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ASYMPTOTICS OF THE MODULE OF MINIMIZERS TO A GINZBURG-LANDAU TYPE FUNCTIONAL

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Abstract The author proves that the module of minimizers for a Ginzburg-Landau type functional converges to 1. And the estimates on the convergent rate are also presented.

Key Words Ginzburg-Landau type functional; module of the minimizers; the rate of convergence.

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1. Introduction

Let $G \subset R^n (n \geq 2)$ be a bounded and simply connected domain with smooth boundary ∂G. g be a smooth map from ∂G into S^{n-1} satisfying $W_g^{1,p}(G, S^{n-1}) \neq \emptyset$, where $W_g^{1,p}(G, S^{n-1}) = \{v \in W^{1,p}(G, S^{n-1}); v|_{\partial G} = g\}$. Consider the Ginzburg-Landau-type functional

$$E_\varepsilon(u, G) = \frac{1}{p}\int_G |\nabla u|^p + \frac{1}{4\varepsilon^p}\int_G (1 - |u|^2)^2, \quad p \geq 2$$

which has been well-studied in [1,2] for $p = n = 2$. For other related papers, we refer to [3–5].

The functional of the form $E_\varepsilon(u, G)$ was introduced in the study of superconductivity. Similar models are also used in superfluids and XY-magnetism. The minimizer u_ε of $E_\varepsilon(u, G)$ represents a complex order parameter and $|u_\varepsilon|$ has physics senses, for example, in superconductivity, $|u_\varepsilon|^2$ is proportional to the density of supercoducting electrons (i.e., $|u_\varepsilon| = 1$ corresponds to the superconducting state and $|u_\varepsilon| = 0$ corresponds to the normal state). In superfluids, $|u_\varepsilon|^2$ is proportional to the density of superfluid. Thus it is interesting to study the asymptotic behavior of $|u_\varepsilon|$ as $\varepsilon \to 0$.

Clearly the functional $E_\varepsilon(u, G)$ achieves its minimum on $W = \{v \in W^{1,p}(G, R^n); v|_{\partial G} = g\}$ by a function u_ε and there exists a subsequence u_{ε_k} of u_ε such that

$$\lim_{\varepsilon_k \to 0} u_{\varepsilon_k} = u_p, \quad \text{in } W^{1,p}(G, R^n) \tag{1.1}$$

where u_p is a map of least p-energy with boundary value g. It is not difficult to prove that the minimizers u_ε solve the following Euler equation

$$-\text{div}\,(|\nabla u|^{p-2}\nabla u) = \frac{1}{\varepsilon^p}u(1-|u|^2) \tag{1.2}$$

in the weak sense, and they also satisfy the maximum principle: $|u_\varepsilon| \le 1$ a.e. on G.

The general minimizers and one class of them which is named the regularizable minimizers, will be both concerned with in this paper. It is not obvious that $|u_\varepsilon|$, the module of the minimizer of $E_\varepsilon(u,G)$, converges to 1 in $C_{\text{loc}}(G,R^n)$ when $p=n$, which is clear as $p>n$ because of (1.1) and the embedding inequality. We shall assert it in Section 2. In the case $p>n$, the rate of convergence for $\nabla|u_\varepsilon|$ will be given in Section 3. Section 4, we shall introduce the regularizable minimizers $\tilde{u}_\varepsilon$. The estimates of their convergent rate which are better than that of general minimizers will be presented in Section 5.

2. C_{loc} Convergence for $|u_\varepsilon|$

From (1.1) and the embedding theorem we can say there exists a subsequence u_{ε_k} of u_ε such that $\lim\limits_{k\to\infty}|u_{\varepsilon_k}|=1$ in $C(\bar{G},R^n)$ when $p>n$. Since the limit 1 is unique, we obtain

$$\lim_{\varepsilon\to 0}|u_\varepsilon| = 1, \quad \text{in } C(\bar{G},R^n) \tag{2.1}$$

We always assume $p=n$ in this section. We shall prove the weaker conclusion in this case:

Theorem 2.1

$$\lim_{\varepsilon\to 0}|u_\varepsilon| = 1, \quad \text{in } C_{\text{loc}}(G,R^n).$$

For this purpose, we prove the following proposition at first.

Proposition 2.2 *Assume $u\in W$ is a weak solution of* (1.2). *For any $\rho>0$, denote $G^{\varepsilon\rho}=\{x\in G;\text{dist}(x,\partial G)>\varepsilon\rho\}$, then there exists a constant $C=C(\rho)$ independent of ε such that*

$$\|\nabla u\|_{L^\infty B(x,\varepsilon\rho/8)} \le C\varepsilon^{-1}, \quad x\in G^{\varepsilon\rho} \tag{2.2}$$

Proof Let $y=x\varepsilon^{-1}$ in (1.2) and denote $v(y)=u(x), G_\varepsilon=\{y=x\varepsilon^{-1}; x\in G\}, G^\rho=\{y\in G_\varepsilon, \text{dist}(y,\partial G_\varepsilon)>\rho\}$. Since u is a weak solution, we have

$$\int_{G_\varepsilon}|\nabla v|^{p-2}\nabla v\nabla\phi = \int_{G_\varepsilon} v(1-|v|^2)\phi, \quad \phi\in W_0^{1,p}(G_\varepsilon,R^n)$$

Taking $\phi=v\zeta^p, \zeta\in C_0^\infty(G_\varepsilon,R)$, we obtain

$$\int_{G_\varepsilon}|\nabla v|^p\zeta^p \le p\int_{G_\varepsilon}|\nabla v|^{p-1}\zeta^{p-1}|\nabla\zeta||v| + \int_{G_\varepsilon}|v|^2(1-|v|^2)\zeta^p$$

Setting $y \in G^\rho, B(y,\rho/2) \subset G_\varepsilon$, and $\zeta = 1$ in $B(y,\rho/4), \zeta = 0$ in $G_\varepsilon \backslash B(y,\rho/2), |\nabla\zeta| \leq C(\rho)$, we have

$$\int_{B(y,\rho/2)} |\nabla v|^p \zeta^p \leq C(\rho) \int_{B(y,\rho/2)} |\nabla v|^{p-1} \zeta^{p-1} + C(\rho)$$

Using Hölder inequality we can derive $\int_{B(y,\rho/4)} |\nabla v|^p \leq C(\rho)$. Combining this with the theorem of [6] yields

$$\|\nabla v\|^p_{L^\infty(B(y,\rho/8))} \leq C(\rho) \int_{B(y,\rho/4)} (1 + |\nabla v|)^p \leq C(\rho)$$

which implies

$$\|\nabla u\|_{L^\infty(B(x,\varepsilon\rho/8))} \leq C(\rho)\varepsilon^{-1}$$

The proof of Theorem 2.1 Noticing the weakly low semicontinuity of the functional $\int_G |\nabla u|^n$ and using (1.1) we have $\underline{\lim}_{\varepsilon_k \to 0} \int_G |\nabla u_{\varepsilon_k}|^n \geq \int_G |\nabla u_n|^n$. Combining this with

$$\frac{1}{n}\int_G |\nabla u_n|^n = E_{\varepsilon_k}(u_n, G) \geq E_{\varepsilon_k}(u_{\varepsilon_k}, G)$$
$$= \frac{1}{n}\int_G |\nabla u_{\varepsilon_k}|^n + \frac{1}{4\varepsilon_k^n}\int_G (1 - |u_{\varepsilon_k}|^2)^2$$

we obtain

$$\frac{1}{n}\int_G |\nabla u_{\varepsilon_k}|^n + \frac{1}{4\varepsilon_k^n}\int_G (1 - |u_{\varepsilon_k}|^2)^2 \to \frac{1}{n}\int_G |\nabla u_n|^n \tag{2.3}$$

as $\varepsilon_k \to 0$. From (1.1) we may conclude that as $\varepsilon_k \to 0, \int_G |\nabla u_{\varepsilon_k}|^n \to \int_G |\nabla u_n|^n$. Substituting this into (2.3) yields

$$\frac{1}{4\varepsilon_k^n}\int_G (1 - |u_{\varepsilon_k}|^2)^2 \to 0 \tag{2.4}$$

as $\varepsilon_k \to 0$. For all subsequence u_{ε_k} of u_ε, there exists a subsequence of u_{ε_k} denoting itself such that (2.4) is always true. So we derive $\frac{1}{4\varepsilon^n}\int_G (1 - |u_\varepsilon|^2)^2 \to 0$, i.e., when $\varepsilon \to 0$,

$$\int_G (1 - |u_\varepsilon|^2)^2 \leq \varepsilon^n o(1) \tag{2.5}$$

For arbitrary K being compact subset of G, there exists ε_0 small enough such that $K \subset G^{2\rho\varepsilon_0}$. We assume $\varepsilon < \varepsilon_0$. For $x_0 \in K$, let $\alpha = |u_\varepsilon(x_0)|$. Proposition 2.2 implies

$$|u_\varepsilon(x) - u_\varepsilon(x_0)| < C\varepsilon^{-1}\tau\varepsilon, \quad \text{if } x \in B(x_0, \tau\varepsilon)$$

where $\tau = (1-\alpha)(NC)^{-1}$, C is the constant of Proposition 2.2 and N is a large constant such that $\tau < \rho/8$. Thus

$$|u_\varepsilon(x)| \leq \alpha + C\tau, \quad \text{if } x \in B(x_0, \tau\varepsilon)$$
$$\int_{B(x_0,\tau\varepsilon)} (1 - |u_\varepsilon(x)|^2)^2 \geq (1 - 1/N)^2(1-\alpha)^{n+2}\pi\varepsilon^n(NC)^{-1}$$

Combining this with (2.5) we obtain $(1-\alpha)^{n+2} \le o(1)$. From this we can complete the proof.

3. The Convergent Rate of $\|u_\varepsilon\|_{W^{1,q}}$

Assume u_ε is the minimizer of $E_\varepsilon(u,G)$ in W. We shall show that there exist constants $C, \lambda > 0$ such that

$$\||u_\varepsilon|\|_{W^{1,q}} \le C\varepsilon^\lambda, \quad \forall q \in (1,p)$$

Noticing that u_ε is a minimizer of $E_\varepsilon(u,G)$, we have $\int_G |\nabla u_p|^p \le \underline{\lim}_{\varepsilon\to 0} \int_G |\nabla \tilde{u}_\varepsilon|^p \le pE_\varepsilon(u_\varepsilon, G) \le \int_G |\nabla u_p|^p$ by using low semicontinuity of $\int_G |\nabla u|^p$. And (1.1) implies $\int_G |\nabla u_\varepsilon|^p \to \int_G |\nabla u_p|^p$ as $\varepsilon \to 0$. Combining these two inequalities we have $\frac{1}{\varepsilon^p}\int_G (1 - |u_\varepsilon|^2)^2 \to 0$, as $\varepsilon \to 0$. Thus the following theorem is only needed.

Theorem 3.1 *If $p > n$, then for any $q \in (1,p)$, there exist constants $C, \lambda > 0$, independent of ε such that*

$$\int_G |\nabla|u_\varepsilon||^q \le C\varepsilon^\lambda$$

for $\varepsilon \in (0,\eta)$ with some small $\eta > 0$.

Proof From (2.1) we can set $u = hw, h = |u|, w = u|u|^{-1}$ in (1.2) as $\varepsilon \in (0,1)$ small enough. Then h, w satisfy

$$\int_G (|\nabla u|^{p-2}(w\nabla h + h\nabla w))\nabla\phi = \frac{1}{\varepsilon^p}\int_G wh(1-h^2)\phi$$

$\forall \phi \in W^{1,p}(G,R^n), \phi|_{\partial G} = 0$. Fix $\beta \in (0, p/2)$ and set $S = \{x \in G; |h(x)| > 1 - \varepsilon^\beta\}, \bar{h} = \max(h, 1-\varepsilon^\beta)$. Since $\bar{h}|_{\partial G} = 1$, taking $\phi = wh(1-\bar{h})$, we have

$$\int_G v^{(p-2)/2}(w\nabla h + h\nabla w)\nabla(wh(1-\tilde{h})) = \frac{1}{\varepsilon^p}\int_G h^2(1-h^2)(1-\bar{h})$$

Noticing that $|w| = 1$ and $2w\nabla w = \nabla(|w|^2) = 0$, we obtain

$$\begin{aligned}\frac{1}{\varepsilon^p}\int_G h^2(1-h^2)(1-\tilde{h}) + \int_S v^{(p-2)/2}h|\nabla h|^2 \\ \le \int_G v^{(p-2)/2}h^2|\nabla w|^2(1-\bar{h}) + \int_G v^{(p-2)/2}|\nabla h|^2(1-\tilde{h})\end{aligned} \quad (3.1)$$

Since u_ε is the minimizer of $E_\varepsilon(u,G)$, we have $E_\varepsilon(u_\varepsilon, G) \le E_\varepsilon(u_p, G) = \frac{1}{p}\int_G |\nabla u_p|^p \le C$, namely

$$\int_G |\nabla u_\varepsilon|^p \le C \quad (3.2)$$

$$\int_G (1 - |u_\varepsilon|^2)^2 \le C\varepsilon^p \quad (3.3)$$

where C is a constant independent of ε. (3.1) implies $\int_S v^{(p-2)/2}h|\nabla h|^2 \le C\varepsilon^\beta$ by using (3.2) and the facts $|\nabla u|^2 = |\nabla h|^2 + h^2|\nabla w|^2$. Sinice $\tilde{h} = h$ on S and $\tilde{h} > 1/2$ for $\varepsilon > 0$ small enough, we have

$$\int_S |\nabla h|^p \le C\varepsilon^\beta \tag{3.4}$$

On the other hand, from the defination of S and (3.3), we have $C\text{mes}(G\backslash S)\varepsilon^{2\beta} \le \int_{G\backslash S}(1-|u|^2)^2 \le C\varepsilon^p$, namely $\text{mes}(G\backslash S) \le C\varepsilon^{p-2\beta}$, using (3.2) again we obtain that for any $q \in (1,p)$

$$\int_{G\backslash S} |\nabla h|^q \le \text{mes}(G-S)^{1-q/p}\left(\int_G |\nabla h|^p\right)^{q/p} \le C\varepsilon^{(p-2\beta)(1-q/p)}$$

The above and (3.4) imply the conclusion of Theorem 3.1.

4. The Regularizable Minimizers $\tilde{u}_\varepsilon$

The minimizers might be un-unique, one of which, denoted by $\tilde{u}_\varepsilon$, can be obtained as the limit of a subsequence $u_\varepsilon^{\tau_k}$ of the minimizers u_ε^τ of the regularized functionals

$$E_\varepsilon^\tau(u,G) = \frac{1}{p}\int_G (|\nabla u|^2 + \tau)^{p/2} + \frac{1}{4\varepsilon^p}\int_G (1-|u|^2)^2, \quad \tau > 0$$

on W as $\tau_k \to 0$, namely

Theorem 4.1 *Assume u_ε^τ to be minimizers of $E_\varepsilon^\tau(u,G)$ in W and $p > 1$. Then there exists a subsequence $u_\varepsilon^{\tau_k}$ of u_ε^τ and $\tilde{u}_\varepsilon \in W$ such that*

$$\lim_{\tau_k \to 0} u_\varepsilon^{\tau_k} = \tilde{u}_\varepsilon, \quad in\ W^{1,p}(G, R^n) \tag{4.1}$$

where $\tilde{u}_\varepsilon$ is the minimizer of $E_\varepsilon(u,G)$ in W.

We call $\tilde{u}_\varepsilon$ the regularizable minimizer of $E_\varepsilon(u,G)$.

It is not difficult to prove that the minimizer u_ε^τ is a classical solution of the equation

$$-\text{div}\,(v^{(p-2)/2}\nabla u) = \frac{1}{\varepsilon^p}u(1-|u|^2) \tag{4.2}$$

and satisfies the maximum principle: $|u_\varepsilon^\tau| \le 1$ on G, where $v = |\nabla u|^2 + \tau$.

Proof First we have $E_\varepsilon^\tau(u_\varepsilon^\tau, G) \le E_\varepsilon^\tau(u_p, G) \le \dfrac{1}{p}\int_G (|\nabla u_p|^2 + 1)^{p/2} = C$ as $\tau \in (0,1)$. This and $|u_\varepsilon^\tau| \le 1$ imply that there exists a subsequence $u_\varepsilon^{\tau_k}$ of u_ε^τ and $\tilde{u}_\varepsilon \in W^{1,p}(G,R^n)$ such that

$$u_\varepsilon^{\tau_k} \xrightarrow{w} \tilde{u}_\varepsilon, \quad \text{in } W^{1,p}(G,R^n) \tag{4.3}$$

$$u_\varepsilon^{\tau_k} \to \tilde{u}_\varepsilon, \quad \text{in } C(\bar{G},R^n), \text{ when } p > n \tag{4.4.1}$$

$$u_\varepsilon^{\tau_k} \to \tilde{u}_\varepsilon, \quad \text{in } L^q(G,R^n), q < \frac{np}{n-p}, \text{ when } 1 < p \le n \tag{4.4.2}$$

as $\tau_k \to 0$. By virtue of (4.3) and the weakly low semicontinuity of the functional $\int_G |\nabla u|^p$, we obtain

$$\int_G |\nabla \tilde{u}_\varepsilon|^p \leq \underline{\lim}_{\tau_k \to 0} \int_G |\nabla u_\varepsilon^{\tau_k}|^p \tag{4.5}$$

We claim $\tilde{u}_\varepsilon \in W$. In fact, (4.4.1) implies this when $p > n$. And when $1 < p \leq n$, it can be deduced from W being the weak, closed subset of $W^{1,p}(G, R^n)$ and (4.3). This means $E_\varepsilon^{\tau_k}(u_\varepsilon^{\tau_k}, G) \leq E_\varepsilon^{\tau_k}(\tilde{u}_\varepsilon, G)$ or

$$\overline{\lim}_{\tau_k \to 0} E_\varepsilon^{\tau_k}(u_\varepsilon^{\tau_k}, G) \leq \lim_{\tau_k \to 0} E_\varepsilon^{\tau_k}(\tilde{u}_\varepsilon, G) \tag{4.6}$$

We can also deduce $\int_G (1 - |u_\varepsilon^{\tau_k}|^2)^2 \to \int_G (1 - |\tilde{u}_\varepsilon|^2)^2$ from (4.4) as $\tau_k \to 0$. This and (4.6) show

$$\overline{\lim}_{\tau_k \to 0} \int_G (|\nabla u_\varepsilon^{\tau_k}|^2 + \tau_k)^{p/2} \leq \lim_{\tau_k \to 0} \int_G (|\nabla \tilde{u}_\varepsilon|^2 + \tau_k)^{p/2} = \int_G |\nabla \tilde{u}_\varepsilon|^p$$

Combining this with (4.5) we obtain $\int_G |\nabla u_\varepsilon^{\tau_k}|^p \to \int_G |\nabla \tilde{u}_\varepsilon|^p$ as $\tau_k \to 0$, which together with (4.3) implies $\nabla u_\varepsilon^{\tau_k} \to \nabla \tilde{u}_\varepsilon$, in $L^p(G, R^n)$. Noticing (4.4) we have the conclusion $u_\varepsilon^{\tau_k} \to \tilde{u}_\varepsilon$, in $W^{1,p}(G, R^n)$ as $\tau_k \to 0$. This is (4.1).

On the other hand, we know

$$E_\varepsilon^{\tau_k}(u_\varepsilon^{\tau_k}, G) \leq E_\varepsilon^{\tau_k}(u, G) \tag{4.7}$$

for all $u \in W$. Noticing the conclusion $\lim_{\tau_k \to 0} E_\varepsilon^{\tau_k}(u_\varepsilon^{\tau_k}, G) = E_\varepsilon(\tilde{u}_\varepsilon, G)$ which had been proved just now we can say $E_\varepsilon(\tilde{u}_\varepsilon, G) \leq E_\varepsilon(u, G)$ when $\tau_k \to 0$ in (4.7), which implies $\tilde{u}_\varepsilon$ is a minimizer of $E_\varepsilon(u, G)$.

Remark Theorem 2.2 in [3] and the proof of Theorem 2.1 imply that if $p = n$, there exists no zero of $\tilde{u}_\varepsilon$, the regularizable minimizer of $E_\varepsilon(u, G)$, in G when ε small enough. Similarly, we can also derive the same conclusion for u_ε^τ which is a minimizer of the regularized functional $E_\varepsilon^\tau(u, G)$ when $p = n$, namely, there exists no zero of u_ε^τ in G when ε, τ small enough.

5. The Rate of the Convergence for $|\tilde{u}_\varepsilon|$

We start our argument with the following

Proposition 5.1 *Suppose $p > n$. Then*

$$\lim_{\varepsilon, \tau \to 0} |u_\varepsilon^\tau| = 1, \quad in\ C(\bar{G}, R^n) \tag{5.1}$$

Proof We have, for $\tau \in (0,1), E_\varepsilon^\tau(u_\varepsilon, G) \leq E_\varepsilon^\tau(u_p, G) = C$. Hence

$$\int_G |\nabla u_\varepsilon^\tau|^p \leq \int_G (|\nabla u_\varepsilon^\tau|^2 + \tau)^{p/2} \leq C \tag{5.2}$$

$$\int_G (1 - |u_\varepsilon^\tau|^2)^2 \leq C\varepsilon^p \tag{5.3}$$

From (5.3) it follows that there exists a subsequence $u_{\varepsilon_k}^{\tau_k}$ of u_ε^τ with $\varepsilon_k \to 0, \tau_k \to 0$ as $k \to \infty$, such that

$$\lim_{k\to\infty} |u_{\varepsilon_k}^{\tau_k}| = 1, \quad \text{a.e. in } G \tag{5.4}$$

(5.2) combined with $|u_\varepsilon^\tau| \le 1$ means that $\|u_\varepsilon^\tau\|_{W^{1,p}(G,R^n)} \le C$ which implies that there exist a function $u_* \in W^{1,p}(G,R^n)$ and a subsequnce of $u_{\varepsilon_k}^{\tau_k}$, supposed to be $u_{\varepsilon_k}^{\tau_k}$ itself, such that

$$\lim_{k\to\infty} u_{\varepsilon_k}^{\tau_k} = u_*, \quad \text{in } C(\bar{G}, R^n) \tag{5.5}$$

Combining (5.5) with (5.4) yields $|u_*| = 1$ in G and hence $\lim_{k\to\infty} |u_{\varepsilon_k}^{\tau_k}| = 1$, in $C(\bar{G}, R^n)$. Since any subsequence of $|u_\varepsilon^\tau|$ contains a uniformly convergent subsequence and the limit is the same number 1, we may assert (5.1) and complete the proof.

Theorem 5.2 *If $p \ge n$, then for any $q \in (1,p)$, there exist constants $C, \lambda > 0$, independent of ε such that*

$$\int_G |\nabla|\tilde{u}_\varepsilon||^q \le C\varepsilon^\lambda$$

for $\varepsilon \in (0,\eta)$ with some small $\eta > 0$.

Proof As a minimizer of $E_\varepsilon^\tau(u,G), u = u_\varepsilon^\tau$ satisfies (4.2). Owing to the Remark in Section 4 and Proposition 5.1 we can set $u = hw, h = |u|, w = u|u|^{-1}$ as $\varepsilon, \tau \in (0,1)$ small enough. Then h, w satisfy

$$-\text{div}\,(v^{(p-2)/2}(w\nabla h + h\nabla w)) = \frac{1}{\varepsilon^p} wh(1-h^2)$$

Multiplying this by wh, we have

$$-\text{div}\,(v^{(p-2)/2}\nabla h)h - \text{div}\,(v^{(p-2)/2}h^2\nabla w)w = \frac{1}{\varepsilon^p} h(1-h^2) \tag{5.6}$$

Fix $\beta \in (0, p/2)$ and set $S = \{x \in G; |h(x)| > 1 - \varepsilon^\beta\}, \tilde{h} = \max(h, 1-\varepsilon^\beta)$. Multiplying (5.6) with $(1-\tilde{h})$, integrating over G and noticing that $h|_{\partial G} = 1$, we have

$$\int_G v^{(p-2)/2} h\nabla h \nabla(h(1-\tilde{h})) + \int_G v^{(p-2)/2} h^2 \nabla w \nabla(w(1-\tilde{h}))$$
$$= \frac{1}{\varepsilon^p}\int_G h^2(1-h^2)(1-\tilde{h})$$

Noticing that $|w| = 1$ and $2w\nabla w = \nabla(|w|^2) = 0$, we obtain

$$\frac{1}{\varepsilon^p}\int_G h^2(1-h^2)(1-\tilde{h}) + \int_S v^{(p-2)/2} h|\nabla h|^2$$
$$\le \int_G v^{(p-2)/2} h^2 |\nabla w|^2 (1-\tilde{h}) + \int_G v^{(p-2)/2} |\nabla h|^2 (1-\tilde{h}) \tag{5.7}$$

By using (5.2), (5.7) and the facts $|\nabla u|^2 = |\nabla h|^2 + h^2|\nabla w|^2$, we have $\int_S v^{(p-2)/2} h|\nabla h|^2 \le C\varepsilon^\beta$. Since $\tilde{h} = h$ on S and $\tilde{h} > 1/2$ for $\varepsilon > 0$ small enough, we derive

$$\int_S |\nabla h|^p \le C\varepsilon^\beta \tag{5.8}$$

On the other hand, from the defination of S and (5.3), we obtain

$$C\text{mes}(G\backslash S)\varepsilon^{2\beta} \le \int_{G\backslash S}(1-|u|^2)^2 \le C\varepsilon^p \tag{5.9}$$

namely $\text{mes}(G\backslash S) \le C\varepsilon^{p-2\beta}$. Using (5.2) again we obtain that for any $q \in (1,p)$

$$\int_{G\backslash S}|\nabla h|^q \le \text{mes}(G\backslash S)^{1-q/p}\left(\int_G |\nabla h|^p\right)^{q/p} \le C\varepsilon^{(p-2\beta)(1-q/p)} \tag{5.10}$$

The above and (5.8), Theorem 4.1 imply the conclusion of Theorem 5.2.

Theorem 5.3 *Assume $p>2$, then there exists a constant C independent of ε, such that*

$$\frac{1}{\varepsilon^p}\int_G(1-|u_\varepsilon|^2) \le C \tag{5.11}$$

Proof First taking the inner product of both the sides of (4.2) with u and integrating over G, we have

$$-\int_G \text{div}\,(v^{(p-2)/2}\nabla u)u = \frac{1}{\varepsilon^p}\int_G |u|^2(1-|u|^2)$$

Integrating by parts, using (5.2) and the Hölder inequality we obtain

$$\begin{aligned}\frac{1}{\varepsilon^p}\int_G |u|^2(1-|u|^2) &\le \int_G v^{(p-2)/2}|\nabla u|^2 + \int_{\partial G} v^{(p-2)/2}|u_n||u| \\ &\le C + \int_{\partial G} v^{(p-2)/2}|u_n| \le C + C\int_{\partial G} v^{(p-2)/2} + C\int_{\partial G} v^{(p-2)/2}|u_n|^2 \\ &\le C + C\int_{\partial G} v^{p/2}\end{aligned} \tag{5.12}$$

where n denotes the unit outward normal to ∂G.

To estimate $\int_{\partial G} v^{p/2}$, we choose a smooth vector field $\nu = (\nu_1, \nu_2, \cdots, \nu_n)$ such that $\nu|_{\partial G} = n$. Taking the inner product of both the sides of (4.2) with $\nu\cdot\nabla u$ and integrating over G we have

$$-\int_G \text{div}\,(v^{(p-2)/2}\nabla u)(\nu\cdot\nabla u) = \frac{1}{2\varepsilon^p}\int_G (1-|u|^2)(\nu\cdot\nabla|u|^2)$$

Integrating by parts and noticing $|u|_{\partial G} = |g| = 1$ and

$$\int_G (1-|u|^2)(\nu\cdot\nabla|u|^2) = -\frac{1}{2}\int_G \nabla(1-|u|^2)^2\cdot\nu = \frac{1}{2}\int_G (1-|u|^2)^2\text{div}\,\nu$$

we obtain

$$\begin{aligned}&-\int_{\partial G} v^{(p-2)/2}|u_n|^2 + \int_G v^{(p-2)/2}\nabla u\cdot\nabla(\nu\cdot\nabla u) \\ &= \frac{1}{4\varepsilon^p}\int_G (1-|u|^2)^2\text{div}\,\nu\end{aligned} \tag{5.13}$$

From the smoothness of ν and (5.2) (5.3) we have

$$\frac{1}{\varepsilon^p}\int_G (1-|u|^2)^2|\mathrm{div}\,\nu| \le C \tag{5.14}$$

$$\begin{aligned}\int_G v^{(p-2)/2}\nabla u\nabla(\nu\cdot\nabla u) &\le C\int_G v^{(p-2)/2}|\nabla u|^2+\frac{1}{2}\int_G v^{(p-2)/2}\nu\cdot\nabla v\\ &\le C+\frac{1}{p}\int_G \nu\cdot\nabla(v^{p/2})\\ &\le C+\frac{1}{p}\int_G \mathrm{div}\,(\nu v^{p/2})-\frac{1}{p}\int_G v^{p/2}\mathrm{div}\,\nu\\ &\le C+\frac{1}{p}\int_{\partial G} v^{p/2}\end{aligned} \tag{5.15}$$

and

$$\begin{aligned}\int_{\partial G} v^{p/2} &= \int_{\partial G} v^{(p-2)/2}(|u_n|^2+|g_t|^2+\tau)\\ &\le \int_{\partial G} v^{(p-2)/2}|u_n|^2+C\int_{\partial G} v^{(p-2)/2}\end{aligned} \tag{5.16}$$

where g_t denotes the derivative of g with respect to the tagent vector t to ∂G. Combining (5.13)–(5.16) we obtain

$$\int_{\partial G} v^{p/2} \le C\int_{\partial G} v^{(p-2)/2}+C+\frac{1}{p}\int_{\partial G} v^{p/2}$$

and derive

$$\int_{\partial G} v^{p/2}\le C \tag{5.17}$$

by using the Young inequality. Substituting (5.17) into (5.12) yields

$$\frac{1}{\varepsilon^p}\int_G |u|^2(1-|u|^2)\le C$$

which together with (5.3) and Theorem 4.1 implies (5.11).

Remark Noticing that $\tilde{u}_{\varepsilon_k}$ is a minimizer of $E_{\varepsilon_k}(u,G)$, we have

$$\int_G |\nabla u_p|^p \le \underline{\lim}_{\varepsilon_k\to 0}\int_G |\nabla \tilde{u}_{\varepsilon_k}|^p \le pE_{\varepsilon_k}(u_{\varepsilon_k},G)\le \int_G |\nabla u_p|^p \tag{5.18}$$

by using low semicontinuity of $\int_G |\nabla u|^p$ and Theorem 4 in [4], which also implies that

$$\int_G |\nabla \tilde{u}_{\varepsilon_k}|^p \to \int_G |\nabla u_p|^p \tag{5.19}$$

as $\varepsilon_k\to 0$. Substituting (5.19) into (5.18) we obtain

$$\frac{1}{4\varepsilon_k^p}\int_G (1-|\tilde{u}_{\varepsilon_k}|^2)^2\to 0, \quad \text{as } \varepsilon_k\to 0 \tag{5.20}$$

For any subsequence $\tilde{u}_{\varepsilon_k}$ of $\tilde{u}_\varepsilon$, we can find a subsequence of $\tilde{u}_{\varepsilon_k}$ denoted itself such that (5.20) is always true. Thus we have

$$\frac{1}{\varepsilon^p}\int_G (1-|\tilde{u}_\varepsilon|^2)^2 \to 0, \quad \text{as } \varepsilon \to 0 \tag{5.21}$$

In the following we shall show that (5.11) implies (5.21) when $p > n$.

We know that

$$|\tilde{u}_{\varepsilon_k}| \to 1, \quad \text{in } C(\bar{G}, R^n) \tag{5.22}$$

as $\varepsilon_k \to 0$ since $E_\varepsilon(\tilde{u}_\varepsilon, G) \leq E_\varepsilon(u_p, G) \leq C$ and the embedding theorem. Noticing that for any subsequence $\tilde{u}_{\varepsilon_k}$ of $\tilde{u}_\varepsilon$, we can find a subsequence of $\tilde{u}_{\varepsilon_k}$ denoted itself such that (5.22) is true, and the limit is always the number 1. This leads to

$$|\tilde{u}_\varepsilon| \to 1, \quad \text{in } C(\bar{G}, R^n) \tag{5.23}$$

as $\varepsilon \to 0$. Thus we have

$$\begin{aligned}\lim_{\varepsilon\to 0}\frac{1}{4\varepsilon^p}\int_G (1-|\tilde{u}_\varepsilon|^2)^2 &\leq \lim_{\varepsilon\to 0}\sup_G |1-|\tilde{u}_\varepsilon|^2|\frac{1}{4\varepsilon^p}\int_G (1-|\tilde{u}_\varepsilon|^2)\\ &\leq C\lim_{\varepsilon\to 0}\sup_G |1-|\tilde{u}_\varepsilon|^2| = 0\end{aligned}$$

by using (5.11) (5.23).

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