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SPECTRAL PROPERTIES OF SECOND ORDER DIFFERENTIAL OPERATORS ON TWO-STEP NILPOTENT LIE GROUPS*

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Abstract In this paper, spectral properties of certain left invariant differential operators on two-step nilpotent Lie groups are completely described by using the theory of unitary irreducible representations and the Plancherel formulae on nilpotent Lie groups.
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1. Introduction

The purpose of this paper is to consider spectral properties of the following operators

$$P = -\sum_{r=1}^{p_1} X_r^2 + \sum_{l=1}^{p_2} C_l Y_l \tag{1.1}$$

where $X_r, Y_l, r = 1, \cdots, p_1, l = 1, \cdots, p_2$, are a basis for the Lie algebra $\mathcal{G}$ of a two-step nilpotent Lie group G, each C_l is a complex constant.

The operator P and its properties, e.g., local solvability, hypoellipticity, have been investigated by many authors. It is well known that (1.1) has not been contained in the class of operators introduced by Hörmander in [1] if the C_l is imaginary. Spectral properties of the Kohn-Laplacian on the Heisenberg group were studied by Luo and Niu in [2]. Also Furutani et al. discussed the spectrum of Laplacian on two-step nilpotent groups (See [3]).

In this paper we will determine the spectrum of P. The main tools here are unitary representations and Plancherel formulae on G.

In order to state our main results we need some notations. We denote the spectrum and the resolvant set of P by $\sigma(P)$ and $\rho(P)$ respectively. Let V_0 be the completion

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of $C_0^\infty(G)$ with the norm (3.2) below and $D(P)$ the class of those $u \in V_0$ such that $Pu \in L^2(G)$. Introduce the set

$$\Gamma_\beta(C) = \Big\{ q \in C : q = \sum_{j=1}^{\frac{p_1-d}{2}} (2\beta_j + 1)\rho_j + \sum_{i=1}^{d} \zeta_i^2 - \sqrt{-1}\sum_{l=1}^{p_2} C_l \eta_l, \quad \rho_j \in \mathbf{R}_+, \zeta_i \in \mathbf{R}, \eta_l \in \mathbf{R} \Big\} \tag{1.2}$$

where C denotes the complex plane, $C_l \in C$ $(l = 1, \cdots, p_2)$, $\frac{p_1 - d}{2}$ is a positive integer, $\beta = (\beta_1, \cdots, \beta_{(p_1-d)/2})$, $\beta_j \in \mathbf{I}_+ = \{0, 1, 2, \cdots\}$, the definition of d will be given in Section 2, and

$$S(P) = \bigcup_{\beta \in I_+^{\frac{p_1-d}{2}}} \Gamma_\beta(C) \cup \mathbf{R}_+ \tag{1.3}$$

where $\mathbf{R}_+$ denotes the set of nonnegative real numbers.

Theorem 1.1 *The spectrum of P is $S(P)$.*

As a consequence, we have

Corollary 1.1 *If C_l $(l = 1, \cdots, p_2)$ is purely imaginary, then $\sigma(P)$ is either $\mathbf{R}$ or $[0, +\infty)$. In particular, if $C_l = 0$, $l = 1, \cdots, p_2$, then $\sigma(P) = [0, +\infty)$.*

Theorem 1.2 *If $d \neq 0$, then P has not any eigenvalue.*

Theorem 1.3 *If $d = 0$ and ρ_j, η_l, C_l, $j = 1, \cdots, \frac{p_1 - d}{2}$; $l = 1, \cdots, p_2$, satisfy*

$$\sum_j (2\beta_j + 1)\rho_j - \sqrt{-1}\sum_l C_l \eta_l = 0 \tag{1.4}$$

for some $\beta \in \mathbf{I}_+^{\frac{p_1-d}{2}}$, then 0 is a unique eigenvalue of P.

The plan of this paper is as follows. In Section 2, we shall recall some basic results on the two-step nilpotent Lie group which will be used later. Section 3 is devoted to the proof of Theorem 1.1. In Section 4, we shall prove Theorems 1.2, 1.3. Finally in Section 5 some applications will be given.

2. Preliminaries

Let G be a connected, simply connected Lie group, whose Lie algebra $\mathcal{G}$ decomposes as a vector space $\mathcal{G} = \mathcal{G}_1 \oplus \mathcal{G}_2$ with $[\mathcal{G}_1, \mathcal{G}_1] \subset \mathcal{G}_2$, $[\mathcal{G}_1, \mathcal{G}_2] = \{0\}$. $\mathcal{G}$ carries a natural family of automorphic dilations given by

$$\delta_s(X) = sX \quad \text{if} \quad X \in \mathcal{G}_1, \quad \delta_s(Y) = s^2 Y \quad \text{if} \quad Y \in \mathcal{G}_2$$

These dilations extend in a natural way to $U(\mathcal{G})$, the universal envoloping algebra of $\mathcal{G}$, which may be identified with the set of all left invariant differential operators on G.

For a nontrivial linear form η on $\mathcal{G}_2$, define an alternative bilinear form B_η on $\mathcal{G}_1 \times \mathcal{G}_1$

$$B_\eta(X, X') = \eta([X, X'])$$

where B_η is a maximal rank. We denote a Zariski open set of $\mathcal{G}_2^*$ by Z. Let d be the dimension of kernal of B_η on Z, p_1, p_2 denote the dimension of $\mathcal{G}_1, \mathcal{G}_2$ respectively, and $(X_1, \cdots, X_{p_1})$, $(Y_1, \cdots, Y_{p_2})$ denote the basis of $\mathcal{G}_1, \mathcal{G}_2$ respectively.

Fix η in Z. There is a basis $\mathcal{B}$ of $\mathcal{G}_1$, obtained by an orthogonal transformation from $(X_1, \cdots, X_{p_1})$, such that

$$\begin{aligned}
&\mathcal{B} = (T_j, Z_j, W_i), \quad j = 1, \cdots, \frac{p_1 - d}{2}, \quad i = 1, \cdots, d \\
&B_\eta(T_j, T_k) = B_\eta(Z_j, Z_k) = B_\eta(W_i, W_j) = B_\eta(W_i, T_j) = B_\eta(W_i, Z_j) = 0 \\
&B_\eta(T_j, Z_k) = \delta_{jk}\rho_j(\eta)
\end{aligned}$$

where $\rho_j(\eta)$ $\left(j = 1, \cdots, \frac{p_1 - d}{2}\right)$ are positive numbers, such that the eigenvalue of $\sqrt{-1}B_\eta$ is $\pm\rho_j(\eta)$. Suppose that $\zeta \in \mathbf{R}^d$ identifies itself with the dual of space generated by (W_i).

Define a representation $\pi_{\zeta,\eta}$ of G in $L^2(\mathbf{R}^{\frac{p_1-d}{2}})$:

$$\pi_{\zeta,\eta}\Big(\exp \sum t_j T_j + \sum z_j Z_j + \sum w_i W_i + y\Big) f(s) = e^{\sqrt{-1}(y\eta + w\zeta + \rho z\cdot\frac{t}{2} + \sqrt{\rho}z\cdot s)} f(s + \sqrt{\rho}t) \tag{2.1}$$

If $\varphi \in \mathcal{S}(G)$ (Schwartz space), π is a unitary irreducible representation of G, then the operator $\pi(\varphi)$ is defined by

$$\pi(\varphi) = \int_G \varphi(g)\pi(g^{-1})dg \tag{2.2}$$

where dg is the image of Lebesgue measure on g *via* the exponential map. From (2.1), (2.2) it follows that

$$\pi_{\zeta,\eta}(f)\varphi(s) = \int f(t, z, w, y)e^{-\sqrt{-1}(y\eta + w\zeta) + \sqrt{-1}\rho z\cdot\frac{t}{2} - \sqrt{-1}\sqrt{\rho}z\cdot s} \cdot \varphi(s - \sqrt{\rho}t)dtdzdwdy \tag{2.3}$$

Given a left invariant vector field W and a unitary representation π on G, then the operator $\pi(W)$ is defined as an unbounded operator

$$\pi(W)f = \frac{d}{d\tau}[\pi(\exp \tau W)f]|_{\tau=0}$$

By (2.3) we obtain

$$\begin{aligned}
&\pi_{\zeta,\eta}(T_j) = \sqrt{\rho_j}\frac{\partial}{\partial s_j}, \quad \pi_{\zeta,\eta}(Z_j) = \sqrt{-1}\sqrt{\rho_j}s_j, \quad j = 1, \cdots, \frac{p_1 - d}{2} \\
&\pi_{\zeta,\eta}(W_i) = \sqrt{-1}\zeta_i, \quad i = 1, \cdots, d
\end{aligned} \tag{2.4}$$

$$\pi_{\zeta,\eta}(Y_l) = \sqrt{-1}\eta_l, \quad l = 1, \cdots, p_2$$

Levy-Bruhl [4] proved the Plancherel formula in polarization form: for $u \in \mathcal{S}(G)$,

$$u(e) = (2\pi)^{-\frac{p_1+d}{2}+p_2} \int_{\mathbf{R}^d \times \mathbf{R}^{p_2}} \operatorname{tr}[\pi_{\zeta,\eta}(u)] D(\eta)^{1/2} d\zeta d\eta \tag{2.5}$$

where $D(\eta)$ is a nonnegative polynomial of η, e is the identity element of G, tr denotes the trace. We let $C_0 = (2\pi)^{-\frac{p_1+d}{2}+p_2}$.

Proposition 2.1 *Let $u, v \in \mathcal{S}(G)$. Then*

$$u(t,z,w,y) = C_0 \int tr[\pi_{\zeta,\eta}(t,z,w,y)^* \pi_{\zeta,\eta}(u)] D(\eta)^{1/2} d\zeta d\eta \tag{2.6}$$

$$(u,v) = C_0 \int tr[\pi_{\zeta,\eta}(u)^* \pi_{\zeta,\eta}(v)] D(\eta)^{1/2} d\zeta d\eta \tag{2.7}$$

Proof Write $u(x) = u(t,z,w,y)$, then $u(x) = u(xe) = u_x(e)$. Since

$$\pi_{\zeta,\eta}(u_x) = \pi_{\zeta,\eta}(x)^* \pi_{\zeta,\eta}(v)$$

(2.6) follows from (2.5). By the fact that both of $\pi_{\zeta,\eta}(u), \pi_{\zeta,\eta}(v)$ are trace class operator, thus Hilbert-Schmidt operator, the right hand side of (2.7) makes sense. Setting $h = u^* * v$, where $u^* = \overline{u(g^{-1})}$, we have $h(e) = (u,v)$ and

$$\pi_{\zeta,\eta}(h) = \pi_{\zeta,\eta}(u)^* \pi_{\zeta,\eta}(v)$$

We conclude (2.7) by (2.5).

Let $u = v$. (2.7) leads to

$$\|u\|_{L^2}^2 = C_0 \int \operatorname{tr}[\pi_{\zeta,\eta}(u)^* \pi_{\zeta,\eta}(u)] D(\eta)^{1/2} d\zeta d\eta = C_0 \int \|\pi_{\zeta,\eta}(u)\|_{HS}^2 D(\eta)^{1/2} d\zeta d\eta \tag{2.8}$$

Since $\mathcal{S}$ is dense in L^2, (2.8) holds for $u \in L^2$.

3. Spectrum

The equation (1.1) can be rewritten as

$$P = -\sum_j (T_j^2 + Z_j^2) - \sum_i W_i^2 + \sum_l C_l Y_l \tag{3.1}$$

By the natural dilation transformations on G, P is a second order homogeneous left invariant differential operator.

Let V_0 be the completion of $C_0^\infty(G)$ by the norm

$$\|\varphi\|_{V_0} = \Big[\|\varphi\|_{L^2}^2 + \sum_j (\|T_j\varphi\|_{L^2}^2 + \|Z_j\varphi\|_{L^2}^2) + \sum_i \|W_i\varphi\|_{L^2}^2\Big]^{1/2} \tag{3.2}$$

Define

$$D(P) = \{u \in V_0,\ Pu \in L^2(G)\} \tag{3.3}$$

where $Pu \in L^2$ in the distributional sense. We understand P as an unbounded operator on L^2 with domain of difinition $D(P)$.

Evidently, the formal adjoint of P is

$$P^* = -\sum_r X_r^2 - \sum_l \overline{C}_l Y_l$$

and the formal transform is

$$P^t = -\sum_r X_r^2 - \sum_l C_l Y_l$$

It is clear that P is closed and densely defined on L^2. If $\mathrm{Re}C_l = 0, l = 1, \cdots, p_2$, then $P^* = P$, namely, P is selfadjoint.

Proposition 3.1 *For $u \in D(P)$, we have*

$$\pi_{\zeta,\eta}(Pu)\varphi = \pi_{\zeta,\eta}(u)\Big[-\sum_j \rho_j\Big(\frac{\partial^2}{\partial s_j^2} - s_j^2\Big) + \sum_i \zeta_i^2 - \sqrt{-1}\sum_l C_l\eta_l\Big]\varphi \tag{3.4}$$

where $\varphi \in \mathcal{S}(\mathbf{R}^{\frac{p_1-d}{2}})$.

Proof Let $|\cdot|$ denote a homogeneous norm (See [5]) and $\psi(x) \in C_0^\infty(G)$ such that $0 \le \psi(x) \le 1$ for $x \in G$ and

$$\psi(x) = \begin{cases} 1, & \text{when } |x| < 1 \\ 0, & \text{when } |x| > 2 \end{cases}$$

Denote

$$\psi_s(x) = \psi(\delta_{1/s}(x)) = \psi\Big(\frac{1}{s}t, \frac{1}{s}z, \frac{1}{s}w, \frac{1}{s^2}y\Big), s > 0$$

For $u \in D(P)$, set $u_s = \psi_s u$, then $u_s \in E' \cap V_0$, where E' denotes the set of all distributions with compact support. One can show that $u_s \to u$ in L^2 and $Pu_s \to Pu$ in L^2 as $s \to \infty$.

It follows that for $\varphi \in \mathcal{S}(\mathbf{R}^{\frac{p_1-d}{2}})$

$$\begin{aligned}\pi_{\zeta,\eta}(Pu)\varphi &= \pi_{\zeta,\eta}(\lim_{s\to\infty} Pu_s)\varphi = \lim_{s\to\infty} \pi_{\zeta,\eta}(Pu_s)\varphi \\ &= \lim_{s\to\infty} \pi_{\zeta,\eta}(u_s)\pi_{\zeta,\eta}(P^t)\varphi = \pi_{\zeta,\eta}(u)\pi_{\zeta,\eta}(P^t)\varphi\end{aligned}$$

By (2.4), the result is proved.

In the sequel we usually take φ as an Hermite function. All Hermite functions $\varphi_\beta(\beta \in \mathbf{I}_+^{\frac{p_1-d}{2}})$ form a normal orthogonal basis of $L^2(\mathbf{R}^{\frac{p_1-d}{2}})$.

Proof of Theorem 1.1 Let us note that $\sigma(P)$ is nonempty (See [6]) and particularly $0 \in \sigma(P)$. We first prove $S(P) \supset \sigma(P)$.

Suppose that $\tau \in C\backslash S(P)$, we shall show that $\tau \in \rho(P)$. It is clear that the distance from τ to $S(P)$ is positive, denoted by $c(\tau)$, hence for all $\beta \in \mathbf{I}_+^{\frac{p_1-d}{2}}$,

$$\Big|\tau - \Big[\sum_j (2\beta_j+1)\rho_j + \sum_i \zeta_i^2 - \sqrt{-1}\sum_l C_l\eta_l\Big]\Big| \geq c(\tau)$$

Now let $\varphi_\beta(\xi)$ be an Hermite function on $\mathbf{R}^{\frac{p_1-d}{2}}$, then

$$\Big(-\frac{\partial^2}{\partial s_j^2} + s_j^2\Big)\varphi_{\beta_j} = (2\beta_j+1)\varphi_{\beta_j}$$

Proposition 3.1 and (2.8) yield that for $u \in D(P)$

$$\begin{aligned}\|Pu-\tau u\|_{L^2}^2 =& C_0\int \|\pi_{\zeta,\eta}(Pu-\tau u)\|_{HS}^2 D(\eta)^{1/2} d\zeta d\eta \\ =& C_0\int \Big\|\pi_{\zeta,\eta}(u)\Big\{\tau - \Big[-\sum_j \rho_j\Big(\frac{\partial^2}{\partial s_j^2} - s_j^2\Big) + \sum_i \zeta_i^2 - \sqrt{-1}\sum_l C_l\eta_l\Big]\Big\}\Big\|_{HS}^2 \\ & \cdot D(\eta)^{1/2} d\zeta d\eta \\ \geq& C_0 c(\tau)^2 \int \|\pi_{\zeta,\eta}(u)\|_{HS}^2 D(\eta)^{1/2} d\zeta d\eta = c(\tau)^2\|u\|_{L^2}^2\end{aligned}$$

which implies that

$$\|Pu-\tau u\|_{L^2} \geq c(\tau)\|u\|_{L^2}$$

and so $P-\tau I : D(P) \to L^2(G)$ is injective.

We can show that $P-\tau I$ is surjective with the same way in [2].

Next we show $S(P) \subset \sigma(P)$.

Suppose that $\mu \in \Gamma_\beta(C)$ for some $\beta \in \mathbf{I}_+^{\frac{p_1-d}{2}}$, we claim that $\mu \in \sigma(P)$. If it is not true, i.e., $\mu \in \rho(P)$, then $\mu I - P$ is a bijective map from $D(P)$ to $L^2(G)$. Therefore for every $f \in L^2(G)$, there exists a unique $u \in D(P)$ such that

$$\mu u - Pu = f$$

By Propositon 3.1, we have

$$\begin{aligned}\pi_{\zeta,\eta}(f)\varphi_\beta =& [\mu\pi_{\zeta,\eta}(u) - \pi_{\zeta,\eta}(u)\pi_{\zeta,\eta}(P^t)]\varphi_\beta \\ =& \pi_{\zeta,\eta}(u)\Big\{\mu - \Big[\sum_j (2\beta_j+1)\rho_j + \sum_i \zeta_i^2 - \sqrt{-1}\sum_l C_l\eta_l\Big]\Big\}\varphi_\beta\end{aligned}$$

It follows from $\mu \in \Gamma_\beta$ that,

$$\pi_{\zeta,\eta}(f)\varphi_\beta = 0, \quad \text{for every } f \in L^2$$

On the other hand, we can find a function $f \in L^2(G)$ below such that $\pi_{\zeta,\eta}(f)\varphi_\beta \neq 0$, which leads to a contradiction.

In fact, set

$$f(t,z,w,y) = f_1(t)f_2(z)f_3(w)f_4(y)$$

where $f_1, f_2 \in \mathcal{S}(\mathbb{R}^{\frac{p_1-d}{2}})$, $f_3 \in \mathcal{S}(\mathbb{R}^d)$, $f_4 \in \mathcal{S}(\mathbb{R}^{p_2})$. From (2.3),

$$\pi_{\zeta,\eta}(f)\varphi_\beta = \int e^{\sqrt{-1}(-y\eta - w\zeta + \rho z\cdot\frac{t}{2} - \sqrt{\rho}z\cdot s}f_1(t)f_2(z)f_3(w)f_4(y)\varphi_\beta(s-\sqrt{\rho}t)dtdzdwdy$$

where $\rho = \rho(\eta)$, and thus

$$\begin{aligned}\pi_{\zeta,\eta}(f)\varphi_\beta|_{s=0} &= \int e^{-\sqrt{-1}(y\eta+w\zeta)+\sqrt{-1}\rho z\cdot\frac{t}{2}}f_1(t)f_2(z)f_3(w)f_4(y)\varphi_\beta(-\sqrt{\rho}t)dtdzdwdy\\ &=\hat{f}_3(\zeta)\hat{f}_4(\eta)\int e^{\sqrt{-1}z\cdot\frac{\rho t}{2}}f_1(t)f_2(z)\varphi_\beta(-\sqrt{\rho}t)dtdz\\ &=\hat{f}_3(\zeta)\hat{f}_4(\eta)\int f_1(t)\hat{f}_2\Big(-\frac{\rho t}{2}\Big)\varphi_\beta(-\sqrt{\rho}t)dt\end{aligned}$$

Now take $f_1(t) = \overline{\varphi_\beta(-\sqrt{\rho}t)}$ and $f_2(z)$, $f_3(w)$, $f_4(y)$ so that $\hat{f}_2\Big(-\frac{\rho t}{2}\Big)$, $\hat{f}_3(\zeta)$, $\hat{f}_4(\eta) > 0$ respectively. Then $\pi_{\zeta,\eta}(f)\varphi_\beta > 0$.

4. Eigenvalue

Proof of Theorem 1.2 Suppose that

$$Pu - \tau u = 0, \quad \text{for some } \tau \in C,\ u \in D(P) \tag{4.1}$$

We will show that $u = 0$ a.e. Applying (2.4) to (4.1) yields

$$\Big\{\tau - \Big[\sum_j(2\beta_j+1)\rho_j + \sum_i \zeta_i^2 - \sqrt{-1}\sum_l C_l\eta_l\Big]\Big\}\pi_{\zeta,\eta}(u)\varphi_\beta = 0 \tag{4.2}$$

where $\beta \in \boldsymbol{I}_+^{\frac{p_1-d}{2}}$, $\zeta \in \mathbb{R}^d, \eta \in \mathbb{R}^{p_2}$. Consider the following three cases on C_l ($l = 1,\cdots,p_2$).

1) Re $C_l \neq 0$, for some l.

Then when only ζ, η satisfy

$$\begin{cases}\text{Re}\,\tau - \Big[\sum_j(2\beta_j+1)\rho_j + \sum_i\zeta_i^2 + \sum_l \text{Im}C_l\cdot\eta_l\Big] = 0, \quad \forall\beta\\ \text{Im}\,\tau + \sum_l \text{Re}C_l\cdot\eta_l = 0\end{cases} \tag{4.3}$$

$\pi_{\zeta,\eta}(u)\varphi_\beta$ may not be equal to zero, otherwise $\pi_{\zeta,\eta}(u)\varphi_\beta$ must be zero. But (4.3) is at most a $d+p_2-1$ dimensional hypersurface in $\mathbb{R}^d\times\mathbb{R}^{p_2}$, so it has zero measure. By (2.8) $u = 0$ a.e.

2) Re $C_l = 0$, $\forall l$, and $\sum_j(2\beta_j+1)\rho_j + \sum_l \text{Im}C_l\cdot\eta_l \neq 0$, for all β.

From (4.2), τ is real and

$$\tau - \Big[\sum_j (2\beta_j+1)\rho_j + \sum_i \zeta_i^2 + \sum_l \mathrm{Im} C_l \cdot \eta_l\Big] = 0 \tag{4.4}$$

determines a $d+p_2-1$ dimensional hypersurface in $\mathbf{R}^d \times \mathbf{R}^{p_2}$, on which $\pi_{\zeta,\eta}(u)\varphi_\beta$ may not be zero. When ζ, η do not satisfy (4.4), $\pi_{\zeta,\eta}(u)\varphi_\beta$ are zero. Also by (2.8), $u = 0$, a.e.

3) Re $C_l = 0$ $(\forall l)$ and $\sum_j (2\beta_j+1)\rho_j + \sum_l \mathrm{Im} C_l \cdot \eta_l = 0$ for some β.

One sees that τ is real and

$$\tau - \sum_{i=1}^d \zeta_i^2 = 0 \tag{4.5}$$

is a $d-1$ dimensional hypersurface in $\mathbf{R}^d$. As before $u = 0$ a.e.

These show that τ is not an eigenvalue of P.

Proof of Theorem 1.3 First, observe that C_l $(l = 1, \cdots, p_2)$ satisfying (1.4) must be purely imaginary. In the case Re $C_l = 0$ $(\forall l)$ and $\sum_j (2\beta_j+1)\rho_j - \sqrt{-1}\sum_l C_l \cdot \eta_l \neq 0$ $(\forall \beta)$, P has not any eigenvalue with the progress in the proof of Theorem 1.2.

In what follows we investigate the case Re $C_l = 0$ $(\forall l)$ and (1.4) holds for some β_0.

It is easy to check that P has not nonzero eigenvalue. It therefore suffices to prove that 0 is a unique eigenvalue of P, i.e., there exists a $u \in \mathcal{S}(G) \subset D(P)$, $u \neq 0$, such that $Pu = 0$.

Indeed, take a nontrivial function

$$u(t,z,y) = C_1 \int g(\eta')\varphi_{\beta_0}(s_1) e^{\sqrt{-1}(y\eta' + \rho_1 z\cdot\frac{t}{2} + \sqrt{\rho_1} z\cdot s_1)} \cdot \varphi_{\beta_0}(s_1 + \sqrt{\rho_1} t) ds_1 D(\eta')^{1/2} d\eta' \tag{4.6}$$

where $C_1 = (2\pi)^{-\frac{p_1}{2}-p_2}$, $\rho_1 = \rho(\eta')$, $g(\eta') \in C_0^\infty(\mathbf{R}^{p_2})$. By integration by parts, we have

$$\int \varphi_{\beta_0}(s_1) e^{\sqrt{-1}(y\eta' + \rho_1 z\cdot\frac{t}{2} + \sqrt{\rho_1} z\cdot s_1)} \varphi_{\beta_0}(s_1 + \sqrt{\rho_1} t) ds_1 \in \mathcal{S}(G)$$

and so $u \in \mathcal{S}(G)$. We also need the following.

Proposition 4.1 *$u(t,z,y)$ in (4.6) satisfies that*

$$\pi_\eta(u)\varphi_\beta(s) = \begin{cases} 0, & \beta \neq \beta_0 \\ g(\eta)\varphi_{\beta_0}(s), & \beta = \beta_0 \end{cases} \tag{4.7}$$

Its proof will be given later. By (4.7) and (1.4),

$$\Big[\sum_j (2\beta_j+1)\rho_j - \sqrt{-1}\sum_l C_l \cdot \eta_l\Big]\pi_\eta(u)\varphi_\beta = 0, \quad \forall \beta, \forall \eta \tag{4.8}$$

this yields that $\pi_\eta(Pu)\varphi_\beta = 0$ and thus $Pu = 0$ from (2.6). The proof is completed.

Proof of Proposition 4.1 From (2.3) one deduces that

$$\pi_\eta(u)\varphi_\beta(s) = C_1 \int g(\eta')\varphi_{\beta_0}(s_1) e^{\sqrt{-1}(y\eta' + \rho_1 z\cdot\frac{t}{2} + \sqrt{\rho_1} z\cdot s_1)} \cdot \varphi_{\beta_0}(s_1 + \sqrt{\rho_1} t) D(\eta')^{1/2} d\eta' ds_1 \cdot e^{\sqrt{-1}(-y\eta + \rho z\cdot\frac{t}{2} - \sqrt{\rho} z\cdot s)} \varphi_\beta(s - \sqrt{\rho} t) dt dz dy$$

Since $\int e^{\sqrt{-1}y(\eta'-\eta)}dy = (2\pi)^{p_2}\delta(\eta-\eta')$, where δ means Dirac function, and $\rho_1 = \rho$, if $\eta' = \eta$,

$$\pi_\eta(u)\varphi_\beta(s) = C_1 \int (2\pi)^{p_2} g(\eta)\varphi_{\beta_0}(s_1) e^{\sqrt{-1}(\rho z\cdot\frac{t}{2} + \sqrt{\rho} z\cdot s_1)} \varphi_{\beta_0}(s_1 + \sqrt{\rho} t) D(\eta)^{1/2} ds_1 \cdot e^{\sqrt{-1}(\rho z\cdot\frac{t}{2} - \sqrt{\rho} z\cdot s)} \varphi_\beta(s - \sqrt{\rho} t) dt dz$$

Again note that

$$\int e^{\sqrt{-1}(\rho z\cdot t + \sqrt{\rho} z\cdot s_1 - \sqrt{\rho} z\cdot s)} dz = (2\pi)^{\frac{p_1}{2}} \delta(-\sqrt{\rho}(s_1 - s) - \rho t)$$

it follows that

$$\begin{aligned}\pi_\eta(u)\varphi_\beta(s) =& C_1 g(\eta)(2\pi)^{\frac{p_1}{2}+p_2} \int \varphi_{\beta_0}(s_1)\varphi_{\beta_0}(s_1 + \sqrt{\rho} t)\varphi_\beta(s - \sqrt{\rho} t) \\ &\cdot \delta(-\sqrt{\rho}(s_1 - s) - \rho t) dt ds_1 \\ =& g(\eta) \int \varphi_{\beta_0}(s_1)\varphi_{\beta_0}(s)\varphi_\beta(s_1) ds_1 = g(\eta)\varphi_{\beta_0}(s) \int \varphi_{\beta_0}(s_1)\varphi_\beta(s_1) ds_1\end{aligned}$$

Applying the orthogonal property of Hermite functions, we conclude the proof.

5. Some Applications

Let G be the Heisenberg group $\mathbf{H}_n$, then $p_1 = 2n$, $p_2 = 1$ and $d = 0$. Consider the Kohn-Laplacian on $\mathbf{H}_n$

$$P = -\sum_{j=1}^{n}(X_j^2 + Y_j^2) + \sqrt{-1}\alpha T$$

where $\{X_j, Y_j, T\}$ ($j = 1, \cdots, n$) is the basis for the Lie algebra of $\mathbf{H}_n$. If $\alpha = \pm(2m+n)$ for some nonnegative integer m, then P has eigenvalue 0, and if $\alpha \neq \pm(2m+n)$ for any nonnegative integer m, then P has not any eigenvalue (See [2]).

Consider the operator

$$P = -\sum_{j=1}^{n}(X_j^2 + Y_j^2) - \sum_{i=1}^{k} Z_i^2 + \sqrt{-1}\alpha T$$

on the two-step nilpotent Lie group $\mathbf{H}_n \times \mathbf{R}^k$, where $\{X_j, Y_j, Z_i, T\}$ ($j = 1, \cdots, n$, $i = 1, \cdots, k$) is the basis of corresponding Lie algebra. Now $p_1 = 2n + k$, $p_2 = 1$, $d = k$. According to Theorem 1.2, P has not any eigenvalue.

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