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UNIQUENESS OF SOLUTIONS FOR SEMICONDUCTOR EQUATIONS WITH AVALANCHE TERM*

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Abstract In this paper, we consider the initial and mixed boundary value problems for the semiconductor equations with avalanche term, the uniqueness of the weak solution for the semiconductor equation has been proved.
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1. Introduction

Let G be a bounded domain in R^n, $1 \le n \le 3$. Set $Q_T = (0,T) \times G$. Suppose that $\partial G = \Gamma_D \cup \Gamma_N$, where Γ_D and Γ_N are pairwise disjoint and Γ_D is closed and possesses positive surface measure. Moreover, $\nu(x_0)$ denotes the outer unit normal at $x_0 \in \partial G$. In this paper we study the following system of nonlinear partial differential equations that describes the transport of electrons and holes in a semiconductor device

$$\frac{\partial u_i}{\partial t} - \operatorname{div} J_i = -R(u_1,u_2) + \alpha_1(\nabla\psi)|J_1| + \alpha_2(\nabla\psi)|J_2|, \quad i = 1,2 \tag{1.1}$$

$$-\nabla \cdot (a\nabla\psi) = f + u_2 - u_1 \tag{1.2}$$

with boundary conditions

$$(u_i,\psi)|_{\Gamma_D} = (\bar{u}_i,\bar{\psi}), \quad \frac{\partial u_i}{\partial \nu}\Big|_{\Gamma_N} = \frac{\partial \psi}{\partial \nu}\Big|_{\Gamma_N} = 0 \tag{1.3}$$

and initial conditions

$$u_i(0,x) = u_{0i}(x), \quad x \in G \tag{1.4}$$

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The unknown functions u_1, u_2 and ψ denote the free electron carrier concentration, the free hole carrier concentration and the electrostatic potential. f is the net density of ionized impurities. $R(u_1, u_2) = r(u_1, u_2)(u_1u_2 - n_i^2)$, $J_i = D_i\nabla u_i + q_iu_iM_i\nabla\psi$, $q_1 = -1$, $q_2 = 1$, $D_i = D_i(x, \nabla\psi)$, $M_i = M_i(x, \nabla\psi)$. The coefficients α_1 and α_2 represent the ionization rates for electrons and holes, respectively. The term $\alpha_1(\nabla\psi)|J_1| + \alpha_2(\nabla\psi)|J_2|$ models the generation of charged particles due to impact ionization (avalanche generation of electrons and holes). In [1, 2], the authors proved the local existence of a weak solutions to (1.1)–(1.4) ($R = 0, n = 3, 2$). In [3], the existence of weak solutions is proved for space dimensions=1, 2, 3, and the uniqueness of solutions is showed in the case of one space dimension. The aim of this paper is to prove the uniqueness of solutions to (1.1)–(1.4) in the case n ($n \le 3$) space dimension.

2. Notations and Assumption

Using the standard notation we denote by $H^1(G)$ the Sobolev Space. A norm in a Banach space E is denoted by $\|\cdot\|_E$. The norm in the space L^p, $p \ge 1$ is, for short denoted by $\|\cdot\|_p$ both for the space $L^p(G)$ and $L^p(G; R^n)$, $\|\cdot\| = \|\cdot\|_2$. We introduce some function spaces $L_+^q(G) = \{v : v \in L^q(G), v \ge 0 \text{ a.e., in } G\}$, $Y = \{v : v \in H^1(G), v|_{\Gamma_D} = 0\}$, Y^* be dual space of Y. We denote by $(\cdot,\cdot)$ the scalar product in $L^2(G)$ and $\langle\cdot,\cdot\rangle$ the duality pairing between Y^* and Y. In this paper we shall work on the following assumption that

(H_1) $\bar{u}_i \in H^1(G) \cap L_+^\infty(G)$, $\bar{\psi} \in H^1(G) \cap L^\infty(G)$;

(H_2) $f = f(x) \in L^3(G)$, $n_i^2 = n_i^2(x) \in L_+^\infty(G), n_1 = n_2$;

(H_3) a is a positive constant; the diffusion coefficients $D_i = D_i(x, y)$ and the mobilities $M_i = M_i(x, y)$ satisfy the following: (i) D_i and M_i are measurable in $x \in G$, continuous in $y \in R^n$, and there are positive constants $\bar{D}_i$ and d_i such that $d_i \le D_i(x, y) \le \bar{D}_i$ for all $(x, y) \in G \times R^n$; (ii) M_i ($i = 1, 2$) are of the form $M_i(x, y) = \mu_i + B_i(x, y)$, $(x, y) \in G \times R^n$; where $\mu_i(i = 1, 2)$ are nonnegative constants, and there exists a constant B_0 such that the functions B_i satisfy

$$|B_i(x, y)y| \le B_0, \quad (x, y) \in G \times R^n$$

(H_4) $r : R_+^2 \to R_+$ is Lipschitzian;

(H_4)$'$ R is the Shockley-Read-Hall term

$R(u_1, u_2) = \dfrac{b}{r_0 + r_1u_1 + r_2u_2}(u_1u_2 - n_i^2)$, where $b \ge 0, r_j > 0$ ($j = 0, 1, 2$) are positive constants;

(H_5) $u_{0i} \in L_+^\infty(G)$, $i = 1, 2$;

(H_6) $\alpha_i(y) \in C(R^n)$, $0 \le \alpha_i(y) \le \alpha_{0i} = \text{const} < +\infty$, $y \in R^n$;

(H_7) Let (H_4) hold, and $\rho_0(u_1 + u_2) \le r(u_1, u_2) \le \rho_1(1 + u_1 + u_2)$, $u_i \in R_+, \rho_0, \rho_1$ are positive constants;

(H_8) Let (H_3), (H_6) hold, and $D_i(x, y)$ are positive constants $D_{0i}, B_i(x, y)y, \alpha_i(y)$ satisfies globally Lipschitz condition.

3. Existence of a Weak Solution to the Problem (1.1)–(1.4)

Using similar methods of the proof in reference [1-4], we can obtain

Theorem 3.1 *Let the hypotheses* (H_1)–(H_6) *be satisfied, and* (i) $\mu_1, \mu_2 > 0$ (H_7) *hold or* (ii) $\mu_1 = \mu_2 = 0$. *Then in the problem* (1.1)-(1.4) *there exists a weak solution* (u_i, ψ), *such that for every* $T > 0$

$$u_i - \bar{u}_i \in L^2(0,T;Y), \psi - \bar{\psi} \in C([0,T];Y)$$
$$u_i - \bar{u}_i \in C([0,T];L^2(G)), u_i(0,x) = u_{0i}(x) \tag{3.1}$$
$$u_i \in L^p(Q_T),\ (2 \le p < 10)\ u_i\nabla\psi \in L^2(Q_T),\ \psi \in L^\infty(0,T;L^\infty(G))$$
$$u_{it} \in L^2(0,T;Y^*) \tag{3.2}$$

and

$$\int_0^T (u_{it}, \varphi_i)dt + \int_{Q_T} (D_i\nabla u_i + q_i u_i M_i \nabla\psi)\nabla\varphi_i dxdt$$
$$= \int_{Q_T} (r(u_1,u_2)(n_i^2 - u_1u_2) + \alpha_1(\nabla\psi)|J_1| + \alpha_2(\nabla\psi)|J_2|)\varphi_i dxdt \tag{3.3}$$

for every $\varphi_i \in L^2(0,T;Y)$,

$$a\int_G \nabla\psi\cdot\nabla\varphi dx = \int_G (f + u_2 - u_1)\varphi dx, \quad \text{for every } \varphi \in Y, t \in [0,T] \tag{3.4}$$

Theorem 3.2 *Let the hypotheses* (H_1)–(H_6) *be satisfied, and* $\mu_1, \mu_2 > 0$, $(H_4)'$ *hold. Then there exists a* $T_1 > 0$, *such that in the problem* (1.1)–(1.4) *there exists a weak solution* (u_i, ψ) *on* $[0, T_1] \times G$.

4. Uniqueness of Weak Solutions

Theorem 4.1 *Under the assumptions of Theorem* 3.1 *or Theorem* 3.2, (H_8) *holds. Then the weak solution of the problem* (1.1)–(1.4) *is unique.*

Proof Suppose that (u_i, ψ) and $(\hat{u}_i, \hat{\psi})$ be two different solutions to the problem (1.1)–(1.4) satisfying (3.1)–(3.4). Then

$$a\int_G \nabla(\hat{\psi} - \psi)\cdot\nabla\varphi dx = \int_G [(\hat{u}_2 - u_2) - (\hat{u}_1 - u_1)]\varphi dx \tag{4.1}$$

for every $\varphi \in Y$, $t \in [0,T]$,

$$\|\nabla(\hat{\psi} - \psi)(t)\| \le C\Sigma\|(\hat{u}_i - u_i)(t)\| \tag{4.2}$$
$$\|(\hat{\psi} - \psi)(t)\|_\infty \le C\Sigma\|(\hat{u}_i - u_i)(t)\| \tag{4.3}$$
$$\frac{1}{2}\Sigma\|(\hat{u}_i - u_i)(t)\|^2 + \Sigma D_{0i}\int_{Q_t} |\nabla(\hat{u}_i - u_i)|^2 dxd\tau$$

$$= \Sigma(-q_i)\int_{Q_t}[(\hat{u}_i(\mu_i + B_i(x,\nabla\psi))\nabla\hat{\psi} - u_i(\mu_i + B_i(x,\nabla\psi))\psi]\cdot\nabla(\hat{u}_i - u_i)dxd\tau$$
$$+ \Sigma\int_{Q_t}(R(u_1,u_2) - R(\hat{u}_1,\hat{u}_2))(\hat{u}_i - u_i)dxd\tau$$
$$+ \Sigma\int_{Q_t}\Big[\Sigma(\alpha_i(\nabla\hat{\psi})|\hat{J}_i| - \alpha_i(\nabla\psi)|J_i|)\Big](\hat{u}_i - u_i)dxd\tau$$
$$\le \Sigma(-q_i)\mu_i\int_{Q_t}(\hat{u}_i - u_i)\nabla\psi\cdot\nabla(\hat{u}_i - u_i)dxd\tau$$
$$+ \Sigma\mu_i\int_{Q_t}\hat{u}_i|\nabla(\hat{\psi} - \psi)|\cdot|\nabla(\hat{u}_i - u_i)|dxd\tau$$
$$+ \Sigma B_0\int_{Q_t}|\hat{u}_i - u_i|\cdot|\nabla(\hat{u}_i - u_i)|dxd\tau$$
$$+ \Sigma L_1\int_{Q_t}\hat{u}_i|\nabla(\hat{\psi} - \psi)|\cdot|\nabla(\hat{u}_i - u_i)|dxd\tau$$
$$+ \Sigma C\int_{Q_t}(1 + u_1^2 + u_2^2 + \hat{u}_1^2 + \hat{u}_2^2)|\hat{u}_i - u_i)|^2dxd\tau$$
$$+ \Sigma\int_{Q_t}\Big\{\Sigma\alpha_{0i}[D_{0i}|\nabla(\hat{u}_i - u_i)| + \mu_i(\hat{u}_i|\nabla(\hat{\psi} - \psi)| + |\hat{u}_i - u_i|\cdot|\nabla\psi|)$$
$$+ B_0|\hat{u}_i - u_i| + L_1\hat{u}_i|\nabla(\hat{\psi} - \psi)|]\Big\}|\hat{u}_i - u_i|dxd\tau$$
$$+ \Sigma\int_{Q_t}\Big(\Sigma|J_i|)L_2(\nabla(\hat{\psi} - \psi)|\cdot|\hat{u}_i - u_i|dxd\tau$$
$$= E_1 + E_2 + E_3 + E_4 + E_5 + E_6 + E_7 \tag{4.4}$$

$$E_2 \le \varepsilon\Sigma\int_{Q_t}|\nabla(\hat{u}_i - u_i)|^2dxd\tau + K_1(\varepsilon)\Sigma\int_{Q_t}\hat{u}_i^2|\nabla(\hat{\psi} - \psi)|^2dxd\tau \tag{4.5}$$

$$E_3 \le \varepsilon\Sigma\int_{Q_t}|\nabla(\hat{u}_i - u_i)|^2dxd\tau + K_2(\varepsilon)\Sigma\int_{Q_t}\|(\hat{u}_i - u_i)|^2dxd\tau \tag{4.6}$$

$$E_4 \le \varepsilon\Sigma\int_{Q_t}|\nabla(\hat{u}_i - u_i)|^2dxd\tau + K_3(\varepsilon)\Sigma\int_{Q_t}\hat{u}_i^2|\nabla(\hat{\psi} - \psi)|^2dxd\tau \tag{4.7}$$

$$E_5 \le C\Sigma\int_0^t\|(1 + u_1^2 + u_2^2 + \hat{u}_1^2 + \hat{u}_2^2)\|_3\|\hat{u}_i - u_i\|\|\hat{u}_i - u_i\|_6d\tau$$
$$\le \varepsilon\Sigma\int_{Q_t}|\nabla(\hat{u}_i - u_i)|^2dxd\tau$$
$$+ K_4(\varepsilon)\int_{Q_t}\|(1 + u_1^2 + u_2^2 + \hat{u}_1^2 + \hat{u}_2^2)\|_3^2\|\hat{u}_i - u_i\|^2d\tau \tag{4.8}$$

$$E_6 \le \varepsilon\Sigma\int_{Q_t}|\nabla(\hat{u}_i - u_i)|^2dxd\tau + \varepsilon\Sigma\int_{Q_t}|\hat{u}_i - u_i|^2|\nabla\psi|^2d\tau$$
$$+ K_5(\varepsilon)\Sigma\int_{Q_t}|(\hat{u}_i - u_i)|^2dxd\tau + K_6(\varepsilon)\Sigma\int_{Q_t}\hat{u}_i^2|\nabla(\hat{\psi} - \psi)|^2dxd\tau \tag{4.9}$$

$$E_7 \le L_2\Sigma\int_0^t\|\big(\Sigma|J_i|)\|\cdot\Big(\int_G|\nabla(\hat{\psi} - \psi)|^2|\hat{u}_i - u_i|^2dx\Big)^{1/2}d\tau \tag{4.10}$$

Let $\varphi = \frac{(\hat{u}_i - u_i)^2}{1+\delta(\hat{u}_i - u_i)^2} \in Y$ in (3.4), ($\delta > 0$), we obtain

$$\begin{aligned}\int_{Q_t} (\hat{u}_i - u_i)\nabla\psi \cdot \frac{\nabla(\hat{u}_i - u_i)}{(1+\delta(\hat{u}_i - u_i)^2)^2} dx d\tau &= \frac{1}{2}\int_{Q_t} \nabla\psi \cdot \nabla \frac{(\hat{u}_i - u_i)^2}{1+\delta(\hat{u}_i - u_i)^2} dx d\tau \\ &= \frac{1}{2a}\int_{Q_t} (f + u_2 - u_1) \cdot \frac{(\hat{u}_i - u_i)^2}{1+\delta(\hat{u}_i - u_i)^2} dx d\tau \end{aligned} \tag{4.11}$$

and taking $\delta \to 0$ in (4.11), we obtain

$$\begin{aligned} E_1 \le& \Sigma\mu_i \Big| \int_{Q_t} (\hat{u}_i - u_i)\nabla\psi \cdot \nabla(\hat{u}_i - u_i) dx d\tau \Big| \\ =& \Sigma \frac{\mu_i}{2a} \Big| \int_{Q_t} (f + u_2 - u_1)(\hat{u}_i - u_i)^2 dx d\tau \Big| \\ \le& \Sigma \frac{\mu_i}{2a} \int_{Q_t} \|(|f| + u_2 + u_1)\|_3 \|(\hat{u}_i - u_i)\| \cdot \|(\hat{u}_i - u_i)\|_6 d\tau \\ \le& \varepsilon \int_{Q_t} |\nabla(\hat{u}_i - u_i)|^2 dx d\tau \\ &+ K_7(\varepsilon)\Sigma \int_0^t \|(|f| + u_2 + u_1)\|_3^2 \|(\hat{u}_i - u_i)(\tau)\|^2 d\tau \end{aligned} \tag{4.12}$$

Let $\varphi = \frac{\hat{u}_i^2}{1+\delta\hat{u}_i^2}(\hat{\psi} - \psi) \in Y$ in (4.1), it is easy to see

$$\begin{aligned}\int_{Q_t} \frac{\hat{u}_i^2}{1+\delta\hat{u}_i^2} |\nabla(\hat{\psi} - \psi)|^2 dx d\tau =& \frac{1}{a}\int_{Q_t} [(\hat{u}_2 - u_2) - (\hat{u}_1 - u_1)] \cdot \frac{\hat{u}_i^2}{1+\delta\hat{u}_i^2}(\hat{\psi} - \psi) dx d\tau \\ &- \int_{Q_t} \nabla(\hat{\psi} - \psi) \cdot \frac{2u_i \nabla u_i}{(1+\delta\hat{u}_i^2)^2}(\hat{\psi} - \psi) dx d\tau \end{aligned} \tag{4.13}$$

taking $\delta \to 0$ in (4.13), we obtain

$$\begin{aligned}\int_{Q_t} \hat{u}_i^2 |\nabla(\hat{\psi} - \psi)|^2 dx d\tau \le& K_8 \int_0^t \|(\hat{\psi} - \psi)(\tau)\|_\infty (\Sigma\|(\hat{u}_i - u_i)\|_6) \|\hat{u}_i\| \cdot \|\hat{u}_i\|_3 d\tau \\ &+ K_9 \int_0^t \|(\hat{\psi} - \psi)(\tau)\|_\infty^2 \|\nabla\hat{u}_i\|^2 d\tau \\ \le& (\sup\|\hat{u}_i(t)\|) K_{10} \int_0^t \|(\hat{\psi} - \psi)\|_\infty (\Sigma\|\nabla(\hat{u}_i - u_i)\|) \|\hat{u}_i\|_3 d\tau \\ &+ K_9 \int_0^t \|(\hat{\psi} - \psi)(\tau)\|_\infty^2 \|\nabla\hat{u}_i\|^2 d\tau \\ \le& \varepsilon\Sigma \int_0^t \|\nabla(\hat{u}_i - u_i)(\tau)\|^2 d\tau + \int_0^t h_1(\tau)(\Sigma\|(\hat{u}_i - u_i)(\tau)\|^2 d\tau \end{aligned} \tag{4.14}$$

where $h_1(\tau) \in L_+^1(0, T)$.

Let $\varphi = \frac{(\hat{u}_i - u_i)^2}{1+\delta(\hat{u}_i - u_i)^2}\psi \in Y$ in (3.4), we obtain

$$\begin{aligned}\int_{Q_t} \frac{(\hat{u}_i - u_i)^2}{1+\delta(\hat{u}_i - u_i)^2}|\psi|^2 dx d\tau =& \frac{1}{a}\int_{Q_t}(f+u_2-u_1)\frac{(\hat{u}_i - u_i)^2}{1+\delta(\hat{u}_i - u_i)^2}\psi dx d\tau \\ & - \int_{Q_t} \nabla\psi \cdot \frac{2(\hat{u}_i - u_i)\nabla(\hat{u}_i - u_i)}{(1+\delta(\hat{u}_i - u_i)^2)^2}\psi dx d\tau\end{aligned} \tag{4.15}$$

taking $\delta \to 0$, we obtain the estimate

$$\begin{aligned}&\varepsilon\Sigma\int_{Q_t}(\hat{u}_i - u_i)^2|\nabla\psi|^2 dx d\tau \\ &\le \varepsilon\Sigma\frac{2}{a}\int_0^t \|(|f|+u_2+u_1)\|_3\|\hat{u}_i - u_i\|\|\hat{u}_i - u_i\|_6\|\psi(\tau)\|_\infty d\tau \\ &\quad + 4\varepsilon\Sigma\int_0^t \|\nabla(\hat{u}_i - u_i)\|^2\|\psi(\tau)\|_\infty^2 d\tau \\ &\le \varepsilon C\Sigma\int_{Q_t}|\nabla(\hat{u}_i - u_i)|^2 dx d\tau + \int_0^t h_2(\tau)(\Sigma\|\hat{u}_i - u_i\|^2)d\tau\end{aligned} \tag{4.16}$$

where $h_2(t) \in L^1_+(0,T)$.

Let $\varphi = \frac{(\hat{u}_i - u_i)^2}{1+\delta(\hat{u}_i - u_i)^2}(\hat{\psi} - \psi) \in Y$, in (4.1), we obtain

$$\begin{aligned}&\int_G \frac{(\hat{u}_i - u_i)^2}{1+\delta(\hat{u}_i - u_i)^2}|\nabla(\hat{\psi} - \psi)|^2 dx \\ &= \frac{1}{a}\int_G[(\hat{u}_2 - u_2) - (\hat{u}_1 - u_1)]\cdot\frac{(\hat{u}_i - u_i)^2}{1+\delta(\hat{u}_i - u_i)^2}(\hat{\psi} - \psi)dx \\ &\quad - \int_G \nabla(\hat{\psi} - \psi)\cdot\frac{2(\hat{u}_i - u_i)\nabla(\hat{u}_i - u_i)}{(1+\delta(\hat{u}_i - u_i)^2)^2}(\hat{\psi} - \psi)dx\end{aligned} \tag{4.17}$$

taking $\delta \to 0$ in (4.17), we obtain the estimate

$$\begin{aligned}\int_G(\hat{u}_i - u_i)^2|\nabla(\hat{\psi} - \psi)|^2 dx &\le \frac{2}{a}\|(\hat{\psi} - \psi)(t)\|_\infty(\Sigma\|\hat{u}_i - u_i\|_3)\|\hat{u}_i - u_i\|\|\hat{u}_i - u_i\|_6 \\ &+ 4\|(\hat{\psi} - \psi)(t)\|_\infty^2\|\nabla(\hat{u}_i - u_i)\|^2 \le C(\Sigma\|\hat{u}_i - u_i\|^2)(\Sigma\|\nabla(\hat{u}_i - u_i)\|^2)\end{aligned} \tag{4.18}$$

Using (4.18) and (4.10), we obtain

$$E_7 \le \varepsilon\Sigma\int_{Q_t}|\nabla(\hat{u}_i - u_i)|^2 dx d\tau + K_{11}\Sigma\int_0^t \|(|J_1|+|J_2|)\|^2\|(\hat{u}_i - u_i)(\tau)\|^2 d\tau \tag{4.19}$$

From (4.4)–(4.10), (4.12), (4.14), (4.16), (4.19), we conclude

$$\Sigma\|(\hat{u}_i - u_i)(t)\|^2 \le \int_0^t h(\tau)\Big(\Sigma\|(\hat{u}_i - u_i)(\tau)\|^2\Big)d\tau \tag{4.20}$$

where $h(t) \in L^1_+(0,T)$. In view of the Gronwall's inequality, we get $\Sigma\|(\hat{u}_i - u_i)(t)\|^2 = 0$, i.e., $\hat{u}_i = u_i$. By (4.3), this implies that $\hat{\psi} = \psi$. Thus, there is at most one solution to the semiconductor equations.

We complete the proof of Theorem 4.1.

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