J. Partial Diff. Eqs. 14(2001), 71–86
Vol.14, No.1

VORTEX MOTION LAW OF AN EVOLUTIONARY GINZBURG-LANDAU EQUATION IN 2 DIMENSIONS*

Liu Zuhan
(Department of Mathematics, Normal College, Yangzhou University, Yangzhou 225002, China)
(Received Mar. 30, 2000; revised Sept. 11, 2000)

Abstract We study the asymptotic behavior of solutions to an evolutionary Ginzburg-Landau equation. We also study the dynamical law of Ginzburg-Landau vortices of this equation under the Neuman boundary conditions. Away from the vortices, we use some measure theoretic arguments used by F.H.Lin in [1] to show the strong convergence of solutions. This is a continuation of our earlier work [2].

Key Words Ginzburg-Landau equations; vortex motion; asymptotic behavior; ε-regularity.

1991 MR Subject Classification 35J55, 35Q40.

Chinese Library Classification O175.29, O175.26.

1. Introduction

We consider the following problem:

$$\frac{\partial u_\varepsilon}{\partial t} = \Delta u_\varepsilon + \frac{1}{\varepsilon^2}(\beta^2(x) - |u_\varepsilon|^2)u_\varepsilon, \quad (x,t) \in \Omega \times R_+ \tag{1.1}$$

$$u_\varepsilon(x,0) = \beta u_\varepsilon^0(x), \quad x \in \Omega \tag{1.2}$$

$$\frac{\partial u_\varepsilon}{\partial \nu}(x,t) = 0, \quad x \in \partial\Omega, \quad t \geq 0 \tag{1.3}$$

where Ω is a smooth bounded domain in R^2, ν the exterior unit normal vector along $\partial\Omega$. $\beta(x) : \Omega \to R$ is a smooth function (say C^3) with positive lower bound. $u_\varepsilon : \Omega \times R_+ \to R^2$.

The initial datum $\beta u_\varepsilon^0(x)$ is smooth and satisfies (1.3). In addition, it also satisfies the following assumptions:

$$\|u_\varepsilon^0(x)\|_{C(\bar{\Omega})} \leq K \tag{1.4}$$

$$\int_\Omega \rho^2(x)[|\nabla u_\varepsilon^0(x)|^2 + \frac{1}{\varepsilon^2}\beta^2(1 - |u_\varepsilon^0|^2)^2]dx \leq K \tag{1.5}$$

* This work supported by National Natural Science Foundation of China.

for a constant K and some m distint points $b_1, \cdots, b_m$ in Ω, where $\rho(x) = \min\{|x - b_j| : j = 1, 2, \cdots, m\}$.

$$E(u_\varepsilon^0) = \frac{1}{2}\int_\Omega [|\nabla u_\varepsilon^0|^2 + \frac{1}{2\varepsilon^2}\beta^2(x)(|u_\varepsilon^0(x)|^2 - 1)^2]dx \\ \leq K[|\ln\varepsilon| + 1] \tag{1.6}$$

The system (1.1)–(1.3) can be viewed as a simplified evolutionary Ginzburg-Landau equation in the theory superconductivity of inhomonogence ([3]).

The aim of this article is to understand the dynamics of vortices, or zeros, of solutions u of (1.1)–(1.3). Its importance to the theory of superconductivity and applications is addressed in many earlier work ([3–7]).

Now we briefly describe some mathematical advances concerning this problem. In $\beta = 1$, the dynamical law for vortices was formally derived in [4,8]. The first rigorous mathematical proof of this dynamical law, which is of form $\frac{\partial}{\partial t}a(t) = -\nabla w(a(t))$, was given by F.H.Lin in [5,9]. See also [10, Lecture 3]. In [5,9], one allows the vortices of degree ± 1 and assumes that they have the same sign. For the vortices of degree ± 1 (possibly of different signs), the same dynamical law was shown later in [11]. We refer to [1] for vortex dynamics under the Neumann boundary conditions or pinning conditions. In the 3-dimensional case, $\beta = 1$, a similar dynamical law was also established in [1] for nearly paralled filaments. The short-time dynamical law for codimension 2 interfaces in higher dimensions was shown in [1]. When $\beta \not\equiv 1$, in the 2-dimensional case, the dynamical law was established in [12] under the first boundary condition. But, here one proves only that u_ε conveges weakly to the limit function in H^1_{loc} as $\varepsilon \to 0^+$.

The main goal of the present paper is to examine the vortex dynamics without topological constraints, and proves that u_ε converges strongly to the limit function in H^1_{loc} as $\varepsilon \to 0^+$.

To understand the behavior of u of (1.1)–(1.3) as $t \to \infty$, one has to look at steady state solutions u_ε, that is, the minimizer of the energy functional

$$E_\varepsilon(u) = \frac{1}{2}\int_\Omega \left[|\nabla u|^2 + \frac{1}{2\varepsilon^2}(\beta^2 - |u|^2)^2\right] dx$$

A complete characterizaton of asymptotic behavier (as $\varepsilon \to 0^+$) of vortices of u_ε is given in the recent work [2].

Now we claim our main theorem.

Theorem 1.1 *Assume that $\beta \in C^3(\bar{\Omega})$ and $\beta_0 = \min_{\bar{\Omega}} \beta(x) > 0$. Under the assumptions (1.4)–(1.6), one has, for any $0 \leq t \leq T$, that*

$$u_\varepsilon(x, t) \to u_*(x, t) \tag{1.7}$$

strongly in $H^1_{\text{loc}}(\bar{\Omega} \times [0, T] \backslash \{(a_j(t), t) : t \in [0, T], j = 1, 2, \cdot, m\})$. Here the convergence is understood in the sense that for any sequence of ε' going to zero, there is a subsequence *for which* (1.7) *is true. Moreover,* u_* *satisfies*

$$\frac{\partial u_*}{\partial t} - \Delta u_* = \beta^{-2}\left(|\nabla u_*|^2 - \Delta\left(\frac{1}{2}\beta^2\right)\right)u_* \tag{1.8}$$

in $\Omega \times (0,T)\backslash\{(a_j(t),t) : t \in [0,T], j = 1,\cdots,m\} \bigcup_{i=1}^{m}\{a_i(t)\}) \times (0,T)$.

The functions $a_j(t) \in \Omega, j = 1,\cdots,m$ *satisfy the following equation:*

$$\begin{cases} \dfrac{d}{dt}a_j(t) = -2\dfrac{\nabla\beta(a_j(t))}{\beta(a_j(t))} \\ a_j(0) = b_j \end{cases} \tag{1.9}$$

for $j = 1,\cdots,m$*, and* $0 \le t \le T$*. Here* T *is chosen so that* $a_j(t)$ *will stay inside* Ω *and* $a_l(t) \ne a_j(t)$*, for all* $0 \le t \le T$ *and for* $l, j = 1,\cdots,m$.

Under some additional technical hypothesis on β, one may take $T = \infty$ in the above theorem ([12]).

The rest of this paper is organized as follows. In Section 2, we prove the weak convergence. In Section 3, we study strong convergence.

2. Weak Convergence

Let $u = \beta v$, then u satisfies

$$v_t = \Delta v + \frac{2}{\beta}\nabla\beta\nabla v + \frac{\Delta\beta}{\beta}v + \frac{1}{\varepsilon^2}\beta^2(1-|v|^2)v, \quad (x,t) \in \Omega \times R^+ \tag{2.1}$$

$$v(x,0) = u_\varepsilon^0, \quad x \in \Omega \tag{2.2}$$

$$\frac{\partial v}{\partial \nu}(x,t) + \left(\frac{1}{\beta}\frac{\partial\beta}{\partial\nu}\right)v = 0, \quad x \in \partial\Omega, \quad t > 0 \tag{2.3}$$

Lemma 2.1 *Let* $M = \max_{\bar\Omega}|\beta(x)|$*, then*

$$|u_\varepsilon(x,t)| \le M(1+K), \quad \text{in } \bar\Omega \times [0,T]$$

Proof Let $\bar u = u_\varepsilon/M$, then

$$\begin{aligned} &\frac{\partial \bar u}{\partial t} = \Delta\bar u + \frac{M^2}{\varepsilon^2}((\beta/M)^2 - |\bar u|^2)\bar u, && \text{in } \Omega \times R^+ \\ &\frac{\partial \bar u}{\partial \nu}(x,t) = 0, && x \in \partial\Omega, t > 0 \\ &\bar u(x,0) = \frac{\beta}{M}u_\varepsilon^0(x), && x \in \Omega \end{aligned}$$

Set $w = |\bar u|^2 - (1+K)^2$, one has

$$\begin{aligned} &\frac{\partial w}{\partial t} - \Delta w + 2M^2\varepsilon^{-2}|\bar u|^2 w \le 0, && \text{in } \Omega \times R^+ \\ &\frac{\partial w}{\partial \nu} = 0, && x \in \partial\Omega, t > 0 \\ &w(x,0) \le 0, && x \in \Omega \end{aligned}$$

Thus, by maximun principle, we have

$$w \leq 0 \ \text{ in } \ \bar{\Omega} \times [0, T]$$

Hence

$$|u_\varepsilon(x,t)| \leq M(1+K) \ \text{ in } \ \bar{\Omega} \times [0, T]$$

Define $\phi_\sigma : R_+ \to R_+$ be a smooth monotone function such that

$$\phi_\sigma(r) = \begin{cases} r^2, & r \leq \sigma, \sigma > 0 \\ 4\sigma^2, & r \geq 2\sigma \end{cases}$$

Lemma 2.2 *Suppose that, for* $0 \leq t \leq T, \min\{|a_j(t) - a_l(t)|, \mathrm{dist}(a_j(t), \partial\Omega),$ *for* $j, l = 1, \cdots, m$ *and* $j \neq l\} \geq 4\sigma, \min(\beta/(2|\nabla\beta| + 1)) \geq \sigma$. *Then, with the above notations, one has*

$$\|u_\varepsilon\|_{H^1(\Omega\times[0,T]\setminus\cup_{j=1}^m \Gamma_\sigma^j)} \leq C(\sigma, T, \beta)$$

where $\Gamma_\sigma^j = \{(x,t) : |x - a_j(t)| \leq \sigma, 0 \leq t \leq T\}$. $a_j(t)$ *is the solution of* (1.9).

Proof Using integration by parts, one gets

$$\begin{aligned} &\frac{d}{dt}\int_\Omega \frac{1}{2}\phi_\sigma(\rho(x,t))\beta^2\left[|\nabla v|^2 + \frac{1}{2\varepsilon^2}\beta^2(1-|v|^2)^2\right]dx \\ &= \int_\Omega \frac{d\phi_\sigma}{dt}\frac{1}{2}\beta^2\left[|\nabla v|^2 + \frac{1}{2\varepsilon^2}\beta^2(1-|v|^2)\right]dx + \int_\Omega -\phi_\sigma\beta^2\left|\frac{dv}{dt}\right|^2 \\ &\quad + \int_\Omega \phi_\sigma\beta^2\frac{\Delta\beta}{\beta}v\frac{dv}{dt} - \int_{\partial\Omega}\phi_\sigma\beta\frac{\partial\beta}{\partial\nu}v\frac{dv}{dt} - \int_\Omega \beta^2\nabla\phi_\sigma\nabla v\frac{dv}{dt} \end{aligned}$$

On the other hand,

$$\left|\int_\Omega \phi_\sigma\beta^2\frac{\Delta\beta}{\beta}v\frac{dv}{dt}\right| \leq \frac{1}{2}\int_\Omega \phi_\sigma\beta^2\left|\frac{dv}{dt}\right|^2 + C\int_\Omega \phi_\sigma\beta^2|v|^2$$

Now we calculate the expression $\beta^2\nabla\phi_\sigma\nabla v\frac{dv}{dt}$. We shall use the summation convention, and simplify notation. We shall also set $\phi_\sigma = \phi, v_\varepsilon = v, e_\varepsilon(v) = \frac{1}{2}\beta^2\Big[|\nabla v|^2 + \frac{1}{2\varepsilon^2}\beta^2(1 - |v|^2)^2\Big]$.

$$\begin{aligned} \beta^2\nabla\phi_\sigma\cdot\nabla v\frac{dv}{dt} =& (\nabla\phi_\sigma\cdot\nabla v)\left[\mathrm{div}\,(\beta^2\nabla v) + (\beta\Delta\beta)v + \frac{\beta^4}{\varepsilon^2}(1-|v|^2)v\right] \\ =& \phi_i(\beta^2 v_i v_j)_j + (\beta\Delta\beta)\phi_i\left(\frac{|v|^2}{2}\right)_i \\ & - \left[\beta^4\frac{1}{4\varepsilon^2}(1-|v|^2)^2\right]_i\phi_i + (\beta^4)_i\frac{(1-|v|^2)^2}{4\varepsilon^2}\phi_i \\ & - \left(\beta^2\frac{|v_j|^2}{2}\right)_i\phi_i + (\beta^2)_i\phi_i\left(\frac{|v_j|^2}{2}\right) \end{aligned}$$

where $\phi = \phi_\sigma, v_i = \dfrac{\partial v}{\partial x_i}, (\beta^2)_i = \dfrac{\partial \beta^2}{\partial x_i}$.

$$\begin{aligned}
&\int_\Omega \beta^2 \nabla\phi \cdot \nabla v \cdot \frac{dv}{dt} \\
&\quad = \int_\Omega -\beta^2 \phi_{ij} v_i v_j + \int_\Omega \Delta\phi \cdot \frac{1}{2}\beta^2 \left[|\nabla v|^2 + \frac{1}{2\varepsilon^2}\beta^2(1-|v|^2)^2\right] \\
&\qquad + \int_\Omega \frac{\nabla\beta^2}{\beta^2}\nabla\phi \cdot \frac{1}{2}\beta^2 \left[|\nabla v|^2 + \frac{\beta^2}{2\varepsilon^2}(1-|v|^2)^2\right] \\
&\qquad + \int_\Omega \beta\nabla\beta\nabla\phi \cdot \frac{1}{2\varepsilon^2}\beta^2(1-|v|^2)^2 + \int_\Omega (\beta\Delta\beta)\nabla\phi \cdot \nabla\left(\frac{|v|^2}{2}\right)
\end{aligned}$$

Note that

$$\int_{\partial\Omega} \phi_\sigma \beta^2 \left(\frac{1}{\beta}\frac{\partial\beta}{\partial\nu}\right) v\frac{dv}{dt} = 2\sigma^2 \int_{\partial\Omega} \beta\frac{\partial\beta}{\partial\nu}\frac{d|v|^2}{dt} = 2\sigma^2 \frac{d}{dt}\int_{\partial\Omega} \beta\frac{\partial\beta}{\partial\nu}|v|^2$$

Hence

$$\begin{aligned}
&\frac{d}{dt}\int_\Omega \phi_\sigma \cdot \frac{1}{2}\beta^2 \left[|\nabla v|^2 + \frac{\beta^2}{2\varepsilon^2}(1-|v|^2)^2\right] dx \\
&\quad \le \int_\Omega \left(\frac{d\phi}{dt} - \frac{\nabla\beta^2}{\beta^2}\nabla\phi\right)\frac{1}{2}\beta^2 \left[|\nabla v|^2 + \frac{\beta^2}{2\varepsilon^2}(1-|v|^2)^2\right] \\
&\qquad + \int_\Omega \left\{\beta^2\phi_{ij}v_iv_j - \Delta\phi\frac{1}{2}\beta^2\left[|\nabla v|^2 + \frac{\beta^2}{2\varepsilon^2}(1-|v|^2)^2\right]\right\} \\
&\qquad + \int_\Omega \beta\nabla\beta\nabla\phi\frac{1}{2\varepsilon^2}\beta^2(1-|v|^2)^2 - \int_\Omega (\beta\Delta\beta)v\nabla\phi\nabla v \\
&\qquad + C\int_\Omega \phi\beta^2|v|^2 - \int_\Omega \phi_\sigma\beta^2\left|\frac{dv}{dt}\right|^2 - 2\sigma^2\frac{d}{dt}\int_{\partial\Omega}\beta\frac{\partial\beta}{\partial\nu}|v|^2
\end{aligned}$$

One observes that on each $B_\sigma(a_l(t)), l = 1,\cdots,m, \Delta\phi = 4, \phi_{ij} = 2\delta_{ij}$. By the above observation, one has

$$\beta^2\phi_{ij}v_iv_j - \Delta\phi \cdot \frac{1}{2}\beta^2|\nabla v|^2 \le 0 \ \text{ if } \ x \in \bigcup_{l=1}^m B_\sigma(a_l(t))$$

Moreover, if $\sigma \le \min(\beta/(2|\nabla\beta|+1))$, one also has

$$\left[\beta\nabla\beta\nabla\phi - \frac{1}{2}\Delta\phi\beta^2\right]\frac{\beta^2}{2\varepsilon^2}(1-|v|^2)^2 \le 0 \ \text{ in } \ x \in \bigcup_{l=1}^m B_\sigma(a_l(t))$$

On the other hand, if $\rho(x,t) \ge \sigma$, then we have

$$\begin{aligned}
&\left|\beta^2\phi_{ij}v_iv_j - \frac{1}{2}\Delta\phi\beta^2|\nabla v|^2\right| \le C_0(\sigma)\phi e_\varepsilon(v) \\
&\left|\left[-\frac{1}{2}\Delta\phi\beta^2 + \beta\nabla\beta\phi\right]\frac{1}{2\varepsilon^2}\beta^2(1-|v|^2)^2\right| \le C_0(\sigma)\phi e_\varepsilon(v)
\end{aligned}$$

Similarly,

$$\left|\phi_t - \frac{\nabla\beta^2}{\beta^2}\nabla\phi\right| \le C_0(\sigma)\sigma \quad \text{if} \quad \rho(x,t)\ge\sigma$$

Finally,

$$\begin{aligned}\left|\phi_t - \frac{\nabla\beta^2}{\beta^2}\nabla\phi\right|(x,t) =& \left|2(x-a_j(t))\frac{d}{dt}a_j(t) - 2(x-a_j(t))\frac{\nabla\beta^2(x)}{\beta^2(x)}\right| \\ \le& 2|x-a_j(t)|\left|\frac{\nabla\beta^2(x)}{\beta^2(x)} - \frac{\nabla\beta^2(a_j(t))}{\beta^2(a_j(t))}\right| \\ \le& C_\sigma|x-a_j(t)|^2 \le C_\sigma\phi^2\end{aligned}$$

for $j=1,\cdots,m$ and for all $x\in B_\sigma(a_j(t))$. Hence

$$\frac{d}{dt}\int_\Omega \phi_\sigma e_\varepsilon(v)dx \le C_\sigma\int_\Omega \phi_\sigma e_\varepsilon(v)dx - 2\sigma^2\frac{d}{dt}\int_{\partial\Omega}\beta\frac{\partial\beta}{\partial\nu}|v|^2 - \int_\Omega \phi_\sigma\beta^2\left|\frac{dv}{dt}\right|^2 + C$$

So

$$\int_\Omega \phi_\sigma e_\varepsilon(v)\bigg|_{t=t} dx \le C_\sigma\int_0^t\int_\Omega \phi_\sigma e_\varepsilon(v)dxdt + Ct + C \tag{2.4}$$

where we have used the fact that $|v|\le 1+K$ in $\bar\Omega\times[0,T]$. Finally we have, by Gronwall's inequality, that

$$\int_0^t\int_\Omega \phi_\sigma e_\varepsilon(v)dxdt \le C(\sigma,T,\beta), \quad \text{for all} \quad 0\le t\le T \tag{2.5}$$

$$\sup_{0\le t\le T}\int_\Omega \phi_\sigma e_\varepsilon(v) \le C(\sigma,T,\beta) \tag{2.6}$$

On the other hand, by calculating the expressions of form:

$$\frac{d}{dt}\int_\Omega \phi^2(x)e_\varepsilon(v) \le -\frac{1}{2}\int_\Omega \phi^2\beta^2\left|\frac{dv}{dt}\right|^2 + C\int_{\Omega\cap\mathrm{spt}(\phi)} e_\varepsilon(v) - \frac{1}{2}\frac{d}{dt}\int_{\partial\Omega}\phi\beta\frac{\partial\beta}{\partial\nu}|v|^2 + C$$

here $\phi(x)$ is a smooth cut-off function which is supported in $\Omega\backslash\bigcup_{j=1}^m B_\sigma(a_j(t))$, one deduces that

$$\int_0^T\int_{\Omega\backslash\bigcup_{j=1}^m B_\sigma(a_j(t))}\left|\frac{dv_\varepsilon}{dt}\right|^2(x,t)dxdt \le C(\sigma,T,\beta) \tag{2.7}$$

Combining (2.4) with (2.7), and noting $v=\beta u$, we get the conclusion of Lemma 2.2.

It is then easy to see that, for any sequence of $\varepsilon_n\to 0$, there is a subsequence $\{u_{\varepsilon_n'}\}$ of $\{u_{\varepsilon_n}\}$ so that

$$u_{\varepsilon_n'}(x,t)\to u_*(x,t)$$

weakly in $H^1_{\mathrm{loc}}(\bar{\Omega}\times(0,T]\backslash\{(a_j(t),t):t\in(0,T],j=1,\cdots,m\})$, and $|u_*(x,t)|=|\beta(x)|$,

$$\frac{\partial u_*}{\partial t}=\Delta u_*+\beta^{-2}\left(|\nabla u_*|^2-\Delta\left(\frac{1}{2}\beta^2\right)\right)u_* \tag{2.8}$$

in $\Omega\times(0,T]\backslash\{(a_j(t),t):t\in(0,T],j=1,\cdots,m\}$.

3. Strong Convergence

In order to prove that u_ε converges strongly, we only need to prove that v_ε converges strongly. The proof of this conclusion is based on the fact that the solutions v_ε to (3.1) satisfy a monotonicity inequality from which the ε-regularity can be proved. Then, it implies the strong convergence of the sequence of $\{v_\varepsilon\}$. Theorem 3.1 is an extension of Theorem 2.1 in [13]. The main task is to find a monotonicity formula and a small energy regularity theorem.

Theorem 3.1 *Let v_ε be a solution of*

$$\frac{\partial v}{\partial t}=\Delta v_\varepsilon+\frac{2\nabla\beta}{\beta}\nabla v_\varepsilon+\frac{\Delta\beta}{\beta}v_\varepsilon+\frac{1}{\varepsilon^2}\beta^2(1-|v_\varepsilon|^2)v_\varepsilon \quad \text{in } B_1\times[-1,0] \tag{3.1}$$

with

$$\int_{-1}^0\int_{B_1}e_\varepsilon(v)dxdt+\int_{-1}^0\int_{B_1}\left|\frac{\partial v_\varepsilon}{\partial t}\right|^2dxdt\le M$$

for $0<\varepsilon<<1$. Here $v_\varepsilon\in C$,

$$B_1=\{x\in R^2;|x|<1\},e_\varepsilon(v_\varepsilon)=\frac{1}{2}\left[|\nabla v_\varepsilon|^2+\frac{1}{2\varepsilon^2}\beta^2(1-|v_\varepsilon|^2)^2\right].$$

Suppose also that $\{v_\varepsilon\}$ converges weakly to a map v_* as $\varepsilon\to0^+$ in $H^1(Q_1)$. Then $\{v_\varepsilon\}$ converges strongly to v_* in $H^1_{\mathrm{loc}}(Q_1)$, where $Q_1=B_1\times(-1,0]$.

In order to prove the above theorem, we need several lemmas.

Lemma 3.2 *Suppose v satisfies*

$$\frac{\partial v}{\partial t}=\Delta v+\frac{2\nabla\beta}{\beta}\nabla v+\frac{\Delta\beta}{\beta}v+\frac{1}{\beta^2}(\beta\Delta\beta-\beta^2|\nabla v|^2)v \quad \text{in } Q_1 \tag{3.2}$$

with

$$\int_{Q_1}\left[|\nabla v|^2+\left|\frac{\partial v}{\partial t}\right|^2\right]dxdt\le M$$

Then, for $0<\alpha<1$, one has

$$v\in C^{2+\alpha,1+\frac{\alpha}{2}}_{\mathrm{loc}}(Q_1)$$

Proof Let $v=e^{i\psi}$. Then $\psi\in H^1(Q_1)$ and

$$\int_{Q_1}|\nabla\psi|^2dxdt\le C$$

$$\psi_t=\Delta\psi+\frac{2\nabla\beta}{\beta}\cdot\nabla\psi \quad \text{in } Q_1$$

One has

$$\psi \in W_2^{2,1}(Q_1)$$

So, by a bootstrap argument, we have

$$\psi \in C_{\text{loc}}^{2+\alpha,1+\frac{\alpha}{2}}(Q_1)$$

Hence

$$v \in C_{\text{loc}}^{2+\alpha,1+\frac{\alpha}{2}}(Q_1)$$

In the sequel, for convenience, we denote v_ε by v.

Lemma 3.3 (Energy inequality)

$$\sup_{0\le t\le T}\left[\int_0^t\int_\Omega |v_t|^2 + E(v(\cdot,t))\right] \le C(E_0+1) \tag{3.3}$$

where $E(v(\cdot,t)) = \frac{1}{2}\int_\Omega\left[|\nabla v|^2 + \frac{\beta^2}{2\varepsilon^2}(1-|v|^2)^2\right]dx, E_0 = E(v(\cdot,0)), C$ *depends only on* β,Ω.

Proof First

$$|v_t|^2 = \frac{1}{\beta^2}\text{div}\,(\beta^2\nabla v)v_t + \frac{\Delta\beta}{\beta}vv_t + \frac{\beta^2}{\varepsilon}(1-|v|^2)vv_t$$

Hence

$$\int_\Omega |v_t|^2 = \int_\Omega \frac{1}{\beta^2}\text{div}\,(\beta^2\nabla v)v_t + \frac{1}{2}\frac{\partial}{\partial t}\left(\frac{\Delta\beta}{\beta}|v|^2\right) - \frac{\partial}{\partial t}\left[\frac{\beta^2}{4\varepsilon^2}(1-|v|^2)^2\right]$$

$$\begin{aligned}\int_0^t\int_\Omega |v_t|^2 &= \int_0^t\int_\Omega \text{div}\,(\beta^2\nabla v)\left(\frac{1}{\beta^2}\right)v_t + \frac{1}{2}\int_0^t\int_\Omega\frac{d}{dt}\left[\frac{\Delta\beta}{\beta}|v|^2 - \frac{1}{2\varepsilon^2}(1-|v|^2)^2\right]\\ &= -\int_0^t\int_\Omega \frac{d}{dt}|\nabla v|^2 + \beta^2 v_t\nabla v\nabla\left(\frac{1}{\beta^2}\right)\\ &\quad + \int_0^t\int_{\partial\Omega} v_t\frac{1}{\beta}\frac{\partial\beta}{\partial\nu}v + \frac{1}{2}\int_0^t\int_\Omega\frac{d}{dt}\left[\frac{\Delta\beta}{\beta}|v|^2 - \frac{1}{2\varepsilon^2}\beta^2(1-|v|^2)^2\right]\end{aligned}$$

By Lemma 2.1, one has

$$\begin{aligned}&\int_0^t\int_\Omega |v_t|^2 + \int_\Omega\left[|\nabla v|^2 + \frac{\beta^2}{2\varepsilon^2}(1-|v|^2)^2\right]\bigg|_{t=t}\\ &\le C\int_\Omega\left[|\nabla v|^2 + \frac{\beta^2}{2\varepsilon^2}(1-|v|^2)^2\right]\bigg|_{t=0} + C\int_0^t\int_\Omega |\nabla v|^2 + C\end{aligned} \tag{3.4}$$

where C depends only on β,Ω. From (3.4), one has

$$\int_\Omega |\nabla v|^2|_{t=t}dx \le C\int_0^t\int_\Omega |\nabla v|^2dxdt + C(E_0+1)$$

By Gronwall's inequality, one has

$$\int_0^t \int_\Omega |\nabla v|^2 dx dt \le C(E_0 + 1) \tag{3.5}$$

From (3.4), (3.5), we have

$$\sup_{0\le t\le T} \left[\int_0^t \int_\Omega |v_t|^2 + E(v(\cdot, t)) \right] \le C(E_0 + 1)$$

The proof of Lemma 3.3 is completed.

Define

$$e_\varepsilon(v) = \frac{1}{2}\left[|\nabla v|^2 + \frac{\beta^2}{2\varepsilon^2}(1 - |v|^2)^2 \right]$$

$$G_{z_0}(x,t) = [4\pi(t_0 - t)]^{-1} \exp\left[-\frac{|x - x_0|^2}{4(t_0 - t)} \right]$$

where $t < t_0, z_0 = (x_0, t_0)$,

$$\begin{aligned}
&G(x,t) = G_0(x_0, t_0) \\
&S_R(z_0) = \{z = (x,t) : t = t_0 - R^2\} \\
&P_R(z_0) = \{z = (x,t) : |x - x_0| < R, |t - t_0| < R^2\} \\
&T_R(z_0) = \{z = (x,t) : x \in R^2, t_0 - 4R^2 \le t \le t_0 - R^2\} \\
&T_1 = T_1(0) \\
&\Psi(R) = \int_{T_R(z_0)} e_\varepsilon(v) G_{z_0} \phi^2 dx dt \\
&\Phi(R) = R^2 \int_{S_R(z_0)} e_\varepsilon(v) G_{z_0} \phi^2 dx dt
\end{aligned}$$

where

$$\phi \in C_0^\infty(B_{\rho_0}(x_0)), 0 \le \phi \le 1, \phi \equiv 1, \quad \text{for } |x - x_0| \le \frac{\rho_0}{2}$$

and

$$|\nabla \phi| \le C_0/\rho_0, x_0 \in Q_1, 0 < \rho_0 < \operatorname{dist}(x_0, \partial Q_1)$$

Lemma 3.4 (Monotonicity formula)

$$\Phi(R) \le \exp(C(R_0 - R))\Phi(R_0) + C(E_0 + 1)(R_0 - R) \tag{3.6}$$

$$\Psi(R) \le \exp(C(R_0 - R))\Psi(R_0) + C(E_0 + 1)(R_0 - R) \tag{3.7}$$

where $E_0 = E(v(\cdot, 0))$.

Proof Let

$$\begin{aligned}
w(x,t) &= v(x+x_0, t+t_0)\\
\alpha(x) &= \beta(x+x_0)\\
\varphi(x) &= \phi(x+x_0)\\
v_R(x,t) &= w(Rx, R^2t)\\
\alpha_R(x) &= \alpha(Rx)\\
\varphi_R(x) &= \varphi(Rx)
\end{aligned}$$

then

$$w_t = \Delta w + \frac{2\nabla\alpha}{\alpha}\nabla w + \frac{\Delta\alpha}{\alpha}w + \frac{\alpha^2}{\varepsilon^2}(1-|w|^2)w$$

$$\begin{aligned}
&\int_{T_R(0)} \frac{1}{2}\left[|\nabla w|^2 + \frac{\alpha^2}{2\varepsilon^2}(1-|w|^2)^2\right] G\varphi^2\\
&\quad = \frac{1}{2}\int_{T_R(z_0)} \left[|\nabla v|^2 + \frac{\beta^2}{2\varepsilon^2}(1-|v|^2)^2\right] G_{z_0}\phi^2
\end{aligned}$$

$$\Psi(R) = \frac{1}{2}\int_{T_1}\left[|\nabla v_R|^2 + \frac{R^2}{2\varepsilon^2}\alpha_R^2(1-|v_R|^2)^2\right] G\varphi_R^2 dxdt$$

$$\begin{aligned}
\frac{d}{dR}\Psi(R)|_{R=1} =& \int_{T_1}\Big\{\Big[-\frac{1}{2t}(x\cdot\nabla v_1) - \Delta v_1 - \frac{\alpha_1^2}{\varepsilon^2}(1-|v_1|^2)v_1\Big](x\cdot\nabla v_1 + 2t\partial_t v_1)\\
&+ \frac{\alpha_1^2}{\varepsilon^2}(1-|v_1|^2)^2 + \frac{\alpha_1}{\varepsilon^2}(x\cdot\nabla\alpha_1)(1-|v_1|^2)^2\Big\}G\varphi_1^2\\
&- 2\int_{T_1} G\varphi_1\nabla v_1\nabla\varphi(x\cdot\nabla v_1 + 2t\partial_t v_1)\\
&+ \int_{T_1}\left[|\nabla v_1|^2 + \frac{\alpha_1^2}{2\varepsilon^2}(1-|v_1|^2)^2\right] G\varphi_1(x\cdot\nabla\varphi_1)\\
=:& I + II + III
\end{aligned}$$

where we have used the fact that $\frac{\partial G}{\partial x_i} = \frac{x_i}{2t}G$. We also denote $\frac{\partial}{\partial t}, (\nabla\varphi_R)|_{R=1}$ by $\partial_t, \nabla\varphi_1$, respectively, etc.

The first term may be estimated

$$\begin{aligned}
I \geq& \frac{1}{2}\int_{T_1}\left\{\frac{1}{2|t|}(x\cdot\nabla v_1 + 2t\partial_t v_1)^2 + \frac{\alpha_1^2}{\varepsilon^2}(1-|v_1|^2)^2\right\}G\varphi_1^2 - C\Psi(1) - C\\
|II| \leq& \frac{1}{8}\int_{T_1}\frac{1}{|t|}|x\cdot\nabla v_1 + 2t\partial_t v_1|^2 G\varphi_1^2 + C\int_{T_1}|\nabla v_1|^2 G|\nabla\varphi_1|^2\\
\leq& \frac{1}{2}|I| + C\Psi(1) + C(E_0+1)
\end{aligned}$$

$$
\begin{aligned}
|III| &\le \frac{1}{2}\Psi(1) + \int_{T_1} \frac{1}{2}\left[|\nabla v_1|^2 + \frac{\alpha_1^2}{2\varepsilon^2}(1-|v_1|^2)^2\right] G|x\cdot\nabla\varphi_1|^2 \\
&\le \frac{1}{2}\Psi(1) + C(E_0+1)
\end{aligned}
$$

where $E_0 = E(v(\cdot,0))$. From the differential inequality

$$
\frac{d}{dR}\Psi(R) \ge \frac{1}{8R}\int_{T_R} \frac{|x\cdot\nabla v_1 + 2t\partial_t v_1|^2}{|t|} G\varphi_1^2 - C\Psi(R) - C(E_0+1)
$$

now (3.7) follows.

The proof of (3.6) is similar to (3.7).

Lemma 3.5 (Small energy regularity theorem) *There exists a constant* $\theta_0 \in (0,1/2)$ *depending only on* β, *such that if for some* $0 < R < \sqrt[2]{t_0}/2$, $z_0 = (x_0,t_0)$, $v = v_\varepsilon$ *satisfies*

$$
\Psi(R) = \frac{1}{2}\int_{T_R(z_0)} \left[|\nabla v|^2 + \frac{1}{2\varepsilon^2}\beta^2(1-|v|^2)^2\right] G_{z_0}dxdt < \theta_0 \tag{3.8}
$$

then

$$
\sup_{P_{\delta R}(z_0)} \left\{|\nabla v|^2 + \frac{1}{2\varepsilon^2}\beta^2(1-|v|^2)^2\right\} \le C(\delta R)^{-2} \tag{3.9}
$$

with a constant $\delta \in (0,1/2)$ *depending only on* E_0 *and* $\inf\{R,1\}$ *and an absolute constant* C.

Proof The proof of Lemma 3.5 is identical to [Lemma 2.1, 14]. For convenience of readers, we sketch it here. Without loss of generality, one considers the case that $z_0 = (0,0)$. Set $r_1 = \delta R, 0 < \delta < 1/2$ to be determined later. For $r,\sigma \in (0,r_1), r+\sigma < r_1$, and $z_0 = (x_0,t_0) \in P_r$, let $e_\varepsilon(v) = e(v)$ in the proof of Lemma 3.5, by using monotonicity inequality, we get ([14])

$$
\sigma^{-2} \le \int_{P_\sigma(z_0)} e(v) \le C\int_{T_R} e(v)(G + \theta R^{-2})dxdt + CR(E_0+1)
$$

for any given $\theta > 0$, if $\delta > 0$ is small enough (For small $R, \delta = O(|\log R|^{-1/2})$). Therefore, from (3.8) it follows that

$$
\sigma^{-2}\int_{P_\sigma(z_0)} e(v)dxdt \le C\theta_0 + C(\theta+\theta_0)E_0 \tag{3.10}
$$

Since v is regular, there exists $\sigma_0 \in (0,r_1)$, such that

$$
(r_1-\sigma_0)^2 \sup_{P_{\sigma_0}} e(v) = \max_{0\le\sigma\le r_1} (r_1-\sigma)^2 \sup_{P_\sigma} e(v)
$$

and there exists $(x_0,t_0) \in \bar{P}_{\sigma_0}$ such that

$$
\sup_{P_{\sigma_0}} e(v) = e(v)(x_0,t_0) = e_0
$$

Set $\rho_0 = (r_1 - \sigma_0)/2$ and $r_0 = \rho_0 \sqrt[2]{e_0}$. By the choice of σ_0, we have

$$\sup_{P_{\rho_0(x_0,t_0)}} e(v) \le \sup_{P_{\sigma_0+\rho_0}} e(v) \le 4e_0$$

We introduce

$$w(x,t) = v\left(\frac{x}{\sqrt[2]{e_0}} + x_0, \frac{t}{e_0} + t_0\right)$$

Thus

(i)

$$w_t - \Delta w + \frac{1}{\sqrt[2]{e_0}}\left(\frac{2}{\beta}\nabla\beta\right)\left(\frac{x}{\sqrt[2]{e_0}} + x_0\right)\nabla w + \frac{1}{e_0}\left(\frac{\Delta\beta}{\beta}\right)\left(\frac{x}{\sqrt[2]{e_0}} + x_0\right)w$$
$$+ \frac{1}{e_0\varepsilon^2}\beta^2\left(\frac{x}{\sqrt[2]{e_0}}\right)(1-|w|^2)w = 0 \ \text{ in } \ P_{r_0}$$

(ii)

$$e(w)(x,t) = \frac{1}{2}|\nabla w|^2 + \frac{1}{4e_0\varepsilon^2}\beta^2\left(\frac{x}{\sqrt[2]{e_0}} + x_0\right)(1-|w|^2)^2 \ \text{ for } \ (x,t) \in P_{r_0}$$

(iii)

$$e(w)(0,0) = 1$$

Now we claim that there exists a constant $C > 0$, such that $r_0 \le C$. In fact, if the constant C does not exist, then we may assume that there is a sequence of solutions $\{w_i\}$ of (i) in P_1 with the following properties:

(iv)

$$\frac{\partial}{\partial t}w_i - \Delta w_i + \delta_i a_i \nabla w_i + \delta_i^2 b_i w_i + \frac{\delta_i^2}{\varepsilon^2}\beta_i^2(1-|w_i|^2)w_i = 0 \ \text{ in } \ P_1$$
$$\text{with } \delta_i \to 0 \ (\text{as } i \to \infty)$$

(v)

$$e(w_i) = \frac{1}{2}|\nabla w_i|^2 + \frac{\delta_i^2}{4\varepsilon_i^2}\beta_i^2(1-|w_i|^2)^2 \le 4 \ \text{ in } \ P_1$$

(vi)

$$e(w_i)(0,0) = 1$$

where $a_i = \left(\frac{2\nabla\beta}{\beta}\right)(\delta_i x + x_0)$, $b_i = \frac{\Delta\beta}{\beta}(\delta_i x + x_0)$, $|a_i| \le C, |b_i| \le C$. Moreover, we have that

$$\begin{aligned}(\partial_t - \Delta)\left(\frac{1}{2}|\nabla w_i|^2\right) =& \nabla(\partial_t w_i - \Delta w_i)\cdot\nabla w_i - |\nabla^2 w_i|^2 \\ \le& \frac{1}{4}\left[\frac{\delta_i^2}{\varepsilon_i^2}\beta_i^2(1-|w_i|^2)^2\right]^2|w_i|^2 + C|\nabla w_i|^2 \\ &+ \frac{\delta_i^2}{\varepsilon_i^2}\beta_i^2(1-|w_i|^2)|\nabla w_i|^2 + \frac{1}{2}|\nabla^2 w_i|^2 + \delta_i^2|w_i|^2 - |\nabla^2 w_i|^2\end{aligned}$$

On the other hand,

$$\begin{aligned}
&\frac{\delta_i^2}{4\varepsilon_i^2}(\partial_t-\Delta)(\beta_i^2(1-|w_i|^2)^2)\\
&=-\left[\frac{\delta_i^2}{\varepsilon_i^2}\beta_i^2(1-|w_i|^2)\right]^2|w_i|^2-\frac{\delta_i^2}{\varepsilon_i^2}\beta_i^2(1-|w_i|^2)w_i\cdot[\delta_i a_i\nabla w_i+\delta_i^2 b_i w_i]\\
&\quad+\frac{1}{4}\left[\frac{\delta_i^2}{\varepsilon_i^2}\beta_i^2(1-|w_i|^2)\right]^2|w_i|^2+C|\nabla w_i|^2-2\frac{\delta_i^2}{\varepsilon^2}\beta_i^2|w_i|^2|\nabla w_i|^2\\
&\quad-\frac{\delta_i^2}{\varepsilon_i^2}\beta_i^2(1-|w_i|^2)|\nabla w_i|^2
\end{aligned}$$

So,

$$\begin{aligned}
&(\partial_t-\Delta)e(w_i)+\frac{1}{2}|\nabla^2 v_i|^2+\frac{1}{4}\left[\frac{\delta_i^2}{\varepsilon_i^2}\beta_i^2(1-|w_i|^2)\right]^2|w_i|^2\\
&\le C|\nabla w_i|^2+\delta_i^2|w_i|^2+2\frac{\delta_i^2}{\varepsilon_i^2}\beta_i^2(1-|w_i|^2)|\nabla w_i|^2\\
&\quad-\frac{\delta_i^2}{\varepsilon_i^2}\beta_i^2(1-|w_i|^2)w_i\delta_i a_i\nabla w_i-\frac{\delta_i^4}{\varepsilon_i^2}\beta_i^2 b_i(1-|w_i|^2)|w_i|^2 \qquad (3.11)
\end{aligned}$$

Note that

$$e(w_i)\le 4 \ \text{ in } \ P_1$$

one has

$$\frac{1}{2}\le|w_i|\le\frac{3}{2}$$

once ε_i/δ_i is small enough. Hence

$$\begin{aligned}
\left|2\frac{\delta_i^2}{\varepsilon_i^2}\beta_i^2(1-|w_i|^2)|\nabla w_i|^2\right|&=\left|\frac{\delta_i^2}{\varepsilon_i^2}\beta_i^2(1-|w_i|^2)|w_i|\frac{|\nabla w_i|^2}{|w_i|}\right|\\
&\le\frac{1}{8}\left[\frac{\delta_i^2}{\varepsilon_i^2}\beta_i^2(1-|w_i|^2)\right]^2|w_i|^2+C|\nabla w_i|^4 \qquad (3.12)
\end{aligned}$$

By Young's inequality, one has

$$\left|\frac{\delta_i^2}{\varepsilon_i^2}\beta_i^2(1-|w_i|^2)w_i\delta_i(a_i\cdot\nabla)w_i\right|\le\frac{1}{8}\left[\frac{\delta_i^2}{\varepsilon_i^2}\beta_i^2(1-|w_i|^2)\right]^2|w_i|^2+C|\nabla w_i|^2 \qquad (3.13)$$

$$\left|\frac{\delta_i^4}{\varepsilon_i^2}\beta_i^2 b_i(1-|w_i|^2)|w_i|^2\right|\le\frac{1}{8}\left[\frac{\delta_i^2}{\varepsilon_i^2}\beta_i^2(1-|w_i|^2)\right]^2|w_i|^2+C\delta_i^4|w_i|^2 \qquad (3.14)$$

Combining (3.11), (3.12), (3.13), with (3.14), we have

$$\begin{aligned}
(\partial_t-\Delta)e(w_i)&\le Ce(w_i)+Ce^2(w_i)+C(\delta_i^4+\delta_i^2)\\
&\le Ce(w_i)+C\delta_i \qquad (3.15)
\end{aligned}$$

where we have used the fact that $|w_i| \le 3/2, \delta_i << 1$. Let $\bar{e} = e(w_i) + C\delta_i$, then (3.15) implies that

$$\partial_t \bar{e} - \Delta \bar{e} \le C\bar{e} \text{ in } P_1$$

Thus the Moser's estimate for the linear heat equations implies that

$$\sup_{P_{1/2}} \bar{e} \le C \int_{P_1} \bar{e} \le C \int_{P_1} e(w_i) + C\delta_i \tag{3.16}$$

Again, by (3.10), one has

$$\int_{P_1} e(w_i) dx dt = \left(\frac{1}{\delta_i}\right)^2 \int_{P_{\delta_i(x_0,t_0)}} e(v_i) \le C\theta_0 + C(\theta + \theta_0)E_0$$

Hence

$$\sup_{P_{1/2}} \bar{e} \le C\theta_0 + C(\theta + \theta_0) + C\delta_i \tag{3.17}$$

(3.17) is an obvious contradiction to (vi) if θ_0 and θ are small enough, and i is large enough. Thus we complete the proof of Lemma 3.5.

Proof of Theorem 3.1 The proof of [1; Theorem 5.2] carries over almost literally. For the sake of completeness, we sketch it here. We suppose v_ε weakly converges as $\varepsilon \to 0^+$ to v in $H^1(Q_1)$. Hence v is smooth inside Q_1 by Lemma 3.2.

Denote by Σ the set of points (x_0, t_0) such that $|x_0| < 1, t_0 \in (-1, 0]$, and such that $e_\varepsilon(v_\varepsilon)(x,t)dxdt$ does not converge weakly to $\frac{1}{2}|\nabla v|^2(x,t)dxdt$ as Radon measures on $P_r(x_0,t_0)$ for any $r \in (0, r_0)$. Here $r_0 = \min\{1 - |x_0|, \sqrt[2]{1+t_0}\}$ and

$$P_r(x_0,t_0) = \{(x,t) : |x - x_0| < r_0, t_0 - r_0^2 \le t \le t_0\}$$

Suppose $\Sigma \ne \emptyset$, and let $(x_0,t_0) \in \Sigma$. After a translation and a scaling, we may assume $r_0 = 1, (x_0,t_0) = (0,0)$. For the scaled sequence

$$w_\varepsilon(x,t) = v_\varepsilon\left(\frac{r_0}{2}x + x_0, \frac{r_0^2}{4}t + t_0\right)$$

one has

$$\begin{aligned} e_\varepsilon(w_\varepsilon) &= \frac{1}{2}|\nabla w_\varepsilon|^2 + \frac{1}{2\varepsilon}\beta^2\left(\frac{2}{r_0}(x - x_0)\right)(1 - |w_\varepsilon|^2)^2 \\ &= \frac{1}{2}|\nabla v_\varepsilon|^2\left(\frac{r_0}{2}\right)^2 + \frac{1}{2\varepsilon^2}\beta^2\left(\frac{2}{r_0}(x - x_0)\right)(1 - |v_\varepsilon|^2)^2 \\ \partial_t w_\varepsilon &= \frac{r_0^2}{4}\partial_t v_\varepsilon \end{aligned}$$

$$\int_{-1}^{0}\int_{B_1} e_\varepsilon(w_\varepsilon)dxdt = \frac{r_0^2}{4}\int_{-\frac{1}{4}r_0^2+t_0}^{t_0}\int_{B(x_0,\frac{r_0}{2})}^{16/r_0^4}\frac{1}{2}\Big[|\nabla v_\varepsilon|^2 + \frac{\beta^2(1-|v_\varepsilon|^2)^2}{2(r_0\varepsilon/2)^2}\Big]dxdt \le C(r_0)$$

$$\int_{-1}^{0}\int_{B_1}|\partial_t w_\varepsilon|^2 dxdt = \frac{r_0^2}{4}\int_{-1}^{0}\int_{B_1}|\partial_t v_\varepsilon|^2 dxdt \le C(r_0)$$

So $w_\varepsilon(x,t)$ satisfies the same hypothesis as v_ε on Q_1 with ε replaced by $2\varepsilon/r_0$.

Since

$$w_\varepsilon(x,t) \to v\left(x_0 + \frac{r_0}{2}x, t_0 + \frac{r_0^2}{2}t\right) = w(x,t)$$

and $w(x,t): Q_1 \to S^1$ satisfies

$$\partial_t w = \Delta w + \frac{2\nabla\alpha}{\alpha}\nabla w + \frac{\Delta\alpha}{\alpha}w + \frac{1}{\alpha^2}(\alpha\Delta\alpha - \alpha^2|\nabla w|^2)w$$

with

$$\int_{Q_1}[|\nabla w|^2 + |\partial_t w|^2]dxdt \le C_0$$

where $\alpha(x) = \beta\left(x_0 + \frac{r_0}{2}x\right)$, we have, for any $\rho_0 \in \left(0, \frac{1}{2}\right), -\frac{1}{4} \le -T_0 = -\rho_0^2 \le 0$, that

$$\int_{B^2_{\rho_0}(0)}|\nabla w|^2(x,T_0)dx \le C_1\rho_0^2$$

for some constant depending only on C_0.

We shall choose ρ_0 so small that if

$$\int_{B^2_{\rho_0}(0)} e_\varepsilon(w_\varepsilon)(x,T_0)dx \le 2C\rho_0^2$$

then

$$e_\varepsilon(w_\varepsilon)dxdt \to \frac{1}{2}|\nabla w|^2(x,t)dxdt \ \text{ in } \ P_{\rho_0/2}(0,0)$$

The last statement follows from the small energy regularity theorem and Lemma 3.5.

Since $(0,0)\in\Sigma$, we see that for a.e. $\rho\in(0,\rho_0)$, $e_\varepsilon(w_\varepsilon)(x,-\rho^2)dx$ does not converge weakly to

$$\frac{1}{2}|\nabla w|^2(x,-\rho^2)dx$$

as Radon measures on $B_{2\rho}(0)$. Since $\int_{Q_1}|\partial_t w_\varepsilon|^2dxdt \le C_0$, we may find a $\rho\in(0,\rho_0)$ such that

$$\int_{B_1}|\partial_t w_\varepsilon|^2(x,-\rho^2)dx \le \frac{2C_0}{\rho_0}$$

Now we look at the sequence $w_\varepsilon(x,-\rho^2)$ that satisfies all the hypotheses of the following lemma ([1, Lemma 5.4]).

Lemma 3.6 *Let $\{v_\varepsilon\}$ be the solutions of*

$$\Delta v_\varepsilon + \frac{\beta^2}{2\varepsilon^2}(1-|v_\varepsilon|^2)v_\varepsilon = f_\varepsilon$$

with $\|\nabla v_\varepsilon\|_{L^\infty(B_1)} \le \frac{M}{\varepsilon}$, $\int_{B_1} e_\varepsilon(v_\varepsilon)dx \le M$, $\|f_\varepsilon\|_{L^2(B_1)} \le M$. *Suppose* $v_\varepsilon \rightharpoonup v_*$ *converges weakly in* $H^1(B_1)$, *then*

$$v_\varepsilon \to v_* \text{ strongly in } H^1_{\text{loc}}(B_1)$$

Since $e_\varepsilon(w_\varepsilon)(x,-\rho^2)dx$ does not converge weakly to $\frac{1}{2}|\nabla w|^2(x,-\rho^2)dx$, we obtain a contradiction. This proves the theorem.

References

[1] Lin F.H., Complex Ginzburg-Landau equations and dynamics of vortices, filaments, and codimension-2 submanifolds, *Comm. Pure Appl. Math.*, **51** (1998), 385–441.

[2] Ding S. and Liu Z., Asymptotics for a class of Ginzburg-Landau functionals, *Chinese Annals of Math.*, **18A**(1997), 437–444.

[3] Rubinstein J., On the equilibrium position of Ginzburg-Landau vortices, *Z. Angew. Math. Phys.*, **46**(1995), 739–751.

[4] E W., Dynamics of vortices in Ginzburg-Landau theories with applications to supconductivity, *Phys. D*, **77** (1994), 384–404.

[5] Lin F.H., Some dynamical properties of Ginzburg-Landau vortices, *Comm. Pure Appl. Math.*, **49** (1996), 323–359.

[6] Neu J. C., Vortex dynamics of the nonlinear wave equation, *Phys. D* **43** (1990), 407–420.

[7] Peres L. and Rubinstein J., Vortex dynamics in U(1) Ginzburg-Landau models, *Phys. D*, **64**(1993), 299–309.

[8] Neu J.C., Vortices in complex scalar fields, *Phys. D*, **43**(1990), 385–406.

[9] Lin F.H., A remark on the previous paper: "Some dynamical properties of Ginzburg-Landau vortices," *Comm. Pure Appl. Math.*, **49** (1996), 361–364.

[10] Lin F.H., Static and moving vortices in Ginzburg-Landau theories, Nonlinear partial differential equations in geometry and physics (Knoxville, TN, 1995), 71–111, Progr. Nonlinear Differential equations Appl., 29, Birkhauser, Basel, 1997.

[11] Jerrard R. and Soner M., Dynamics of Ginzburg-Landau vortices, *Arch. Rational Mech. Anal.*, **142**(1998), 99–125.

[12] Jian H.Y. and Song B.H., Vortex dynamics of Ginzburg-Landau equations in inhomogeneous superconductors, Preprint, 1999.

[13] Chen Y.M. and Struwe M., Existence and partial regularity for the heat flow for harmonic maps, *Math. Z.*, **201** (1989), 83–103.

[14] Chen Y.M., Dirichlet problems for heat flows of harmonic maps in higher dimensions, *Math. Z.*, **208** (1991), 557–565.