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TIME-ASYMPTOTIC BEHAVIOR OF SOLUTIONS FOR GENERAL NAVIER-STOKES EQUATIONS IN EVEN SPACE-DIMENSION*

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Abstract We study the time-asymptotic behavior of solutions to general Navier-Stokes equations in even and higher than two space-dimensions. Through the pointwise estimates of the Green function of the linearized system, we obtain explicit expressions of the time-asymptotic behavior of the solutions. The result coincides with weak Huygan's principle.

Key Words Compressible flow; conservation laws; general Navier-Stokes equation; space-dimension; Green's function; time-asymtotic behavior.
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1. Introduction

In this paper, we derive a detailed description of the asymptotic behavior of solutions of Cauchy problem for the general Navier-Stokes systems of conservation laws in n-dimension, where $n > 2$ is even. General Navier-Stokes equation is

$$\begin{cases} \rho_t + \mathrm{div}\rho v = 0 \\ (\rho v^j)_t + \mathrm{div}(\rho v^j v) + P(\rho, e)_{x_j} = \varepsilon \triangle v^j + \eta \mathrm{div} v_{x_j}, \quad j = 1, \cdots, n \\ (\rho E)_t + (\rho E v + P(\rho, e) v) = \triangle \left(k \left(T(e) + \frac{1}{2}\varepsilon |v|^2 \right) \right. \\ \qquad \left. + \varepsilon \mathrm{div}((\nabla v) v) + (\eta - \varepsilon) \mathrm{div}((\mathrm{div} v) v) \right) \end{cases} \tag{1.1}$$

Here $\rho(x,t), v(x,t), e(x,t), P = P(\rho, e)$ and $T(e)$ represent respectively the fluid density, velocity, specific internal energy, pressure and normalized tempreture, and $E = e + \frac{1}{2}|v|^2$ is the specific total energy. $k > 0$ is the heat conductivity, $\varepsilon > 0$ and $\eta \geq 0$ are viscosity constants, and div and $\triangle$ are the usual spatial divergence and Laplace operator. We assume throughout that $P(\rho, e)$ and $T(e)$ are smooth in a neighborhood of constant state (ρ^*, e^*) and $P_\rho = P_\rho(\rho^*, e^*) > 0, P_e = P_e(\rho^*, e^*) > 0, p = P(\rho^*, e^*)$ and $d^2 = kT'(\rho^*) > 0$.

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For the equation (1.1) Liu and Zeng (see [1]) studied general hyperbolic-parabolic systems in one-dimension and obtained pointwise estimate. In several space variables, the asymptotic behavior of the solution of the Cauchy problem for Navier-Stokes equations has been studied in [2] and [3] but only in L^p space. As for pointwise estimate, for isentropic Navier-Stokes equations in odd space dimension Liu and Wang gave a pointwise estimate in [4], and Xu gave a pointwise estimate for linearized system in even space-dimension in [5].

The plan of this paper is as follows. The linearized system of (1.1) around the constant state $(\rho^*, v^*, e^*)^\tau = (1, 0, e^*)^\tau, (e^* > 0)$ is

$$\begin{cases} \rho_t + \mathrm{div} v = 0 \\ v_t + p_\rho \nabla \rho + p_e \nabla e = \varepsilon \Delta v + \eta \nabla \mathrm{div} v \\ e_t + p \mathrm{div} v = d^2 \Delta e \end{cases} \tag{1.2}$$

We first get the pointwise estimate of Green function G of (1.2), then get the asymptotic behavior of the solution of (1.1) by using Duhamel's principle. Comparing with [5], our main difficulty is that we can't get the explit expression for G. So in Section 2 we will introduce a method which allows us to get the estimate without explict representation of the matrix.

In this article, $n > 2$ is space dimension, which is even. C, ε are positive constants.

2. Pointwise Estimate of Green Function

The Green matrix G is defined as the solution of the following problem:

$$\begin{cases} (\partial_t + A_1(D_x) + B_1(D_x))G(x,t) = 0 \\ G(x,0) = \delta(x) I \end{cases} \tag{2.1}$$

The symbols of $A_1(D_x)$ and $B_1(D_x)$ are $\sqrt{-1}A(\xi)$ and $|\xi|B(\xi)$ respectively, where

$$A(\xi) = \begin{pmatrix} 0 & \xi^\tau & 0 \\ P_\rho \xi & 0 & P_e \xi \\ 0 & P\xi & 0 \end{pmatrix}, \quad B(\xi) = |\xi|^{-1} \begin{pmatrix} 0 & 0 & 0 \\ 0 & \varepsilon |\xi|^2 I + \eta \xi \xi^\tau & 0 \\ 0 & 0 & d^2 |\xi|^2 \end{pmatrix}$$

$\xi = (\xi_1, \cdots, \xi_n)^\tau$, and $\delta(x)$ is the Dirac function and I the $(n+2) \times (n+2)$ identity matrix. We apply Fourier transformation to the x variables and get

$$\begin{cases} \hat{G}_t(\xi, t) = -\sqrt{-1} E(\xi) \hat{G}(\xi, t) \\ \hat{G}(\xi, 0) = I \end{cases} \tag{2.2}$$

where $E(\xi) = A(\xi) - \sqrt{-1}|\xi| B(\xi)$.

Let $E_{\alpha,\beta}(\xi) = \beta A(\xi) - \sqrt{-1}\alpha B(\xi)$. From simple calculation we know that it has four diffrent eigenvalues. Arrange them as $\lambda_j^{\alpha,\beta} (j = 1, 2, 3, 4)$. They have multiplicity 1,1,1 and $n-1$ respectively. Let the left and right eigenvectors associated with $\lambda_j^{\alpha,\beta}$

be $r_j^{\alpha,\beta}, l_j^{\alpha,\beta}$. From simple calculation, we know that $\lambda_4^{\alpha,\beta}$ has $n-1$ different set of eigenvectors $l_j^{\alpha,\beta}, r_j^{\alpha,\beta}$. Let $l_i^{\alpha,\beta} r_j^{\alpha,\beta} = \delta_{i,j}$, then $E_{\alpha,\beta}(\xi) = \sum_{j=1}^{4} \lambda_j^{\alpha,\beta} P_j^{\alpha,\beta}$, where $P_j^{\alpha,\beta} = r_j^{\alpha,\beta} l_j^{\alpha,\beta}$. Especially $E = \sum_{j=1}^{4} \lambda_j^{|\xi|,1}$.

Before the estimate, we first give a lemma, which is proved by Shizuta and Kawashima (see Theorem 1.1 of [6]).

Lemma 2.1 The following statements are equivalent.

(1) The system (1.2) is dissipative.

(2) For all $\xi \in \boldsymbol{R}^n \backslash 0$, the eigenvector of $A(\xi)$ does not lie in the null space of $B(\xi)$.

(3) $\mathrm{Im}\left(\lambda_j^{|\xi|,1}(\xi)\right) \le -C\frac{|\xi|^2}{1+|\xi|^2}, j=1,2,3,4$ for real ξ.

We can check easily that no eigenvector of $A(\xi)$ lies in the null space of $B(\xi)$ for any $\xi \in \boldsymbol{R}^n \backslash 0$, so we have

$$\mathrm{Im}\left(\lambda_j^{|\xi|,1}(\xi)\right) \le -C\frac{|\xi|^2}{1+|\xi|^2} \tag{2.3}$$

$j = 1,2,3,4$ for real ξ.

Lemma 2.2 *If $f(\xi,t)$ satisfies*

$$|D^\beta(\xi^\alpha \hat{f}(\xi,t))| \le C(\min(1, |\xi|^{|\alpha|+k-|\beta|}) + |\xi|^{|\alpha|+k} t^{\frac{\beta}{2}})(1+t|\xi|^2)^m e^{-\theta|\xi|^2 t}$$

for any integer m and $k > -n$, then

$$|D^\alpha f(x,t)| \le Ct^{-\frac{n+|\alpha|+k}{2}} B_N(x,t)$$

where N is any fixed interger, and

$$B_N(x,t) = \left(1 + \frac{|x|^2}{1+t}\right)^{-N}$$

Proof If $|\beta| < k + |\alpha|$, then

$$\begin{aligned} |x^\beta D^\alpha f(x,t)| &\le C\int |\xi|^{|\alpha|+k}\left(|\xi|^{-|\beta|} + t^{\frac{|\beta|}{2}}\right)(1+|\xi|^2 t)^m e^{-\theta|\xi|^2 t} d\xi \\ &\le Ct^{-\frac{|\alpha|+n+k-|\beta|}{2}} \end{aligned}$$

If $|\beta| \ge k + |\alpha|$, then

$$\begin{aligned} |x^\beta D^\alpha f(x,t)| &\le C\int \left(1 + |\xi|^{|\alpha|+k} t^{\frac{|\beta|}{2}}\right)(1+|\xi|^2 t)^m e^{-\theta|\xi|^2 t} d\xi \\ &\le Ct^{-\frac{|\alpha|+n+k-|\beta|}{2}} \end{aligned}$$

Let $|\beta| = 0$ when $|x|^2 \le t+1$, and $|\beta| = N$ when $|x|^2 > t+1$, then we obtain the result.

Taking Taylor expansion at $\alpha = 0$ for $\lambda_k^{\alpha,1}(\xi)$ and $P_k^{\alpha,1}(\xi)$, noting that $P_k^{\alpha,1}(\xi)$ is 0-homogeneous in $|\xi|$, we get

Lemma 2.3 *For $|\xi| < \varepsilon$ and ε small enough, we have*

$$\lambda_k^{|\xi|,1}(\xi) = \lambda_k^{0,1}(\xi) + |\xi|(\partial_\alpha \lambda_k^{\alpha,1}(\xi))_{\alpha=0} + O(|\xi|^3) \tag{2.4}$$

$$P_k^{|\xi|,1}(\xi) = P_k^{0,1}(\xi) + O(|\xi|) \tag{2.5}$$

By simple calculation, we know that $\lambda_1^{0,1} = 0, \lambda_2^{0,1} = c|\xi|, \lambda_3^{0,1} = -c|\xi|, \lambda_4^{0,1} = 0$. Taking Taylor expansion at $\alpha = 0$ for

$$P_j^{\alpha,1}\left(A - \sqrt{-1}\alpha B\right)P_j^{\alpha,1} = P_j^{\alpha,1}\left(\lambda_j^{\alpha,1}P_j^{\alpha,1}\right)$$

and comparing the coefficients of α on both sides, we have

$$-\sqrt{-1}P_j^{0,1}BP_j^{0,1} = \left(\partial_\alpha \lambda_j^{\alpha,1}(\xi)\right)_{\alpha=0} P_j^{0,1}$$

Since $B, P_j^{0,1}$ are real for real $\xi, \sqrt{-1}(\partial_\alpha \lambda_j^{\alpha,1}(\xi))_{\alpha=0}$ is real. From (2.3) and that $(\partial_\alpha \lambda_j^{\alpha,1}(\xi))_{\alpha=0}$ is 1-homogeneous in ξ, we see that there exists positive constant ε such that

$$\left(\sqrt{-1}\partial_\alpha \lambda_k^{\alpha,1}(\xi)\right)_{\alpha=0} > \varepsilon|\xi| \tag{2.6}$$

Let $\hat{w} = \dfrac{\sin(c|\xi|t)}{c|\xi|}$, then for small $|\xi|$, we have

$$\begin{aligned}
\hat{G}(\xi,t) =& \cos(c|\xi|t)e^{-\sqrt{-1}(|\xi|(\partial_\alpha\lambda_2^{\alpha,1}(\xi))_{\alpha=0}+O(|\xi|^3))t}(P_2^{0,1}(\xi) + O(|\xi|)) \\
&- \sqrt{-1}\frac{\sin(c|\xi|t)}{c|\xi|}e^{-\sqrt{-1}(|\xi|(\partial_\alpha\lambda_2^{\alpha,1}(\xi))_{\alpha=0}+O(|\xi|^3))t}(P_2^{0,1}(\xi) + O(|\xi|)) \cdot c|\xi| \\
&+ \cos(c|\xi|t)e^{-\sqrt{-1}(|\xi|(\partial_\alpha\lambda_3^{\alpha,1}(\xi))_{\alpha=0}+O(|\xi|^3))t}(P_3^{0,1}(\xi) + O(|\xi|)) \\
&+ \sqrt{-1}\frac{\sin(c|\xi|t)}{c|\xi|}e^{-\sqrt{-1}(|\xi|(\partial_\alpha\lambda_3^{\alpha,1}(\xi))_{\alpha=0}+O(|\xi|^3))t}(P_3^{0,1}(\xi) + O(|\xi|)) \cdot c|\xi| \\
&+ e^{-\sqrt{-1}\lambda_1^{|\xi|,1}t}P_1^{|\xi|,1}(\xi) + e^{-\sqrt{-1}\lambda_4^{|\xi|,1}t}P_4^{|\xi|,1}(\xi) \\
=& \hat{w}_t\hat{F}_1 + \hat{w}\hat{F}_2 + \hat{w}_t\hat{F}_3 + \hat{w}\hat{F}_4 + \hat{F}_5
\end{aligned}$$

From simple calculation and (2.6), we can get

Lemma 2.4 *If $|\xi|$ is small enough, there exists a positive constant b, such that*

$$\begin{aligned}
&|D_\xi^\beta(\xi^\alpha\hat{F}_j(\xi,t))| \\
&\le C\left(\min\left(1, |\xi|^{1+|\alpha|-|\beta|}\right) + |\xi|^{|\alpha|+1}t^{\frac{|\beta|}{2}}\right)(1+|\xi|^2t)^{|\beta|+1}e^{-b|\xi|^2t}, \quad j = 2,4
\end{aligned} \tag{2.7}$$

$$\begin{aligned}
&|D_\xi^\beta(\xi^\alpha\hat{F}_j(\xi,t))| \\
&\le C\left(\min\left(1, |\xi|^{|\alpha|-|\beta|}\right) + |\xi|^{|\alpha|}t^{\frac{|\beta|}{2}}\right)(1+|\xi|^2t)^{|\beta|+1}e^{-b|\xi|^2t}, \quad j = 1,3,5
\end{aligned} \tag{2.8}$$

From Lemma 2.2, we get

Lemma 2.5 *If $|\xi| < \varepsilon$ and ε is small enough, we have*

$$|D^\alpha F_j(x,t)| \le C_N t^{-\frac{n+|\alpha|+1}{2}} B_N(x,t), \quad j = 2,4 \tag{2.9}$$
$$|D^\alpha F_j(x,t)| \le C_N t^{-\frac{n+|\alpha|}{2}} B_N(x,t), \quad j = 1,3,5 \tag{2.10}$$

In order to estimate $G(x,t)$, we also need the following two lemmas which can be found in [5].

Lemma 2.6 *For $f(x) \in C^{\frac{n}{2}}$, there exist constants a_α, b_α, such that*

$$w * f(x) = \sum_{0 \le |\alpha| \le \frac{n-2}{2}} a_\alpha t^{|\alpha|+1} \int_{|y|\le 1} \frac{D^\alpha f(x+cty)}{\sqrt{1-|y|^2}} y^\alpha dy$$
$$w_t * f(x) = \sum_{0 \le |\alpha| \le \frac{n}{2}} b_\alpha t^{|\alpha|} \int_{|y|\le 1} \frac{D^\alpha f(x+cty)}{\sqrt{1-|y|^2}} y^\alpha dy$$

Lemma 2.7 *For any positive integer N, there exists constant C, such that*

$$\left| \int_{|y|\le 1} \frac{B_{2N}(x+cty,t)}{\sqrt{1-|y|^2}} y^\alpha dy \right| \le C(1+t)^{-\frac{n-1}{2}} A_N(x,t)$$

where $A_N(x,t) = \int_0^1 \frac{1}{\sqrt{1-r^2}} \left(1 + \frac{(|x|-ctr)^2}{1+t}\right)^{-N} dr.$

Let

$$\chi_1(\xi) = \begin{cases} 1 & \text{if } |\xi| < \varepsilon \\ 0 & \text{if } |\xi| > 2\varepsilon \end{cases}, \quad \chi_3(\xi) = \begin{cases} 1 & \text{if } |\xi| > R+1 \\ 0 & \text{if } |\xi| < R \end{cases}$$

be cut-off functions, with $2\varepsilon < R, \chi_2 = 1 - \chi_1 - \chi_3$. Since $\hat{G}_j = \chi_j \hat{G} (j = 1,2,3)$, from (2.3) we know that

$$|D_x^\alpha G_1| \le C \int_{|\xi|\le 2\varepsilon} \xi^\alpha \hat{G} d\xi \le C \tag{2.11}$$
$$|D_x^\alpha G_2| \le C \int_{|\xi|\le R+1} \xi^\alpha \hat{G} d\xi \le C \tag{2.12}$$

From Lemmas 2.5–2.7 and (2.11), we have

Lemma 2.8 *For $\varepsilon > 0$ small enough, we have*

$$|D_x^\alpha G_1(x,t)| \le C(1+t)^{-\frac{3n}{4}+\frac{1}{2}-\frac{|\alpha|}{2}} A_N(x,t) + C(1+t)^{-\frac{n+|\alpha|}{2}} B_N(x,t)$$

By the method similar to the proof of Proposition 3.3 in [4], we get $|D_x^\alpha G_2(x,t)| \le Ct^{-\frac{n+|\alpha|}{2}} B_N(x,t)$. From (2.12), we get the following result:

Lemma 2.9 *For fixed ε and R, we have*

$$|D_x^\alpha G_2(x,t)| \le C(1+t)^{-\frac{n+|\alpha|}{2}} B_N(x,t)$$

Take Taylor expansion for $\lambda_j^{1,\beta}(\xi)$ in β near $\beta=0$, we get

$$\lambda_j^{1,\beta}(\xi)=\lambda_j^{1,0}(\xi)+\beta(\partial_\beta\lambda_j^{1,\beta}(\xi))_{\beta=0}+\cdots+\frac{1}{m!}\beta^m(\partial_\beta^m\lambda_j^{1,\beta}(\xi))_{\beta=0}+r(\beta,\xi)$$

where $(\partial_\beta^m\lambda_j^{1,\beta}(\xi))_{\beta=0}$ is 0-homogeneous in ξ, thus

$$\lambda_j^{|\xi|,1}=|\xi|\lambda_j^{1,\frac{1}{|\xi|}}=|\xi|\lambda_j^{1,0}(\xi)+(\partial_\beta\lambda_j^{1,\beta}(\xi))_{\beta=0}+\sum_{j=1}^{m}a_j|\xi|^{-j}+O(|\xi|^{-m-1})$$

Thus we can write

$$\begin{aligned}e^{-\sqrt{-1}\lambda_j^{|\xi|,1}t}=&e^{-\sqrt{-1}(|\xi|\lambda_j^{1,0}(\xi)+(\partial_\beta\lambda_j^{1,\beta}(\xi))_{\beta=0})t}\\&\times\left(1+\left(\sum_{i=1}^{m}a_i|\xi|^{-i}\right)t+\cdots+\frac{1}{m!}\left(\sum_{i=1}^{m}a_i|\xi|^{-i}\right)^m t^m+R(t,\xi)\right)\end{aligned}$$

where $R(t,\xi)\le C(1+t)^{m+1}(1+|\xi|)^{-(m+1)}$, and we have

$$\begin{aligned}\hat{G}_3(\xi,t)&=\sum_{j=1}^{4}e^{-\sqrt{-1}\lambda_j^{|\xi|,1}t}P_j^{|\xi|,1}\\&=\sum_{j=1}^{4}e^{-\sqrt{-1}(|\xi|\lambda_j^{1,0}(\xi)+(\partial_\beta\lambda_j^{1,\beta}(\xi))_{\beta=0})t}\left(q_{j,0}+\sum_{i=1}^{m}p_{j,i}(t)q_{j,i}(\xi)+R_j(t,\xi)\right)\end{aligned}$$

where $p_{j,i},q_{j,i},R_j$ are matrices, and

$$|p_{j,i}(t)|\le C(1+t)^i,|q_{j,i}(\xi)|\le C(1+|\xi|)^{-i},|R_j(t,\xi)|\le C(1+t)^{m+1}(1+|\xi|)^{-(m+1)}$$

Let

$$\begin{aligned}L_{j,0}(t,\xi)&=e^{-\sqrt{-1}(|\xi|\lambda_j^{1,0}(\xi)+(\partial_\beta\lambda_j^{1,\beta}(\xi))_{\beta=0})t}q_{j,0}\\L_{j,i}(t,\xi)&=e^{-\sqrt{-1}(|\xi|\lambda_j^{1,0}(\xi)+(\partial_\beta\lambda_j^{1,\beta}(\xi))_{\beta=0})t}p_{j,i}(t)q_{j,i}(\xi)\end{aligned}$$

From (2.3) we know that $\mathrm{Re}\left(-\sqrt{-1}\lambda^{|\xi|,1}\right)\le -b$ for some $b>0$. By the definitions of G and $L_{j,l}$, we have

Lemma 2.10 *For R sufficiently large, there exist distributions*

$$K_{|\alpha|}(x,t)=\sum_{j=0}^{4}\sum_{i=0}^{n+|\alpha|}\overline{L_{j,i}}$$

such that

$$\left|D_x^\alpha(G_3-\chi_3(D)K_{|\alpha|}(x,t))\right|\le Ce^{-bt}B_N(x,t)$$

wehre $b>0,\widehat{\overline{L_{j,i}}}=L_{j,i}(t,\xi)$.

Summing up Lemma 2.8, Lemma 2.9 and Lemma 2.10, we can get

Theorem 2.1 *For $x\in R^n$, we have*

$$|D_x^\alpha(G(x,t)-\chi_3(D)K_{|\alpha|}(x,t))|\le C(1+t)^{-\frac{n+|\alpha|}{2}}\left(B_N(x,t)+(1+t)^{-\frac{n}{4}+\frac{1}{2}}A_N(x,t)\right)$$

3. Asymptotic Behavior for Nonlinear System

We denote $u = (\rho-\rho^*, v-v^*, e-e^*)^\tau = (\rho-1, v, e-e^*)^\tau, u_0 = (\rho_0-1, v_0, e_0-e^*)^\tau$ and rewrite (1.1) as

$$\partial_t u + A_1(D_x)u + B_1(D_x)u = Q(u) \tag{3.1}$$

When $|u|$ is small enough, we can write

$$Q(u) = \sum_j D_{x_j}\left(r_j(u) + \sum_l D_{x_l} r_{j,l}(u)\right)$$

where $r_j(u) = O(|u|^2), r_{j,l}(u) = O(|u|^2)$. In this section, we will use this fact to consider the Cauchy problem of (3.1)

$$\begin{cases} \partial_t u + A_1(D_x)u + B_1(D_x)u = Q(u) \\ u|_{t=0} = u_0 \end{cases} \tag{3.2}$$

As in [2] and [4], we have

Proposition 3.1 *Suppose that $u_0 \in H^{s+l}(\boldsymbol{R}^n)$, where $s = \frac{n}{2}+1, l$ is a nonnegative integer, and that $\|u_0\|_{H^{s+l}}$ is sufficiently small. Then there exists a unique global classical solution $u \in H^{s+l}$ of (1,1) satisfying*

$$\|D_x^\alpha u\|_{L^2}(t) = O(1)\|u_0\|_{H^{s+l}}, \quad 0 \le |\alpha| \le s+l$$

$$\left(\int_0^\infty \|D_x^\alpha u\|_{L^2}^2(t)dt\right)^{\frac{1}{2}} = O(1)\|u_0\|_{H^{s+l}}, \quad 1 \le |\alpha| \le s+l$$

$$\|D_x^\alpha u\|_{L^\infty} = O(1)\|u_0\|_{H^{s+l}}, \quad 0 \le |\alpha| \le l$$

Let $E = \max\{\|u_0\|_{H^{s+l}}, \|u_0\|_{W^{l,1}}\}$, by Proposition 3.1 we have $\|u_0\|_{W^{l,\infty}} \le CE$.

Now we will give a pointwise estimate for the solution u of (3.2). Take D_x^α on (3.1) and apply Duhamel's principle, we obtain

$$D_x^\alpha u = D_x^\alpha G(t) * u_0 + \int_0^t G(t-s) * D_x^\alpha Q(u(s))ds = R_1^\alpha + R_2^\alpha \tag{3.3}$$

First we give three lemmas.

Lemma 3.1 *If $|y| \le M, t > 4M^2, p \ge 0$ and $N > 0$, we have*

$$(1+(|x-y|-pt)^2/t)^{-N} \le C_N(1+(|x|-pt)^2/t)^{-N}$$

The proof can be found in [4].

Lemma 3.2 *If $\text{supp}\hat{f}(\xi) \subset O_R = \{\xi, |\xi| > R\}$ and satisfies the estimate*

$$|D_\xi^\beta \hat{f}(\xi)| \le C|\xi|^{-|\beta|}$$

then there exist distributions $f_1(x), f_{2,\gamma}(x)$ and a constant C_0, such that

$$f(x) = f_1(x) + \sum_{|\gamma|\le 1} D_x^\gamma f_{2,\gamma}(x) + C_0\delta(x)$$

where $\delta(x)$ is the Dirac function, and for $f_1(x)$ and $f_{2,\gamma}(x)$ we have the following estimate: there exists constant $C > 0$ such that for any positive integer N,

$$|D_x^\alpha f_1(x)| \le C(1+|x|^2)^{-N}$$

and

$$\|f_{2,\gamma}\|_{L_1} \le C, \mathrm{supp} f_{2,\gamma}(x) \subset \{x; |x| < 2\varepsilon_0\}$$

with ε_0 small enough.

Proof Since $\mathrm{supp}\hat{f}(\xi) \subset O_R$, one has

$$J = |x^\beta D_x^\alpha f(x)| \le C \int_{|\xi|>R} |\xi|^{|\alpha|-|\beta|} d\xi$$

Taking $|\beta| = 2N > |\alpha| + n$, we see $J \le C$. Thus for $|x| \ne 0$,

$$|D_x^\alpha f(x)| \le C|x|^{-2N}$$

Let $f_1(x) = \Psi(x) f(x)$ with $\Psi(x) \in C^\infty$ and

$$\Psi(x) = \begin{cases} 1, & |x| \ge 2\varepsilon_0 \\ 0, & |x| < \epsilon_0 < 1 \end{cases}$$

we know that $|D^\alpha f_1(x)| \le C(1+|x|^2)^{-N}$.

Next set

$$f_{3,\gamma}(x) = (1-\Psi(x)) R_\gamma(D_x) f(x), \quad f_{4,\gamma}(x) = (D_x^\gamma \Psi(x))(R_\gamma(D_x) f(x))$$

with $R_\gamma(D_x)$ to be the singular integral operator with symbol $R_\gamma(\xi) = \xi^\gamma/|\xi|^2 (|\gamma| = 1)$. Since $\sum_{|\gamma|=1} D_x^\gamma R_\gamma(D_x) = I$, we have

$$f(x) = f_1(x) + \sum_{|\gamma|=1} (D_x^\gamma f_{3,\gamma}(x) + f_{4,\gamma}(x))$$

Let $f_{2,\gamma}(x) = x^{-\beta_1}\left(x^{\beta_1} f_{3,\gamma}(x)\right)$ and $p = 2n, q = 2n/(2n-1)$. We have

$$\|f_{2,\gamma}\|_{L_1} \le C\|x^{\beta_1} f_{3,\gamma}\|_{L_p} \|x^{-\beta_1}\|_{L_q(|x|<1)}$$

Taking $|\beta_1| = n-1$, we have $\|x^{-\beta_1}\|_{L_q(|x|<1)} \le C$. Let $\eta(\xi)$ be the Fourier transform of $(1-\Psi(x))$, we have

$$\|x^{\beta_1} f_{3,\gamma}\|_{L_p} \le C\|D^{\beta_1}\eta * R_\gamma(\xi)\hat{f}\|_{L_q} \le C\|\eta\|_{L_\infty}\|D^{\beta_1} R_\gamma(D_x)\hat{f}\|_{L_q} \le C$$

Therefore $\|f_{2,\gamma}\|_{L_1} \le C$. Let $h_\gamma(x) = f_{3,\gamma}(x) - f_{2,\gamma}(x)$, then $h_\gamma(x)$ is supported at the point $x = 0$ and its Fourier transform is bounded:

$$\|\hat{h}_\gamma\|_{L^\infty} \le \|\hat{f}_{3,\gamma}\|_{L_\infty} + \|f_{2,\gamma}\|_{L_1} \le C$$

It follows that $\sum_{|\gamma|=1} h_\gamma(x) = C_0\delta(x)$.

Let $f_{2,0}(x) = x^{-\beta}\left(x^\beta \sum_{|\gamma|=1} f_{4,\gamma}(x)\right)$. Use the same method, we get $\|f_{2,0}\|_{L_1} \le C$, and $h_0 = \sum_{|\gamma|=1} f_{4,\gamma}(x) - f_{2,0}(x) = C_0\delta(x)$. Thus the lemma is proved.

Now we came to a very technical lemma

Lemma 3.3 *If functions $H(x,t)$ and $S(x,t)$ satisfy*

$$|D_x^\alpha S(x,t)| \le C(1+t)^{-\frac{n+|\alpha|+1}{2}}(B_N(x,t) + (1+t)^{-\frac{n}{4}+\frac{1}{2}}A_N(x,t))$$

and

$$|D_x^\alpha H(x,t)| \le C(1+t)^{-n-\frac{|\alpha|}{2}}(B_n(x,t) + (1+t)^{-\frac{n}{2}+1}A_n(x,t))$$

then

$$\left|D_x^\alpha \int_0^t S(t-s) * H(s)ds\right| \le C(1+t)^{-\frac{n+|\alpha|}{2}}(B_{\frac{n}{2}}(x,t) + (1+t)^{-\frac{n}{4}+\frac{1}{2}}A_n(x,t))$$

Since its proof is very technical, we leave it to Section 4.

Now we continue to study R_1^α in (3.3).

Theorem 3.1 *Suppose u_0 has compact support, t is large enough, we have*

$$|R_1^\alpha| \le CE(1+t)^{-\frac{n+|\alpha|}{2}}(B_N(x,t) + (1+t)^{-\frac{n}{4}+\frac{1}{2}}A_N(x,t))$$

where $|\alpha| \le l+s$.

Proof $|R_1^\alpha| = |D_x^\alpha(G - \chi_3(D)K_{|\alpha|}) * u_0 + D_x^\alpha \chi_3(D)K_{|\alpha|} * u_0|$.

From Lemma 3.1 and Theorem 2.1, we have

$$\begin{aligned}&|D_x^\alpha(G - \chi_3(D)K_{|\alpha|}) * u_0| \\ &\le CE(1+t)^{-\frac{n+|\alpha|}{2}}(B_N(x,t) + (1+t)^{-\frac{n}{4}+\frac{1}{2}}A_N(x,t))\end{aligned}$$

From the definition of $K_{|\alpha|}$, we get

$$|D_x^\alpha \chi_3(D)K_{|\alpha|} * u_0| \le C(1+|x|)^{-2N}e^{-\frac{bt}{2}} \le C(1+t)^{-(n+|\alpha|)/2}B_N(x,t)$$

Thus

$$R_1^\alpha \le CE(1+t)^{-(n+|\alpha|)/2}(B_N(x,t) + (1+t)^{-\frac{n}{4}+\frac{1}{2}}A_N(x,t))$$

Next we consider R_2^α in (3.3). We write R_2^α as follows

$$\begin{aligned}R_2^\alpha =& \int_0^t \chi_3(D)K_{|\alpha|}(t-s) * D_x^\alpha Q(u(s))ds \\ &+ \int_0^t (G - \chi_3(D)K_{|\alpha|})(t-s) * D_x^\alpha Q(u(s))ds \\ =& R_{2,1}^\alpha + R_{2,2}^\alpha\end{aligned}$$

Set $\varphi_\alpha(x,t) = (1+t)^{n/2+\nu(|\alpha|)}(B_{\frac{n}{2}}(x,t) + (1+t)^{-\frac{n}{4}+\frac{1}{2}}A_n(x,t))^{-1}$.

$$M(t) = \sup_{0\le\tau\le t, |\alpha|\le l-3} \max |D_x^\alpha u(x,\tau)| \varphi_\alpha(x,\tau)$$

where

$$\nu(k) = \frac{1}{2}\begin{cases} k, & k \le l-3 \\ 0, & k > l-3 \end{cases}$$

Theorem 3.2 *If* $|\alpha| \le l-3, l \le n+3$, *then*

$$\begin{aligned}|R_{2,1}^\alpha| \le & C(M^{|\alpha|+3} + M^2 + EM) \\ & \times (1+t)^{-\frac{n}{2}-\nu(|\alpha|)}(B_{\frac{n}{2}}(x,t) + (1+t)^{-\frac{n}{4}+\frac{1}{2}}A_n(x,t))\end{aligned} \tag{3.4}$$

where $CE < 1$.

Proof From the definition of $K_{|\alpha|}$, we know that

$$|R_{2,1}^\alpha| \le C\sum_{j=1}^4 \sum_{i=0}^{n+|\alpha|} \int_0^t \bar{L}_{ji} * D^\alpha D_{x_l}\left(\sum_l r_l + \sum_k D_{x_k} r_{k,l}\right) ds$$

By Lemma 3.2, we have

$$\begin{aligned}|R_{2,1}^\alpha| \le & C\sum_{i=0}^{n+|\alpha|} \int_0^t \left(f_1(x) + \sum_{|\gamma|\le 1} D_x^\gamma f_{2,\gamma}(x) + C_0\delta(x)\right) \\ & * \left(\sum_l D_x^\alpha D_{x_l}\left(r_l + \sum_k D_{x_k} r_{k,l}\right)\right) e^{-b(t-s)}\end{aligned}$$

because

$$\left|D_x^\gamma D_x^\alpha D_{x_l}\left(r_l + \sum_k D_{x_k} r_{k,l}\right)\right| \le C \sum_{|\alpha_j|\le|\alpha|+3} |D^{\alpha_1}u||D^{\alpha_2}u| \prod_{j\ge 3} |D^{\alpha_j}u| \tag{3.5}$$

From the definition of M and Prop. 3.1, we get

$$\begin{aligned}&\left|D_x^\gamma D_x^\alpha D_{x_l}\left(r_l + \sum_k D_{x_k} r_{k,l}\right)\right|(x,t) \\ & \le C(M^{|\alpha|+3}(t) + M^2(t))(1+t)^{-n-\nu(|\alpha|)}(B_n(x,t) + (1+t)^{-\frac{n}{2}+1}A_n(x,t)), \\ & \qquad \text{if } |\alpha| \le l-6\end{aligned}$$

$$\begin{aligned}&\left|D_x^\gamma D_x^\alpha D_{x_l}\left(r_l + \sum_k D_{x_k} r_{k,l}\right)\right|(x,t) \\ & \le C(M^{|\alpha|+3}(t) + M^2(t))(1+t)^{-n}(B_n(x,t) + (1+t)^{-\frac{n}{2}+1}A_n(x,t)),\end{aligned}$$

$$\text{if } l-5 \le |\alpha| \le l-3$$

$$|D_x^\gamma D_x^\alpha D_{x_l}\left(r_l + \sum_k D_{x_k} r_{k,l}\right)|(x,t)$$
$$\le C(M^{|\alpha|+3}(t) + M(t))(1+t)^{-\frac{n}{2}}(B_{\frac{n}{2}}(x,t) + (1+t)^{-\frac{n}{4}+\frac{1}{2}}A_n(x,t))A,$$
$$\text{if } l \ge |\alpha| \ge l-2$$

where $\|A\|_{L_\infty} \le CE$. By the same method, we get a similar estimate for

$$D_x^\alpha D_{x_l}\left(r_l + \sum_k D_{x_k} r_{k,l}\right)(x,t)$$

Because $e^{-bt}D_x^\alpha f_1(x) \le Ce^{-bt}B_N(x,t)$, by similar method as that of the proof of Lemma 3.3 and $l \le n+3$, we get

$$\int_0^t e^{-b(t-s)} f_1(x) * \left(\sum_l D_x^\alpha D_{x_l}\left(r_l + \sum_k D_{x_k} r_{k,l}\right)\right) dt$$
$$\le C(M^{|\alpha|+3} + M^2 + EM)(1+t)^{-\frac{n}{2}-\nu(|\alpha|)}(B_{\frac{n}{2}}(x,t) + (1+t)^{-\frac{n}{4}+\frac{1}{2}}A_n(x,t))$$

$$D_x^\gamma f_{2,\gamma}(x) * D_x^\alpha D_{x_l}\left(r_l + \sum_k D_{x_k} r_{k,l}\right)$$
$$\le C\|f_{2,\gamma}\|_{L_1}\left\|D_x^\gamma D_x^\alpha D_{x_l}\left(r_l + \sum_k D_{x_k} r_{k,l}\right)(y)\right\|_{L_\infty(|x-y|<2\varepsilon_0)}$$
$$\le C(M^{|\alpha|+3} + M^2 + EM)(1+t)^{-\frac{n}{2}-\nu(|\alpha|)}(B_{\frac{n}{2}}(x,t) + (1+t)^{-\frac{n}{4}+\frac{1}{2}}A_n(x,t))$$

$$C_0\delta(x) * D_x^\alpha D_{x_l}\left(r_l + \sum_k D_{x_k} r_{k,l}\right) = C_0 D_x^\alpha D_{x_l}\left(r_l + \sum_k D_{x_k} r_{k,l}\right)(x)$$
$$\le C(M^{|\alpha|+3} + M^2 + EM)(1+t)^{-\frac{n}{2}-\nu(|\alpha|)}(B_{\frac{n}{2}}(x,t) + (1+t)^{-\frac{n}{4}+\frac{1}{2}}A_n(x,t))$$

Thus (3.4) is proved.

Theorem 3.3 *If* $|\alpha| \le l-3$

$$|R_{2,2}^\alpha| \le C(M^2 + M^{|\alpha|})(1+t)^{-\frac{n}{2}-\nu(|\alpha|)}(B_{\frac{n}{2}}(x,t) + C(1+t)^{-\frac{n}{4}+\frac{1}{2}}A_n(x,t)) \quad (3.6)$$

Proof If $|\alpha| \le l-6$,

$$R_{2,2}^\alpha = \int_0^t \left(\sum_{j=1}^n D^\alpha(D_{x_j}\left(G - \chi_3 K_{|\alpha|}\right)(t-s) * r_j(s)\right)$$
$$+ \sum_{i=1}^n D^\alpha(D_{x_j} D_{x_i}(G - \chi_3 K_{|\alpha|})(t-s) * r_{j,i}(s)))ds$$

By Theorem 2.1, we have

$$|D_x^\alpha D_{x_j}(G-\chi_3 K_{|\alpha|})| \le C(1+t)^{-\frac{n+|\alpha|+1}{2}}(B_{\frac{n}{2}}(x,t)+(1+t)^{-\frac{n}{4}+\frac{1}{2}}A_n(x,t))$$

From $r_j = O(|u|^2)$, we have

$$\begin{aligned}|D_x^\alpha r_j| &\le C \sum_{\sum|\alpha_j|\le|\alpha|} |D^{\alpha_1}u||D^{\alpha_2}u|\prod_{j\ge 3}|D^{\alpha_j}u| \\ &\le C(M^2+M^{|\alpha|})(1+t)^{-n-\frac{|\alpha|}{2}}(B_n(x,t)+(1+t)^{-\frac{n}{2}+1}A_n(x,t))\end{aligned}$$

For $D_{x_i}D_{x_j}(G-\chi_3K_{|\alpha|})$ and $r_{j,i}$, we have similar estimates. Using Lemma 3.3, we get (3.6).

If $l-5\le|\alpha|\le l-3$, we can rewrite

$$\begin{aligned}R_{2,2}^\alpha = \int_0^t \Big(&\sum_{j=1}^n D^\beta D_{x_j}\left(G-\chi_3K_{|\alpha|}\right)(t-s)*(D^{\alpha_1}r_j(s)) \\ &+\sum_{i=1}^n D^\beta D_{x_j}D_{x_i}(G-\chi_3K_{|\alpha|})(t-s)*(D^{\alpha_1}r_{j,i}(s))\Big)ds\end{aligned}$$

Here $|\alpha_1| = l-6$ and $|\beta| = |\alpha|-|\alpha_1|$. In Lemma 3.3, we replace H by $D^\beta D_{x_j}(G-F_{|\alpha|})$ or $D^\beta D_{x_j}D_{x_i}(G-F_{|\alpha|})$, S by $D^{\alpha_1}r_j$ or $D^{\alpha_1}r_{j,i}$. We can also get (3.9) for $l-5\le|\alpha|\le l-3$. Combining the three theorems above, we get

$$\begin{aligned}&|D_x^\alpha u(x,t)| \\ &\le C(E+M^2+M^{|\alpha|+3}+M^2)(1+t)^{-\frac{n+\nu(|\alpha|)}{2}}(B_{\frac{n}{2}}(x,t)+(1+t)^{-\frac{n}{4}+\frac{1}{2}}A_n(x,t))\end{aligned}$$

Then $D_x^\alpha u(x,t)\varphi_\alpha(x,t)\le C(E+M(t)^2+M^{|\alpha|+3}(t)+EM(t))$. Thus we have

$$M(t)\le C(E+M^2(t)+M^{|\alpha|+3}(t)+EM(t))$$

Because E is small enough, using continuity of $M(t)$ and induction, we obtain $M(t)\le CE$.

Thus we obtain the main result in this paper.

Theorem 3.4 *Suppose that $u_0\in H^{s+l}(R^n)$ has compact support, $s=n/2+1, 3\le l\le n+3$ with $E=\max\{\|u_0\|_{H^{s+l}}, \|u_0\|_{W^{1,l}}\}$ small enough, and $|\alpha|\le l-3$, then the solution u of (1.1) satisfies*

$$|D_x^\alpha u(x,t)|\le C(1+t)^{-\frac{n}{2}-\nu(|\alpha|)}(B_{\frac{n}{2}}(x,t)+(1+t)^{-\frac{n}{4}+\frac{1}{2}}A_n(x,t))$$

Remark The solution of the nonlinear system (1.1) has the decay factor $B_{\frac{n}{2}}$ and A_n depending on the space dimension n, while the Green function has the decay factor B_N and A_N of arbitrary order $N>0$. In other words, due to the effect of nonlinearity, the solution exhibits a much weaker form of Huygen's principle.

4. The Proof of Lemma 3.3

First we give some Lemmas.

Lemma 4.1 *Let $a(t,s,x), b(t,x,s) \geq 0$, we have*

$$\int_{R^n} \left(1+\frac{(|y|-a)^2}{1+b}\right)^{-n} dy \leq C((1+b)^{\frac{n}{2}} + (1+b)^{\frac{1}{2}}|a|^{n-1}) \tag{4.1}$$

The Proof can be found in [4].

Lemma 4.2 *If $ct \leq |x| \leq ct+\sqrt{t}$, then $A_n(x,t) > Ct^{-\frac{1}{2}}$.*

Proof Set $|x| = kct$, then $1 < k < 1+\frac{1}{c\sqrt{t}}$.

$$\begin{aligned} I &\geq C\int_0^1 \frac{1}{(1+c^2t(k-r)^2)^n}dr = \int_{c\sqrt{t}(k-1)}^{c\sqrt{t}k} \frac{1}{(1+r^2)^n c\sqrt{t}}dr \\ &\geq \int_C^{C+1} \frac{1}{(1+r^2)^n c\sqrt{t}}dr \geq Ct^{-\frac{1}{2}} \end{aligned}$$

Lemma 4.3 *If $|x| \leq ct$, then $A_n(x,t) \geq Ct^{-\frac{1}{2}}$.*

Proof Set $|x| = kct$, then $0 \leq k \leq 1$.

$$\begin{aligned} I &\geq \int_{c\sqrt{t}(k-1)}^{c\sqrt{t}k} \frac{1}{(1+r^2)^n c\sqrt{t}}dr \\ &= \int_0^{c\sqrt{t}k} \frac{1}{(1+r^2)^n c\sqrt{t}}dr + \int_0^{c\sqrt{t}(1-k)} \frac{1}{(1+r^2)^n c\sqrt{t}}dr \\ &\geq \int_0^{\frac{1}{2}} \frac{1}{(1+r^2)^n c\sqrt{t}}dr \geq Ct^{-\frac{1}{2}} \end{aligned}$$

Now we set

$$\begin{aligned} \theta_1 &= (1+t-s)^{-\frac{n+1}{2}}(1+s)^{-n}, & p_1 &= B_N(y-x,t-s)B_n(y,s) \\ \theta_2 &= (1+t-s)^{-\frac{n+1}{2}}(1+s)^{-\frac{3n}{2}+1}, & p_2 &= B_N(y-x,t-s)A_n(y,s) \\ \theta_3 &= (1+t-s)^{-\frac{3n}{4}}(1+s)^{-n}, & p_3 &= A_N(y-x,t-s)B_n(y,s) \\ \theta_4 &= (1+t-s)^{-\frac{3n}{4}}(1+s)^{-\frac{3n}{2}+1}, & p_4 &= A_N(y-x,t-s)A_n(y,s) \end{aligned}$$

By the conditions of Lemma 3.3, we know that

$$\left|D_x^\alpha \int_\Omega H(t-s)*S(s)ds\right| \leq C(1+t)^{-\frac{|\alpha|}{2}} \left|\sum_i \int_\Omega \theta_i p_i dsdy\right|$$

where $\Omega = [0,t]\times IR^n$. Evidently, Lemma 3.3 will follow from the estimate below:

$$\left|\int_\Omega \theta_i p_i dsdy\right| \leq C(1+t)^{-\frac{n}{2}}(B_{\frac{n}{2}} + (1+t)^{-\frac{n}{4}+\frac{1}{2}}A_n(x,t)).$$

Set $\Omega_1 = R^n \cap \left[\frac{t}{2}, t\right], \Omega_2 = R^n \cap \left[0, \frac{t}{2}\right]$. Now we begin from the estimate of $\prod_i = \left|\int_\Omega \theta_i p_i dsdy\right|$. We omit the estimate of $\prod_1$, because it is very similar to Case 4.1 and Case 4.2 in [4].

The estimate of $\prod_2$ is carried out in four diffrent cases.

Case 2.1 $|x|^2 \le t$,

$$\prod_2 \le \int_{\Omega_1} \theta_2 B_N(y-x, t-s)dsdy + \int_{\Omega_2} \theta_2 A_n(y,s)dsdy$$
$$\le C(1+t)^{-\frac{3n}{2}+1} + C(1+t)^{-\frac{n}{2}} \le C(1+t)^{-\frac{n}{2}} B_n(x,t)$$

Case 2.2 $ct - \sqrt{t} \le |x| \le ct + \sqrt{t}$. By Lemma 4.2 and 4.3,

$$\prod_{2,1} \le \int_{\Omega_1} \theta_2 p_2 dsdy \le C(1+t)^{-\frac{3n}{2}+\frac{3}{2}} \le C(1+t)^{-\frac{3n}{4}+\frac{1}{2}} A_n(x,t)$$

for $\prod_{2,2}$, we need a beter estimate.

If $|y| \ge \frac{|x|+csr}{2}$, we have $|y| - csr \ge \frac{|x|-csr}{2} \ge \frac{t}{8}$, and

$$\prod_{2,2} \le \int_0^{\frac{t}{2}} \theta_2 \left(\frac{1+s}{1+t}\right)^n (1+t-s)^{n/2} ds \le C(1+t)^{-\frac{3n}{4}+\frac{1}{2}} A_n(x,t)$$

If $|y| \le \frac{|x|+csr}{2}$, we have $|y-x| \ge |x| - |y| \ge \frac{|x|-csr}{2} \ge \frac{t}{8}$.

$$\prod_{2,2} \le \int_0^{\frac{t}{2}} \theta_2 (1+t)^{-n} (1+s)^{n-\frac{1}{2}} ds \le C(1+t)^{-\frac{3n}{4}+\frac{1}{2}} A_n(x,t)$$

Case 2.3 $\sqrt{t} \le |x| \le ct - \sqrt{t}$.

If $s \ge \frac{t}{2}$,

$$\prod_2 \le C(1+t)^{-\frac{3n}{2}+\frac{3}{2}} \le C(1+t)^{-\frac{3n}{4}+\frac{1}{2}} A_n(x,t)$$

If $\frac{|x|}{2cr} \le s < \frac{t}{2}$,

$$\prod_2 \le C(1+t)^{-\frac{n+1}{2}} \int_{\frac{|x|}{2cr}}^{\frac{t}{2}} \left(1+\frac{|x|}{2cr}\right)^{-\frac{3n}{2}+1} \frac{1}{\sqrt{1-r^2}} \frac{\sqrt{1+t}}{cr} \frac{dydr}{\left(1+\frac{|y-x|^2}{1+t}\right)^N}$$
$$\le C(1+|x|)^{-\frac{3n}{2}+1}$$
$$\le C(1+|x|)^{-n} \le C(1+t)^{-\frac{n}{2}} B_{\frac{n}{2}}(x,t)$$

If $s \le \frac{|x|}{2cr}$

$$\left(1+\frac{|y-x|^2}{1+t-s}\right)^{-N} \le C\left(1+\frac{|x|^2}{1+t-s}\right)^{-N}, \quad \text{if } |y-x| \ge \frac{|x|}{4},$$

$$\left(1+\frac{(|y|-csr)^2}{1+s}\right)^{-n} \le C\left(1+\frac{|x|^2}{1+s}\right)^{-n}, \quad \text{if } |y-x| \le \frac{|x|}{4},$$

$$\begin{aligned}\prod_2 &\le \int_0^{\frac{|x|}{2cr}} \theta_2(1+s)^{n-\frac{1}{2}} B_N(x,t)ds + \int_0^{\frac{|x|}{2cr}} \theta_2 B_n(x,t)\left(\frac{1+s}{1+t}\right)^n (1+t-s)^{\frac{n}{2}} ds \\ &\le C(1+t)^{-\frac{n}{2}} B_{\frac{n}{2}}(x,t)\end{aligned}$$

Case 2.4 $|x| \ge ct+\sqrt{t}$.

Then $|x|-ctr \ge \sqrt{t} \ge 0$.

$$p_2 \le \begin{cases} B_N(y-x,t-s)A_n(x,s), & \text{if } |y| \ge \frac{|x|+csr}{2} \\ \left(1+\frac{(|x|-csr)^2}{1+t-s}\right)^{-N} A_n(y,s), & \text{if } |y| \le \frac{|x|+csr}{2}\end{cases}$$

$$(|x|-csr)^2 \ge \begin{cases} C|x|^2, & s \le \frac{t}{2} \\ C(|x|-ctr)^2, & s \ge \frac{t}{2}\end{cases}$$

$$\begin{aligned}\prod_2 \le& C\int_{\frac{t}{2}}^t \theta_2(1+t-s)^{\frac{n}{2}} A_n(x,t)ds + \int_{\frac{t}{2}}^t \theta_2 A_N(x,t)\left(\frac{1+t-s}{1+t}\right)^N (1+s)^{n-\frac{1}{2}} ds \\ &+ \int_0^{\frac{t}{2}} \theta_2 B_n(x,t)(1+s)^{n-\frac{1}{2}} ds + \int_0^{\frac{t}{2}} \theta_2 (1+t-s)^{\frac{n}{2}} \left(\frac{1+s}{1+t}\right)^n B_n(x,t) \\ \le& C(1+t)^{-\frac{3n}{4}+\frac{1}{2}} A_n(x,t) + C(1+t)^{-\frac{n}{2}} B_n(x,t)\end{aligned}$$

Now we estimate $\prod_3$.

Case 3.1 $|x|^2 \le t$,

$$\begin{aligned}\prod_3 &\le C\int_0^{\frac{t}{2}} \theta_3(1+s)^{\frac{n}{2}} ds + C\int_{\frac{t}{2}}^t \theta_3(1+t-s)^{n-\frac{1}{2}} ds \\ &\le C(1+t)^{-\frac{n}{2}} \le C(1+t)^{-\frac{n}{2}} B_n(x,t)\end{aligned}$$

Case 3.2 $(|x|-ctr)^2 \le t$,

$$\begin{aligned}\prod_3 &\le C\int_{\frac{t}{2}}^t \theta_3(1+t-s)^{n-\frac{1}{2}} ds + C\int_0^{\frac{t}{2}} \theta_3(1+s)^{\frac{n}{2}} ds \\ &\le C(1+t)^{-\frac{3n}{4}+\frac{1}{2}} \le C(1+t)^{-\frac{3n}{4}+\frac{1}{2}} A_n(x,t)\end{aligned}$$

Case 3.3 $\sqrt{t} \le |x| \le ctr - \sqrt{t}$. Set $t_1 = \frac{t}{2} - \frac{|x|}{2cr}, t_2 = t - \frac{|x|}{4cr}$.

If $s \le t_1$, when $|y| \le \frac{ctr - |x|}{4}$, we can get $-|y-x| + c(t-s)r \ge \frac{ctr-|x|}{4}$, thus we have

$$\begin{aligned}\prod_3 \le & C\int_0^{t_1} \theta_3 A_n(x,t)\left(\frac{1+s}{1+t}\right)^n (t-s)^{n-\frac{1}{2}}ds \\ & + C\int_0^{t_1}\theta_3 A_N(x,t)(1+s)^{\frac{n}{2}}ds \\ \le & C(1+t)^{-\frac{3n}{4}+\frac{1}{2}}A_n(x,t)\end{aligned}$$

If $s \ge t_2$, when $|y| \le \frac{|x|}{2}$, we can get $|y-x| - c(t-s)r \ge \frac{|x|}{4}$, thus we have

$$\begin{aligned}\prod_3 \le & C\int_{t_2}^{t} \theta_3 B_N(x,t)\left(\frac{t-s}{1+t}\right)^N (1+s)^{\frac{n}{2}}ds \\ & + C\int_{t_2}^{t}\theta_3 B_n(x,t)(t-s)^{n-\frac{1}{2}}ds \\ \le & C(1+t)^{-\frac{n}{2}}B_n(x,t)\end{aligned}$$

If $t_1 < s < t_2$, we have

$$\theta_3(t,s) \le \begin{cases} C(1+(ctr-|x|))^{-n}(1+t)^{-\frac{3n}{4}}, & 0 \le s \le \frac{t}{2} \\ C(1+t)^{-\frac{3n}{4}}(1+|x|)^{-n}, & \frac{t}{2} \le s \le t \end{cases}$$

and

$$\int_{\Omega\cap(t_1<s<t_2)} P_3 dsdy \le C\sqrt{1+t}\int_{\boldsymbol{R}^n}\left(1+\frac{|y|^2}{1+t}\right)^{-n}dy \le C(1+t)^{\frac{n+1}{2}}$$

Thus $\prod_3 \le C(1+t)^{-\frac{n}{2}}B_{\frac{n}{2}} + C(1+t)^{-\frac{3n}{4}+\frac{1}{2}}A_n(x,t)$.

The estimate of $\prod_4$ is as follows.

We write p_4 as

$$\begin{aligned}p_4 = & \int_0^1\int_0^{r_1}\frac{1}{\sqrt{1-r_1^2}}\frac{1}{\sqrt{1-r_2^2}}\frac{1}{\left(1+\frac{(|y-x|-c(t-s)r_1)^2}{1+t-s}\right)^N}\frac{1}{\left(1+\frac{(|y|-csr_2)^2}{1+s}\right)^n}dr_2dr_1 \\ & + \int_0^1\int_0^{r_2}\frac{1}{\sqrt{1-r_1^2}}\frac{1}{\sqrt{1-r_2^2}}\frac{1}{\left(1+\frac{(|y-x|-c(t-s)r_1)^2}{1+t-s}\right)^N}\frac{1}{\left(1+\frac{(|y|-csr_2)^2}{1+s}\right)^n}dr_1dr_2 \\ = & p_{4,1} + p_{4,2}\end{aligned}$$

and $\prod_4 = \int_\Omega \theta_4 p_{4,1}dsdy + \int_\Omega \theta_4 p_{4,2}dsdy$.

Case 4.1 $|x| \le ct + \sqrt{t}$. Because of Lemma 4.2, 4.3,

$$\prod_4 \le \int_{\Omega_1} \theta_4 A_N(y-x, t-s)dyds + \int_{\Omega_2} \theta_4 A_n(y,s)dyds$$
$$\le C(1+t)^{-\frac{3n}{4}} \le C(1+t)^{-\frac{3n}{4}+\frac{1}{2}} A_n(x,t)$$

Case 4.2 $|x| \ge ct + \sqrt{t}$.

When $|y| - csr_1 \ge \frac{|x| - ctr_1}{2}$, we can get $|y| - csr_2 \ge |y| - csr_1 \ge \frac{|x| - ctr_1}{2} \ge 0$.

When $|y| - csr_1 \le \frac{|x| - ctr_1}{2}$, we can get $|y - x| - c(t-s)r_1 \ge \frac{|x| - ctr_1}{2} \ge 0$. Thus

$$\begin{aligned}\int_\Omega \theta_4 p_{4,1} dsdy = &\int_{\frac{t}{2}}^t \theta_4 \int_0^1 \int_0^{r_1} \frac{1}{\sqrt{1-r_1^2}} \frac{1}{\sqrt{1-r_2^2}} \left(1 + \frac{(|x| - ctr_1)^2}{1+s}\right)^{-n} \\ &\cdot (1+t-s)^{n-\frac{1}{2}} dr_2 dr_1 ds \\ &+ \int_{\frac{t}{2}}^t \theta_4 \int_0^1 \int_0^{r_1} \frac{1}{\sqrt{1-r_1^2}} \frac{1}{\sqrt{1-r_2^2}} \left(1 + \frac{(|x| - ctr_1)^2}{1+t-s}\right)^{-N} \\ &\cdot (1+s)^{n-\frac{1}{2}} \left(\frac{t-s}{1+t}\right)^N dr_2 dr_1 ds \\ &+ \int_0^{\frac{t}{2}} \theta_4 \int_0^1 \int_0^{r_1} \frac{1}{\sqrt{1-r_1^2}} \frac{1}{\sqrt{1-r_2^2}} \left(1 + \frac{(|x| - ctr_1)^2}{1+t}\right)^{-n} \\ &\cdot \left(\frac{1+s}{1+t}\right)^n (1+t-s)^{n-\frac{1}{2}} dr_2 dr_1 ds \\ &+ \int_0^{\frac{t}{2}} \theta_4 \int_0^1 \int_0^{r_1} \frac{1}{\sqrt{1-r_1^2}} \frac{1}{\sqrt{1-r_2^2}} \left(1 + \frac{(|x| - ctr_1)^2}{1+t}\right)^{-N} \\ &\cdot (1+s)^{n-\frac{1}{2}} dr_2 dr_1 ds \\ \le &C(1+t)^{-\frac{3n}{4}+\frac{1}{2}} A_n(x,t)\end{aligned}$$

For $\int \theta_4 p_{4,2} dsdy$, the method of proof is similar. Thus Lemma 3.3 is proved.

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