Nishida [6]. Also, Aregba and Hanouzet [7] and Tartar [8] have considered a class of semilinear hyperbolic system and proved some global existence and blow-up results.

In this paper, we study the system (1.1) together with initial data and show that a result similar to the one in [1], [2] can be obtained. The proof will be based on the use of characteristics and the theory of linear first-order partial differential equations.

2. Local Existence

We consider the following problem

$$u_t(x,t) = a(u(x,t), v(x,t))v_x(x,t) \tag{2.1}$$

$$v_t(x,t) = b(u(x,t), v(x,t))u_x(x,t), \quad x \in \mathbb{R}, t \geq 0 \tag{2.2}$$

$$u(x,0) = u_0(x), v(x,0) = v_0(x), \quad x \in \mathbb{R} \tag{2.3}$$

Proposition 2.1 *Assume that a and b are C^2 strictly positive functions and let u_0 and v_0 in $H^2(\mathbb{R})$ be given. Then the problem (2.1)–(2.3) has a unique local solution (u, v), on a maximal time interval $[0, T)$, satisfying*

$$u, v \in C([0,T), H^2(\mathbb{R})) \cap C^1([0,T), H^1(\mathbb{R})) \tag{2.4}$$

This result can be proved by either using a classical energy argument [9] or the nonlinear semigroup theory [10].

Remark 2.1 A similar result can also be established for a and b strictly negative.

Remark 2.2 u, v are in $C^1(\mathbb{R} \times [0,T))$ by the Sobolev embedding theorem.

Remark 2.3 A local existence result is also available for higher-dimensional hyperbolic systems (See [9, 10]).

Remark 2.4 If a and b are smooth enough and u_0 and v_0 are in $H^k(\mathbb{R})$, then the solution

$$u, v \in \bigcap_{i=1}^{k} C^i([0,T), H^{k-i}(\mathbb{R})) \tag{2.5}$$

3. Formation of Singularities

In this section, we state and prove our main result. We first begin with a lemma that gives uniform bounds on the solution.

Lemma 3.1 *Assume that a and b are as in the proposition 2.1. Then for any $\varepsilon > 0$, there exists $\delta > 0$ such that for any u_0, v_0 in $H^2(\mathbb{R})$ satisfying*

$$|u_0(x)| < \delta, \quad |v_0(x)| < \delta, \quad x \in \mathbb{R} \tag{3.1}$$

the solution satisfies

$$|u(x,t)| < \varepsilon, \quad |v(x,t)| < \varepsilon, \quad x \in \mathbb{R}, \quad t \in [0,T) \tag{3.2}$$

Proof We introduce the quantities

$$\begin{aligned} r(x,t) &:= u(x,t) + \int_0^{v(x,t)} \alpha(u(x,t),\xi)d\xi \\ s(x,t) &:= u(x,t) - \int_0^{v(x,t)} \beta(u(x,t),\xi)d\xi \end{aligned} \tag{3.3}$$

where α and β are the solutions of the linear problems:

$$\begin{cases} \alpha_y(y,z) - \mu(y,z)\alpha_z(y,z) = (\mu(y,z))_z \cdot \alpha(y,z) \\ \alpha(y,0) = \dfrac{1}{\mu(y,0)} \end{cases} \tag{3.4}$$

and

$$\begin{cases} \beta_y(y,z) + \mu(y,z)\beta_z = -(\mu(y,z))_z \cdot \beta(y,z) \\ \beta(y,0) = \dfrac{1}{\mu(y,0)} \end{cases} \tag{3.5}$$

where

$$\mu(y,z) = \sqrt{\frac{b(y,z)}{a(y,z)}} \tag{3.6}$$

These are linear first order partial differential equations. The solution can be obtained by using the method of characteristics (See e.g. [11, 12]). Since

$$\frac{1}{\mu(0,0)} = 2\gamma_1 > 0 \tag{3.7}$$

one can choose $\lambda > 0$ such that

$$\begin{cases} \gamma_1 \le \alpha(y,z) \le \gamma_2 \\ \gamma_1 \le \beta(y,z) \le \gamma_2 \end{cases} \tag{3.8}$$

for any (y,z) satisfying

$$|y| \le \lambda, \quad |z| \le \lambda \tag{3.9}$$

We also introduce the differential operators:

$$\begin{aligned} \partial_t &:= \frac{\partial}{\partial t} - \rho(u,v)\frac{\partial}{\partial x} \\ D_t &:= \frac{\partial}{\partial t} + \rho(u,v)\frac{\partial}{\partial x} \end{aligned} \tag{3.10}$$

where

$$\rho(u,v) = \sqrt{ab(u,v)} \tag{3.11}$$

and compute

$$\partial_t r = r_t - \rho r_x = \Big(1 + \int_0^v \alpha_y(u,\xi)d\xi\Big)u_t + \alpha v_t$$

$$-\rho\Big(1+\int_0^v \alpha_y(u,\xi)d\xi\Big)u_x - \rho\alpha v_x \tag{3.12}$$

By using (1.1) and the fact that α satisfies

$$1+\int_0^v \alpha_y(u,\xi)d\xi = \mu\alpha$$

we arrive at

$$\partial_t r = \alpha(\mu u_t - \rho v_x + v_t - \mu\rho u_x) = \alpha\mu(u_t - a v_x) + \alpha(v_t - b u_x) = 0 \tag{3.13}$$

Also, by using the fact that β satisfies

$$1-\int_0^v \beta_y(u,\xi)d\xi = \mu\beta$$

similar computations yield

$$D_t s = 0 \tag{3.14}$$

We thus conclude, from (3.13) and (3.14), that r and s remain constant along backward and forward characteristics respectively. Consequently; as long as a smooth solution continues to exist, r and s satisfy

$$\max|r(x,t)| = \max|r_0(x)|, \max|s(x,t)| = \max|s_0(x)| \tag{3.15}$$

We then exploit (3.3), (3.9), and (3.15) to estimate u and v as follows

$$|v(x,t)| \le \frac{\|r_0\|_\infty + \|s_0\|_\infty}{2\gamma_1}$$
$$|u(x,t)| \le \frac{(2\gamma_1+\gamma_2)\|r_0\|_\infty + \gamma_2\|s_0\|_\infty}{2\gamma_1} \tag{3.16}$$

whenever (u,v) satisfies

$$|u(x,t)| \le \lambda, \quad |v(x,t)| \le \lambda \tag{3.17}$$

We choose $\delta > 0$ small enough so that

$$\frac{\|r_0\|_\infty + \|s_0\|_\infty}{2\gamma_1} \le \min\Big(\varepsilon, \frac{\lambda}{2}\Big); \quad \frac{(2\gamma_1+\gamma_2)\|r_0\|_\infty + \|\gamma_2\|s_0\|_\infty}{2\gamma_1} \le \min\Big(\varepsilon, \frac{\lambda}{2}\Big) \tag{3.18}$$

and set

$$T_0 := \sup_{\tau \le T}\{\tau : |u(x,t)| \le \lambda,\ |v(x,t)| \le \lambda,\ t\in[0,\tau)\} \tag{3.19}$$

We have two cases. Either $T_0 = T$ which implies that (3.17) – hence (3.2) – holds for all $t \in [0,T)$, or $T_0 < T$; in this case we have

$$|u(x,t)| \le \frac{\lambda}{2}, \quad |v(x,t)| \le \frac{\lambda}{2}, \ t\in[0,T_0) \tag{3.20}$$

by virtue of (3.18). The continuity of u and v then yields the existence of T_1 such that $T_0 < T_1 \leq T$ and

$$|u(x,t)| \leq \lambda, \ |v(x,t)| \leq \lambda, \ t \in [0,T_1) \tag{3.21}$$

This contradicts the maximality of T_0. Therefore $T_0 = T$, hence (3.2) holds for all t in $[0,T)$, for the above choice of δ. This completes the proof of the lemma 3.1.

Remark 3.1 The functions a, b need not be positive on $\mathbb{R}^2$. The lemma can be established even for a, b satisfying $a(0,0)b(0,0) > 0$.

In order to state and prove our main theorem, we set

$$P(u,v) := \frac{(ab)_u}{b} + \frac{(ab)_v}{\sqrt{ab}} \tag{3.22}$$

Theorem 3.1 *Assume that a and b are as in the proposition 2.1. Assume further that*

$$P(0,0) \neq 0 \tag{3.23}$$

Then there exist initial data u_0, v_0 in $H^2(\mathbb{R})$ for which the solution of (2.1)–(2.3) blows up in finite time.

Remark 3.2 We note that (3.23) is exactly the genuine nonlinearity condition for hyperbolic systems (See e.g. [13]) and it is reduced to $a'(0) \neq 0$ for the systems studied in [1–3].

Proof We suppose that $P(0,0) > 0$. Similar proof can be established for $P(0,0) < 0$. We take an x-partial derivative of (3.13) to have

$$(\partial_t r)_x = r_{xt} - \rho r_{xx} - r_x \rho_x = 0 \tag{3.24}$$

which, in turn, implies

$$\partial_t (r_x) = r_x \rho_x = r_x(\rho_u u_x + \rho_v v_x) \tag{3.25}$$

By using

$$\begin{cases} u_x = \sqrt{\dfrac{a}{b}} \cdot \dfrac{\beta r_x + \alpha s_x}{2\alpha\beta} \\ v_x = \dfrac{\beta r_x - \alpha s_x}{2\alpha\beta} \end{cases}$$

and substituting in (3.25), we obtain

$$\partial_t r_x = \frac{\rho_u \sqrt{\dfrac{a}{b}} + \rho_v}{2\alpha} r_x^2 + \frac{\rho_u \sqrt{\dfrac{a}{b}} - \rho_v}{2\beta} r_x s_x \tag{3.26}$$

We then set

$$W := \rho^{1/2} r_x$$

and use (3.26), to get

$$\begin{aligned}\partial_t W =& \rho^{1/2}\partial_t r_x + \frac{1}{2}\rho^{-1/2} r_x \partial_t \rho \\ =& \frac{\rho_u\sqrt{\frac{a}{b}}+\rho_v}{2\alpha\rho^{1/2}} W^2 + \rho^{1/2}\frac{\rho_u\sqrt{\frac{a}{b}}-\rho_v}{2\beta} r_x s_x + \frac{1}{2}\rho^{-1/2} r_x(\rho_u\partial_t u + \rho_v \partial_t v)\end{aligned} \tag{3.27}$$

We estimate the last term in (3.27) as follows

$$\begin{aligned}\rho_u\partial_t u + \rho_v\partial_t v =& \rho_u(u_t - \rho u_x) + \rho_v(v_t - \rho v_x) \\ =& \rho_u(a v_x - \rho u_x) + \rho_v(b u_x - \rho v_x) = a\rho_u(v_x - \mu u_x) - \rho\rho_v(v_x - \mu u_x) \\ =& -\rho(\mu u_x - v_x)\left(\rho_u\sqrt{\frac{a}{b}} - \rho_v\right) = -\rho\frac{s_x}{\beta}\left(\rho_u\sqrt{\frac{a}{b}} - \rho_v\right)\end{aligned} \tag{3.28}$$

By substituting in (3.27) we obtain

$$\partial_t W = \frac{P(u,v)}{4\alpha\rho^{1/2}} W^2 \tag{3.29}$$

Therefore (3.29) shows that W (hence r_x) blows up in a finite time, if we choose initial data small enough in L^∞ norm, with derivatives satisfying

$$b(u_0,v_0)u_0' + \sqrt{a(u_0,v_0)b(u_0,v_0)}v_0' > 0 \tag{3.30}$$

Remark 3.3 The same result holds for

$$b(u_0,v_0)u_0' - \sqrt{a(u_0,v_0)b(u_0,v_0)}v_0' > 0 \tag{3.31}$$

In this case consider the evolution of s_x on the forward characteristics.

Remark 3.4 Note that the blow up occurs even for initial data with small gradient satisfying (3.30) or (3.31).

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