J.Partial Diff. Eqs. 13(2000), 11–20
©International Academic Publishers Vol.13, No.1

THE HAMILTONIAN SYSTEMS OF THE LCZ HIERARCHY BY NONLINEARIZATION

Li Lu
(Department of Physics, Shanxi University, Taiyuan 030006, China)
(Received Dec. 29, 1997; revised Mar. 19, 1999)

Abstract In this paper, we first search for the Hamiltonian structure of LCZ hierarchy by use of a trace identity. Then we determine a higher-order constraint condition between the potentials and the eigenfunctions of the LCZ spectral problem, and under this constraint condition, the Lax pairs of LCZ hierarchy are all nonlinearized into the finite-dimensional integrable Hamiltonian systems in Liouville sense.
Key Words LCZ hierarchy; Hamiltonian structure; the higher-order constraint condition; Hamiltonian systems.
1991 MR Subject Classification 35Q53, 70H05.
Chinese Library Classification O175.29.

1. Introduction

It is well known that finding new finite-dimensional completely integrable Hamiltonian systems in Liouville sense is very important [1]. Cao [2] devoloped an approach to produce the finite-dimensional integrable systems for AKNS hierarchy by the nonlinearization of Lax pair of evolution equations under certain constraints between the potentials and the eigenfunctions. Recently, on the basis of Cao's work, Zeng-Li [3] proposed the so-called higher-order symmetric constraint to get the finite-dimensional integrable Hamiltonian systems. According to this approach, many finite-dimensional integrable Hamiltonian systems are obtained [4–6].

For the following LCZ spectral problem

$$\Phi_x = \begin{pmatrix} -i\lambda + r & q+r \\ q-r & i\lambda - r \end{pmatrix} \Phi \tag{1}$$

Qiao [7], Mu [8] et al. presented respectively the commutator representation and a complete integrable Hamiltonian system in the Liouville sense under Bargmann constraint condition. In this paper, the Hamiltonian structure of the LCZ hierarchy is given by using a trace identity, and the so-called higher-order constraint condition between the potentials and the eigenfunctions is obtained by the Hamiltonian structure. Under this constraint, a completely integrable Hamiltonian system is obtained.

The layout of this paper is as follows. In Section 2, we will give the LCZ integrable hierarchy by zero-curvature representation and search for its Hamiltonian structure by use of a trace identity. Then, in Section 3, we determine a higher-order constraint between the potentials and the eigenfunctions of LCZ spectral problem, and under this constraint, the Lax pair of LCZ hierarchy are all nonlinearized into finite-dimensional integrable Hamiltonian systems in Liouville sense.

2. LCZ Hierarchy and Its Hamiltonian Structure

We consider LCZ spectral problem

$$\Phi_x = U(u,\lambda)\Phi, \quad U(u,\lambda) = \begin{pmatrix} -i\lambda + r & q+r \\ q-r & i\lambda - r \end{pmatrix} \tag{2}$$

where $u = (q, r)^T$ is the potential function, λ is the spectral parameter, $-i^2 = 1$. In order to derive LCZ hierarchy of evolution equation by using zero-curvature equation, we first solve the following adjoint representation equation of (2)

$$V_x = [U, V] \tag{3}$$

Let us choose

$$V = \begin{pmatrix} a & b \\ c & -a \end{pmatrix} = \sum_{m=0}^{\infty} V_m \lambda^{-m}, \; V_m = \begin{pmatrix} a_m & b_m \\ c_m & -a_m \end{pmatrix}$$

and on setting $a = \sum_{m=0}^{\infty} a_m \lambda^{-m}$, $b = \sum_{m=0}^{\infty} b_m \lambda^{-m}$, $c = \sum_{m=0}^{\infty} c_m \lambda^{-m}$, from (3) we can obtain the following recursion relations to determine a_m, b_m, c_m

$$\begin{cases} b_0 = c_0 = 0 \\ a_m = \partial^{-1} r(b_m + c_m) - \partial^{-1} q(b_m - c_m), & m \geq 0 \\ b_{m+1} = i(q+r)a_m - irb_m + \dfrac{1}{2} i\partial b_m, & m \geq 0 \\ c_{m+1} = i(q-r)a_m - irc_m - \dfrac{1}{2} i\partial c_m, & m \geq 0 \end{cases} \tag{4}$$

in which we choose $a_0 = 1$ and assume that $a_m|_{u=0} = b_m|_{u=0} = c_m|_{u=0} = 0$ $(m \geq 1)$, which means to select constants of integration to be zero when $m \geq 1$. In this way, the recursion relations (4) uniquely determine a series of polynomial functions with respect to $u, u_x, u_{xx}, \cdots$. By using (4), for example, we can work out

$$b_0 = c_0 = 0, \quad a_0 = 1$$
$$b_1 = i(q+r), \quad c_1 = i(q-r), \quad a_1 = 0$$
$$b_2 = r(q+r) - \frac{1}{2}(q+r)_x, c_2 = r(q-r) + \frac{1}{2}(q-r)_x, \; a_2 = \frac{1}{2}(q^2 - r^2)$$

$$b_3 = \frac{i}{2}(q+r)(q^2-3r^2) + \frac{i}{2}(q+r)r_x + ir(q+r)_x + \frac{i}{4}(q+r)_{xx}$$
$$c_3 = \frac{i}{2}(q-r)(q^2-3r^2) - \frac{i}{2}(q-r)r_x - ir(q-r)_x - \frac{i}{4}(q-r)_{xx}$$
$$a_3 = ir^3 - iq^2r - \frac{i}{2}(q_xr - qr_x)$$

Let us now associate with LCZ spectral problem (2) the following auxiliary problem

$$\begin{cases} \Phi_{t_n} = V^{(n)}\Phi = V^{(n)}(u,\lambda)\Phi \\ V^{(n)} = \sum\limits_{m=0}^{n} V_m\lambda^{n-m} + \begin{pmatrix} \frac{1}{2}(b_n-c_n)-a_n & 0 \\ 0 & -\frac{1}{2}(b_n-c_n)+a_n \end{pmatrix}, \quad n \geq 0 \end{cases} \tag{5}$$

then by use of (4), zero-curvature equation $U_{t_n} - V_x^{(n)} + [U, V^{(n)}] = 0$ $(n \geq 0)$ becomes

$$\begin{cases} q_{t_n} = \frac{1}{2}(b_n+c_n)_x, & n \geq 0 \\ r_{t_n} = -\frac{1}{2}(2a_n - b_n + c_n)_x, & n \geq 0 \end{cases} \tag{6}$$

and thus (6) is easily written as

$$u_{t_n} = \begin{pmatrix} q \\ r \end{pmatrix}_{t_n} = \begin{pmatrix} \frac{1}{2}\partial & 0 \\ 0 & -\frac{1}{2}\partial \end{pmatrix} \begin{pmatrix} b_n+c_n \\ 2a_n-b_n+c_n \end{pmatrix} = \frac{1}{2i}J\begin{pmatrix} b_n+c_n \\ 2a_n-b_n+c_n \end{pmatrix} \tag{7}$$

where

$$J = i\begin{pmatrix} \partial & 0 \\ 0 & -\partial \end{pmatrix}, \quad \partial = \frac{\partial}{\partial x} \tag{8}$$

which is a symplectic operator, and further making use of the recursion relations (4), we have

$$\begin{pmatrix} b_n+c_n \\ 2a_n-b_n+c_n \end{pmatrix} = L\begin{pmatrix} b_{n-1}+c_{n-1} \\ 2a_{n-1}-b_{n-1}+c_{n-1} \end{pmatrix} \tag{9}$$

where L is a recursion operator

$$L = \frac{i}{2}\begin{pmatrix} 0 & -\partial+2q \\ -2\partial^{-1}q\partial - \partial & -2\partial^{-1}r\partial - 2r \end{pmatrix}, \quad \partial\partial^{-1} = \partial^{-1}\partial = 1 \tag{10}$$

which with J satisfies $JL = -(JL)^* = L^*J$. Then substituting (9) into (7), we lead to isospectral LCZ hierarchy

$$u_{t_n} = \begin{pmatrix} q \\ r \end{pmatrix}_{t_n} = \frac{1}{2i}J\begin{pmatrix} b_n+c_n \\ 2a_n-b_n+c_n \end{pmatrix} = JL^{n-1}\begin{pmatrix} q \\ -r \end{pmatrix}, \quad n \geq 1 \tag{11}$$

in which J and L are given in (8) and (10).

Example For $n=1$ and $n=2$, (11) is respectively reduced to the following two systems:

$$\begin{pmatrix} q \\ r \end{pmatrix}_t = i\begin{pmatrix} q \\ r \end{pmatrix}_x$$

and

$$\begin{pmatrix} q \\ r \end{pmatrix}_t = \begin{pmatrix} -\frac{1}{2}r_{xx} + q_x r + q r_x \\ -\frac{1}{2}q_{xx} + 3 r r_x - q q_x \end{pmatrix}$$

here when $q = r$, the second equation is the famous Bergurs equation

$$q_t + \frac{1}{2}q_{xx} - 2qq_x = 0$$

Now we proceed to search for the Hamiltonian structure of LCZ hierarchy of evolution equation (11) which may be established by applying the so-called trace identity [9]. As is usual, we first need the following quantities which are easy to calculate

$$\left\langle V, \frac{\partial U}{\partial \lambda} \right\rangle = -2ia, \quad \left\langle V, \frac{\partial U}{\partial q} \right\rangle = b + c, \quad \left\langle V, \frac{\partial U}{\partial r} \right\rangle = 2a - b + c$$

in which $\langle \cdot, \cdot \rangle$ stands for the Kiling-Cartan form of matrices: $\langle A, B \rangle = \operatorname{tr} AB$. Then by use of the trace identity [9]

$$\begin{cases} \frac{\delta}{\delta q}\left\langle V, \frac{\partial U}{\partial \lambda} \right\rangle = \lambda^{-\gamma}\frac{\partial}{\partial \lambda}\lambda^{\gamma}\left\langle V, \frac{\partial U}{\partial q} \right\rangle \\ \frac{\delta}{\delta r}\left\langle V, \frac{\partial U}{\partial \lambda} \right\rangle = \lambda^{-\gamma}\frac{\partial}{\partial \lambda}\lambda^{\gamma}\left\langle V, \frac{\partial U}{\partial r} \right\rangle \end{cases}$$

in which γ is a constant, we arrive at

$$\begin{cases} -2i\frac{\delta a}{\delta q} = \lambda^{-\gamma}\frac{\partial}{\partial \lambda}\lambda^{\gamma}(b+c) \\ -2i\frac{\delta a}{\delta r} = \lambda^{-\gamma}\frac{\partial}{\partial \lambda}\lambda^{\gamma}(2a - b + c) \end{cases} \tag{12}$$

Noticing $a = \sum\limits_{m=0}^{\infty} a_m \lambda^{-m}$, $b = \sum\limits_{m=0}^{\infty} b_m \lambda^{-m}$, $c = \sum\limits_{m=0}^{\infty} c_m \lambda^{-m}$, after comparing the coefficients of λ^{-n} on two sides of (12), we obtain

$$\begin{cases} -2i\frac{\delta a_n}{\delta q} = (b_{n-1} + c_{n-1})(-n + 1 + \gamma), & n \geq 1 \\ -2i\frac{\delta a_n}{\delta r} = (2a_{n-1} - b_{n-1} + c_{n-1})(-n + 1 + \gamma), & n \geq 1 \end{cases} \tag{13}$$

To fix the constant γ, we simply set $n = 2$ in (13), then lead to the constant $\gamma = 0$, thus (13) gives rise to an important formula

$$\begin{cases} b_n + c_n = 2i\frac{\delta H_{n+1}}{\delta q}, \\ 2a_n - b_n + c_n = 2i\frac{\delta H_{n+1}}{\delta r}, \end{cases} \quad H_{n+1} = \frac{a_{n+1}}{n}\ (n \geq 1),\ H_1 = -ir \tag{14}$$

which shows that the LCZ hierarchy (11) possesses the following Hamiltonian structure

$$u_{t_n} = \frac{1}{2i}J\begin{pmatrix} b_n + c_n \\ 2a_n - b_n + c_n \end{pmatrix} = JL^{n-1}\begin{pmatrix} q \\ -r \end{pmatrix} = J\frac{\delta H_{n+1}}{\delta u},\ n \geq 0 \tag{15}$$

where J, L are given in (8) and (10). Therefore by Propositon 3 of [9], we conclude that the $\{H_n\}$ are conserved densities for the whole LCZ hierarchy of evolution equation (15) and they are in involutive in pairs.

3. LCZ Finite-Dimensional Hamiltonian Systems

For distinct λ_j, $j = 1, 2, \cdots, N$, we consider the systems

$$\Phi_{j,x} = U(u, \lambda_j)\Phi_j, \ U(u, \lambda_j) = \begin{pmatrix} -i\lambda_j + r & q + r \\ q - r & i\lambda_j - r \end{pmatrix} \tag{16}$$

$$\Phi_{j,t_n} = V^{(n)}(u, \lambda_j)\Phi_j, \quad j = 1, 2, \cdots, N \tag{17}$$

$$V^{(n)}(u, \lambda_j) = \begin{pmatrix} \sum\limits_{m=0}^{n} a_m \lambda_j^{n-m} + \frac{1}{2}(b_n - c_n) - a_n & \sum\limits_{m=0}^{n} b_m \lambda_j^{n-m} \\ \sum\limits_{m=0}^{n} c_m \lambda_j^{n-m} & -\sum\limits_{m=0}^{n} a_m \lambda_j^{n-m} - \frac{1}{2}(b_n - c_n) + a_n \end{pmatrix}$$

in which $u = (q, r)^T$, $\Phi_j = (\phi_{1j}, \phi_{2j})^T$, $j = 1, 2, \cdots, N$. If we assume that $u, u_x, \cdots$ tend to zero as $|x|$ tends to ∞, λ_j are the discrete eigenvalue for which eigenfunctions $(\phi_{1j}, \phi_{2j})^T$ vanish as $|x| \to \infty$, then we have [7] for every λ_j $(j = 1, 2, \cdots, N)$

$$\begin{cases} \dfrac{\delta\lambda_j}{\delta q} = \dfrac{1}{E_j}(\phi_{2j}^2 - \phi_{1j}^2) \\ \dfrac{\delta\lambda_j}{\delta r} = \dfrac{1}{E_j}(\phi_{1j} + \phi_{2j})^2 \end{cases} \tag{18}$$

where $E_j = 2i\int \phi_{1j}\phi_{2j}dx$, and if Φ_j satisfies (16), then we can show that $\dfrac{\delta\lambda_j}{\delta u} = \left(\dfrac{\delta\lambda_j}{\delta q}, \dfrac{\delta\lambda_j}{\delta r}\right)^T$ given by (18) satisfies

$$L\frac{\delta\lambda_j}{\delta u} = \lambda_j \frac{\delta\lambda_j}{\delta u}, \quad j = 1, 2, \cdots, N \tag{19}$$

in which L is the recursion operator given by (10). therefore $\dfrac{\delta\lambda_j}{\delta u}$ belongs to the invariant space of the recursion operator L.

By [3], we have for a given k

$$\frac{\delta H_k}{\delta u} - \sum_{j=1}^{N} \alpha_j \frac{\delta\lambda_j}{\delta u} = 0 \tag{20}$$

which is invariant with respect to the action of integrable Hamiltonian flows (15). This property shows that (16), (17), (20) are compatible, thus (20) gives an infinite constraint which is compatible with (16), (17) (we choose $\dfrac{\alpha_j}{E_j} = -\dfrac{i}{4}$)

$$\frac{\delta H_k}{\delta u} = -\frac{i}{4}\sum_{j=1}^{N} \begin{pmatrix} \phi_{2j}^2 - \phi_{1j}^2 \\ (\phi_{1j} + \phi_{2j})^2 \end{pmatrix} = -\frac{i}{4} \begin{pmatrix} \langle\Phi_2, \Phi_2\rangle - \langle\Phi_1, \Phi_1\rangle \\ \langle\Phi_1 + \Phi_2, \Phi_1 + \Phi_2\rangle \end{pmatrix} \tag{21}$$

where $\Phi_1=(\phi_{11},\phi_{12},\cdots,\phi_{1N})^T$, $\Phi_2=(\phi_{21},\phi_{22},\cdots,\phi_{2N})^T$, $\langle\cdot,\cdot\rangle$ denotes the standard inner product in R^N.

For $k=2$, (21) gives an explicit expression of u in terms of λ_j and Φ_j

$$\frac{\delta H_2}{\delta u}=\begin{pmatrix} q \\ -r \end{pmatrix}=-\frac{i}{4}\begin{pmatrix} \langle\Phi_2,\Phi_2\rangle-\langle\Phi_1,\Phi_1\rangle \\ \langle\Phi_1+\Phi_2,\Phi_1+\Phi_2\rangle \end{pmatrix} \tag{22}$$

and by substituting (22) into (16), we can reduce a completely intergrable finite-dimensional Hamiltonian system [8].

In the following, we consider the integrable systems associated with the higher-order symmetric constraints. As an example, we obtain a higher-order constraint from (21) for $k=3$

$$\frac{\delta H_3}{\delta u}=\frac{i}{2}\begin{pmatrix} r_x-2qr \\ -q_x-q^2+3r^2 \end{pmatrix}=-\frac{i}{4}\begin{pmatrix} \langle\Phi_2,\Phi_2\rangle-\langle\Phi_1,\Phi_1\rangle \\ \langle\Phi_1+\Phi_2,\Phi_1+\Phi_2\rangle \end{pmatrix}$$

that is

$$\begin{pmatrix} r_x-2qr \\ -q_x-q^2+3r^2 \end{pmatrix}=-\frac{1}{2}\begin{pmatrix} \langle\Phi_2,\Phi_2\rangle-\langle\Phi_1,\Phi_1\rangle \\ \langle\Phi_1+\Phi_2,\Phi_1+\Phi_2\rangle \end{pmatrix} \tag{23}$$

in which we used $H_3=\frac{1}{2}a_3=\frac{i}{2}r^3-\frac{i}{2}q^2r-\frac{i}{4}(q_xr-qr_x)$. Obviously, (23) can not give the explicit constraint of u in terms of λ_j and Φ_j, therefore we can not make use of the approach of substituting the explicit constraint into (16) to construct the integrable system associated with the higher-order constraint (23). But because the properties of (20) ensure that (20) is compatible with (16), (17), we consider the following systems by combining (23) with (16)

$$\begin{pmatrix} r_x-2qr \\ -q_x-q^2+3r^2 \end{pmatrix}=-\frac{1}{2}\begin{pmatrix} \langle\Phi_2,\Phi_2\rangle-\langle\Phi_1,\Phi_1\rangle \\ \langle\Phi_1+\Phi_2,\Phi_1+\Phi_2\rangle \end{pmatrix} \tag{24a}$$

$$\Phi_{1x}=-i\Lambda\Phi_1+r\Phi_1+(q+r)\Phi_2$$

$$\Phi_{2x}=(q-r)\Phi_1+i\Lambda\Phi_2-r\Phi_1 \tag{24b}$$

Note that the stationary equation system of (24a)

$$\begin{pmatrix} r_x-2qr \\ -q_x-q^2+3r^2 \end{pmatrix}=0$$

can be transformed into

$$q_x=\frac{\partial\overline{H}}{\partial r},\quad r_x=-\frac{\partial\overline{H}}{\partial q}$$

where $\overline{H}=r^3-q^2r$, then (24a) reads equivalently

$$q_x=\frac{\partial\overline{H}}{\partial r}+\frac{1}{2}\langle\Phi_1+\Phi_2,\Phi_1+\Phi_2\rangle$$

$$r_x=-\frac{\partial\overline{H}}{\partial q}-\frac{1}{2}\langle\Phi_2,\Phi_2\rangle+\frac{1}{2}\langle\Phi_1,\Phi_1\rangle$$

Thus (24) can be evidently expressed as the following Hamiltonian system

$$Q_x = \frac{\partial H}{\partial P}, \quad P_x = -\frac{\partial H}{\partial Q} \tag{25}$$

with the Hamiltonian function

$$H = \overline{H} - i\langle \Lambda\Phi_1, \Phi_2\rangle + r\langle \Phi_1, \Phi_2\rangle - \frac{1}{2}(q-r)\langle \Phi_1, \Phi_1\rangle + \frac{1}{2}\langle \Phi_2, \Phi_2\rangle \tag{26}$$

in which $Q = (\phi_{11}, \phi_{12}, \cdots, \phi_{1N}, q)$, $P = (\phi_{21}, \phi_{22}, \cdots, \phi_{2N}, r)$, $\Lambda = \mathrm{diag}\,\{\lambda_1, \lambda_2, \cdots, \lambda_N\}$. In the following we want to show that (25) is a finite-dimensional integrable system in the Liouville sense and that under the control of (25), systems (17) and (15) for $n \geq 0$ are also a hierarchy of finite-dimensional integrable systems in the Liouville sense.

First under the higher-order constraint condition (23), by use of (19) we have

$$\begin{pmatrix} b_n + c_n \\ 2a_n - b_n + c_n \end{pmatrix} = 2iL^{n-2}\frac{\delta H_3}{\delta u} = \frac{1}{2}\begin{pmatrix} \langle \Lambda^{n-2}\Phi_2, \Phi_2\rangle - \langle \Lambda^{n-2}\Phi_1, \Phi_1\rangle \\ \langle \Lambda^{n-2}(\Phi_1 + \Phi_2), \Phi_1 + \Phi_2\rangle \end{pmatrix}, \quad n \geq 2 \tag{27}$$

and by substituting (27) into (4), we obtain

$$\begin{aligned} a_{n,x} + 2qa_n =& \frac{1}{2}r\langle \Lambda^{n-2}\Phi_2, \Phi_2\rangle - \frac{1}{2}r\langle \Lambda^{n-2}\Phi_1, \Phi_1\rangle \\ &+ \frac{1}{2}q\langle \Lambda^{n-2}(\Phi_1 + \Phi_2), \Phi_1 + \Phi_2\rangle, \quad n \geq 2 \end{aligned}$$

Besides, from (16), we have

$$\begin{aligned} \frac{1}{2}\langle \Lambda^{n-2}\Phi_1, \Phi_2\rangle_x =& \frac{1}{2}q(\langle \Lambda^{n-2}\Phi_2, \Phi_2\rangle + \langle \Lambda^{n-2}\Phi_1, \Phi_1\rangle) \\ &+ \frac{1}{2}r(\langle \Lambda^{n-2}\Phi_2, \Phi_2\rangle - \langle \Lambda^{n-2}\Phi_1, \Phi_1\rangle), \quad n \geq 2 \end{aligned} \tag{28}$$

Thus we obtain

$$\begin{cases} a_n = \frac{1}{2}\langle \Lambda^{n-2}\Phi_1, \Phi_2\rangle, & n \geq 2 \\ b_n = -\frac{1}{2}\langle \Lambda^{n-2}\Phi_1, \Phi_1\rangle, & n \geq 2 \\ c_n = \frac{1}{2}\langle \Lambda^{n-2}\Phi_2, \Phi_2\rangle, & n \geq 2 \end{cases} \tag{29}$$

and for $n = 0, 1$, respectively

$$a_0 = 1, b_0 = c_0 = 0; a_1 = 0, b_1 = i(q + r), c_1 = i(q - r)$$

We see also that $(V^2)_x = [U, V^2]$ since $V_x = [U, V]$ and letting $F = \frac{1}{2}\mathrm{tr}\, V^2 = a^2 + bc$,

$$F_x = \left(\frac{1}{2}\mathrm{tr}\, V^2\right)_x = \frac{d}{dx}(a^2 + bc) = 0$$

i.e. F is a generating function of integrals of motion of (25), after letting $F = \sum_{m=0}^{\infty} F_n \lambda^{-n}$ and noticing $\langle \Phi_1, \Phi_2 \rangle = q^2 - r^2$, we obtain the following expressions of F_n

$$\begin{aligned}
F_0 =&1, \ F_1 = 0, \ F_2 = 0 \\
F_3 =&i\Big(- i\langle \Lambda\Phi_1, \Phi_2\rangle - \frac{1}{2}(q-r)\langle \Phi_1, \Phi_1\rangle + \frac{1}{2}(q+r)\langle \Phi_2, \Phi_2\rangle\Big) = iH \\
F_n =& \sum_{m=0}^{n} (a_m a_{n-m} + b_m c_{n-m}) \\
=& \sum_{m=2}^{n-2} (a_m a_{n-m} + b_m c_{n-m}) + 2a_n + b_1 c_{n-1} + b_{n-1} c_1 \\
=&\frac{1}{4}\sum_{m=2}^{n-2} (\langle \Lambda^{m-2}\Phi_1, \Phi_2\rangle\langle \Lambda^{n-m-2}\Phi_1, \Phi_2\rangle - \langle \Lambda^{m-2}\Phi_1, \Phi_1\rangle\langle \Lambda^{n-m-2}\Phi_2, \Phi_2\rangle) \\
&+ \langle \Lambda^{n-2}\Phi_1, \Phi_2\rangle + \frac{i}{2}(q+r)\langle \Lambda^{n-3}\Phi_2, \Phi_2\rangle - \frac{i}{2}(q-r)\langle \Lambda^{n-3}\Phi_1, \Phi_1\rangle \\
&+ \frac{1}{4}\langle \Lambda^{n-4}(\Phi_1 + \Phi_2), \Phi_1 + \Phi_2\rangle(q^2 - r^2 - \langle \Phi_1, \Phi_2\rangle), \quad n \geq 4
\end{aligned} \tag{30}$$

in which the polynomial functions F_n $(n \geq 0)$ include $2N+2$ dependent variables.

Let us now consider the temporal part (17) of constrained Lax pairs. A direct calculation gives that systems (17) with (15) for $n \geq 0$ are cast into

$$Q_{t_n} = \frac{\partial F_{n+2}}{\partial P}, \quad P_{t_n} = -\frac{\partial F_{n+2}}{\partial Q}, \quad n \geq 1 \tag{31}$$

when Q, P satisfy the spatial part (25). In fact, from (17) and (29), we have

$$\begin{aligned}
\Phi_{1t_n} =& \sum_{m=0}^{n} (a_m \Lambda^{n-m}\Phi_1 + b_m \Lambda^{n-m}\Phi_2) + \Big[\frac{1}{2}(b_n - c_n) - a_n\Big]\Phi_1 \\
=&\frac{1}{2}\sum_{m=2}^{n} (\langle \Lambda^{m-2}\Phi_1, \Phi_2\rangle \Lambda^{n-m}\Phi_1 - \langle \Lambda^{m-2}\Phi_1, \Phi_1\rangle \Lambda^{n-m}\Phi_2) + \Lambda^n\Phi_1 \\
&+ i(q+r)\Lambda^{n-1}\Phi_2 - \frac{1}{4}\langle \Lambda^{n-2}(\Phi_1 + \Phi_2), \Phi_1 + \Phi_2\rangle\Phi_1 \\
=&\frac{\partial F_{n+2}}{\partial \Phi_2}, \quad n \geq 1
\end{aligned} \tag{32}$$

$$\begin{aligned}
\Phi_{2t_n} =& \sum_{m=0}^{n} (c_m \Lambda^{n-m}\Phi_1 - a_m \Lambda^{n-m}\Phi_2) + \Big[- \frac{1}{2}(b_n - c_n) + a_n\Big]\Phi_2 \\
=&\frac{1}{2}\sum_{m=2}^{n} (\langle \Lambda^{m-2}\Phi_2, \Phi_2\rangle \Lambda^{n-m}\Phi_1 - \langle \Lambda^{m-2}\Phi_1, \Phi_2\rangle \Lambda^{n-m}\Phi_2) - \Lambda^n\Phi_2 \\
&+ i(q-r)\Lambda^{n-1}\Phi_1 + \frac{1}{4}\langle \Lambda^{n-2}(\Phi_1 + \Phi_2), \Phi_1 + \Phi_2\rangle\Phi_2 \\
=& - \frac{\partial F_{n+2}}{\partial \Phi_1}, \quad n \geq 1
\end{aligned} \tag{33}$$

and from (30), (6) and (4), we have

$$\begin{aligned}\frac{\partial F_{n+2}}{\partial r} &= \frac{i}{2}\langle \Lambda^{n-1}\Phi_2, \Phi_2\rangle + \frac{i}{2}\langle \Lambda^{n-1}\Phi_1, \Phi_1\rangle - \frac{1}{2}r\langle \Lambda^{n-2}(\Phi_1+\Phi_2), \Phi_1+\Phi_2\rangle \\ &= i(c_{n+1}-b_{n+1}) - r(2a_n - b_n + c_n) \\ &= \frac{1}{2}(b_n + c_n)_x = q_{t_n}, \quad n \geq 1 \end{aligned} \tag{34}$$

$$\begin{aligned}\frac{\partial F_{n+2}}{\partial q} &= \frac{i}{2}\langle \Lambda^{n-1}\Phi_2, \Phi_2\rangle - \frac{i}{2}\langle \Lambda^{n-1}\Phi_1, \Phi_1\rangle + \frac{1}{2}q\langle \Lambda^{n-2}(\Phi_1+\Phi_2), \Phi_1+\Phi_2\rangle \\ &= i(c_{n+1}+b_{n+1}) + q(2a_n - b_n + c_n) \\ &= \frac{1}{2}(2a_n - b_n + c_n)_x = -r_{t_n}, \quad n \geq 1 \end{aligned} \tag{35}$$

Similarly, because we have $V_{t_n} = [V^{(n)}, V]$, $n \geq 0$, we may see with the same argument that $F = \frac{1}{2}\mathrm{tr}\, V = a^2 + bc$ is also a generating function of integrals of motion for (31), that is $\frac{\partial F_m}{\partial t_n} = 0$, $n, m \geq 0$. Defining the following Poisson bracket [6]

$$\{f, g\} = \left\langle \frac{\partial f}{\partial Q}, \frac{\partial g}{\partial P}\right\rangle_1 - \left\langle \frac{\partial f}{\partial P}, \frac{\partial g}{\partial Q}\right\rangle_1$$

where $\langle\cdot,\cdot\rangle_1$ denotes the standard inner product in R^{N+1}, and by use of (31), we find

$$\begin{aligned}\{F_m, F_{n+2}\} &= \left\langle \frac{\partial F_m}{\partial Q}, \frac{\partial F_{n+2}}{\partial P}\right\rangle_1 - \left\langle \frac{\partial F_m}{\partial P}, \frac{\partial F_{n+2}}{\partial Q}\right\rangle_1 \\ &= \left\langle \frac{\partial F_m}{\partial Q}, Q_{t_n}\right\rangle_1 + \left\langle \frac{\partial F_m}{\partial P}, P_{t_n}\right\rangle_1 = \frac{\partial F_m}{\partial t_n} = 0, \quad n \geq 1, \quad m \geq 0\end{aligned}$$

and because F_n is the constant when $n < 3$, then $\{F_m, F_n\} = 0$ for $n < 3$. Therefore we obtain

$$\{F_m, F_n\} = 0, \quad \text{for } n, m \geq 0 \tag{36}$$

This shows that polynomial functions F_n, $n = 0, 1, 2, \cdots$ constitute an involutive systems. On the other hand, from (31), we directly have

$$\left.\frac{\partial F_{n+2}}{\partial \Phi_1}\right|_{\Phi_1=0, q=0} = \Lambda^n \Phi_2, \quad \left.\frac{\partial F_{n+2}}{\partial q}\right|_{\Phi_1=0, q=0} = \frac{i}{2}\langle \Lambda^{n-1}\Phi_2, \Phi_2\rangle$$

$$\left.\frac{\partial F_{n+2}}{\partial \Phi_2}\right|_{\Phi_2=0, r=0} = \Lambda^n \Phi_1, \quad \left.\frac{\partial F_{n+2}}{\partial r}\right|_{\Phi_2=0, r=0} = \frac{i}{2}\langle \Lambda^{n-1}\Phi_1, \Phi_1\rangle$$

and because of the Vandermonde determinant $V(\lambda_1, \lambda_2, \cdots, \lambda_N) \neq 0$ for N distinct λ_j, $j = 1, 2, \cdots, N$,

$$\nabla F_n = \left(\frac{\partial F_n}{\partial \Phi_1}, \frac{\partial F_n}{\partial q}, \frac{\partial F_n}{\partial \Phi_2}, \frac{\partial F_n}{\partial r}\right), \quad 3 \leq n \leq N+3$$

are linearly independent. This shows that (25) and (31) are the finite-dimensional integrable Hamiltonian systems in the Liouville sense.

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