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THE BOUNDARY REGULARITY OF PSEUDO-HOLOMORPHIC DISKS*

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Abstract In this paper we will prove the continuity, the C^k-regularity after deforming suitably the domain, and the Hölder continuity, of the weakly pseudo-holomorphic disk with its boundary in a singular totally-real subvariety with only corners as its singularities.

Key Words Pseudo-holomorphic disk; totally-real submanifold; continuity; Hölder continuity; corner.
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1. Introduction

Since M. Gromov introduced pseudo-holomorphic curves into the symplectic geometry in 1985 [1], the application of pseudo-holomorphic curves to the symplectic geometry has become a main tool in the study of symplectic manifolds and achieved great success. Pseudo-holomorphic disks play a great role in the study of lagrangian submanifolds in a symplectic manifold [2–4].

The regularity of pseudo-holomorphic curves at the interior points was established by several authors [1,5,6] and the boundary regularity was established by Ye[5] with a slightly different setting and by Sikorav[6] who assumed the continuity of those curves in the smooth boundary case. M. Gromov suggested to deal with the regularity and the gradient estimates at the boundary points by making a reflection across the boundary and reducing this problem to the interior point case. The reflection argument indeed was carried out by Pansu[7] under the assumptions which require that the boundary manifold be real analytic and the almost complex structure be integrable near the boundary manifold.

In many applications one needs the corresponding regularity results for the pseudo-holomorphic disks with their boundaries in a totally-real subvariety with corners, for examples, such as in defining Floer homology for the Lagrangian intersections and

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in defining the invariants of Gromov-Witten type [8,9] in above situations, in both cases one needs the compactness of the moduli spaces of pseudo-holomorphic disks, so one has to deal with the bubblings of a family of weakly pseudo-holomorphic disks at the corners. We cannot expect C^1-regularity at the corners because the pseudo-holomorphic maps are conformal, but we can change the domain suitably and obtain the C^k-regularity for any positive integer k and discuss the bubblings at the corners under the C^k topology.

Let (M, J) be a closed oriented smooth almost complex manifold of dimension $2n$ with a smooth almost complex structure J. On M a Riemannian metric $<,>$ is assumed and J is compatible with this metric in the sense that J is an isometry. Let L be a totally-real submanifold which may have corner points as its singular points. Let D^2 denote the open unit disk in the complex plane with the standard complex structure.

First we give a formulation of the weakly pseudo-holomorphic disks with the natural boundary conditions. Let $u \in W^{1,2}(D^2, \partial D^2; M, L)$ be smooth away from the singularities, and $X \in W^{1,2}(D^2, \partial D^2; u^*(TM), u^*(TL)) \cap L^\infty$ be a vector field along u with its boundary values tangent to L at the smooth points. It is easy to see that $< \partial u/\partial \nu, X >= 0$, where ν denotes the inward unit vector normal to ∂D^2 and X a vector tangent to L at the smooth point of L. So by differentiating the Cauchy-Riemann equations, we have

$$\frac{\partial^2 u}{\partial s^2} + \frac{\partial^2 u}{\partial t^2} + \nabla_s J(u)\frac{\partial u}{\partial t} - \nabla_t J(u)\frac{\partial u}{\partial s} = 0$$

Integrating this equality by parts gives ii) in following definition.

Definition 1 *We call map $u \in W^{1,2}(D^2, \partial D^2; M, L)$ a weakly pseudo-holomorphic disk with its boundary in L if*

i)
$$\bar{\partial}_J u = \frac{\partial u}{\partial s} + J(u)\frac{\partial u}{\partial t} = 0 \tag{1}$$

a.e. on D^2;

ii)
$$\iint_{D^2} \langle \frac{\partial u}{\partial s}, \frac{\partial \phi}{\partial s}\rangle + \langle \frac{\partial u}{\partial t}, \frac{\partial \phi}{\partial t}\rangle + \iint_{D^2} \langle \nabla_s J(u)\frac{\partial u}{\partial t}, \phi\rangle - \langle \nabla_t J(u)\frac{\partial u}{\partial s}, \phi\rangle = 0 \tag{2}$$

for any $W^{1,2} \cap L^\infty$-vector field ϕ along u with its boundary values tangential to L a.e.. Here that $u(\partial D^2)$ is included in L means the L^2-trace of u is included in L, where (s, t) denotes the coordinate variable on D^2.

Readers may compare the definition of the weakly pseudo-holomorphic disk in this paper with Ye's definition of weakly pseudo-holomorphic disks, especially the normal boundary conditions in Ye [5].

Now we give the exact meaning of the angles at the corner points.

Definition 2 *Let M be a smooth almost complex manifold with a smooth almost complex structure $J(\cdot)$ and L be a totally-real subvariety with respect to $J(\cdot)$. The angle at $p \in L$ is said to be α if the followings hold:*

1) *$L-\{p\}$ near p consists of two smooth totally-real submanifolds L_1, L_2 of M which can be extended smoothly across p to be totally-real submanifolds near p;*

2) *there exits a coordinate system (D^{2n}, Ψ) near p of M so that*

$$\Psi(p) = 0, \Psi(L_1) = \{(x_1, \cdots, x_n) : 0 < x_d < \delta, d = 1, \cdots, n\} \subset R^n \subset C^n$$
$$\Psi(L_2) = \{(x_1 + ix_1\tan\alpha, \cdots, x_n + ix_n\tan\alpha) : (x_1, \cdots, x_n) \in \Psi(L_1)\}$$

for some $\delta > 0$.

3) $(\Psi^{-1})^* J(0) = i$.

We call such a coordinate system a canonical one.

Remark In this definition, it is restrictive for L at the corner points to form an angle α, however, since our aim is to apply the results in this paper to symplectic manifolds, it should be reasonable to require the angle at the corner points as given in above definition because we can deform L to form the angle as required near the corner points by using diffeomorphisms on M, then choose the tamed almost complex structure.

Now our main theorems are

Theorem 1 *The weakly pseudo-holomorphic disks defined above are continuous on $\overline{D^2}$.*

Theorem 2 *Let u be a weakly pseudo-holomorphic disk in (M, L), and $x \in \partial D^2$. If $u(x)$ is a singular point of L, and the angle at $u(x)$ of L is $\frac{m\pi}{l}$, here m, l are two coprime positive integers. Then under suitable coordinate charts $(D^+_{\delta^l}(0), \phi), (D^{2n}, \Phi)$ around x and $u(x)$ the map $\tilde{u}(z) = u(z^l)$ is smooth on $\overline{D^+_{1,l;\delta}(0)}$ for small $\delta > 0$, where D^{2n} denotes the open unit ball in R^{2n} and $D^+_\tau(0) = \{z \in C : |z| < \tau, \mathrm{Im}(z) \geq 0$ for a real number $\tau > 0, z \in \overline{D^+_{1,l;\delta}(0)} = \{z = re^{i\theta} \in C : 0 \leq r \leq \delta$ and $0 \leq \theta \leq \frac{\pi}{l}\}$.*

As a corollary of Theorem 2, we have

Theorem 3 *The weakly pseudo-holomorphic disks defined above are locally Hölder continuous on $\overline{D^2}$.*

We will give some applications of these results to discussing the Gromov's compactness of the moduli space of the weakly pseudo-holomorphic disks in a furthercoming paper.

In this paper $D^+, D^+_\delta, D^+_{k,l;\delta}$ denote open subsets in the complex plane, and we take all constants $c_i > 0$.

2. The Proofs of the Main Theorems

The interior regularity results follow from the regularity theory for the standard elliptic systems. So we only discuss the boundary regularity, and for the continuity we will follow the discussion of free boundary problems, especially the Jost's argument [10]. First we assume M is closed for simplicity in the following, otherwise we assume the Riemannian metric on M has bounded geometry.

It is well-known that the continuity of a minimal surface follows the famous monotonicity lemma, so we just give the statement of this lemma and outline its proof, we refer the details to Jost's paper [10].

Lemma 1 *Let $u \in W^{1,2}(D^2, M)$ be a weakly pseudo-holomorphic disk with its boundary in L, and $q \in M$. Then there exist $\rho(q) > 0$ and constant C_1 such that*

$$\frac{\sigma(\rho_1)}{\rho_1^2} I(q, \rho_1) \leq \frac{\sigma(\rho_2)}{\rho_2^2} I(q, \rho_2)$$

for $0 < \rho_1 < \rho_2 < \rho(q)$. Where $\sigma(\rho) = (1 + C_1\rho)e^{2C_1\rho}$ and $I(q, \rho) = \iint_{D^2} \|du\|^2 \cdot \psi\left(\frac{d(u(z), q)}{\rho}\right)$, where $\psi(\cdot)$ is defined below.

Proof First, we define a cutoff function as follows:

Let $\psi : R^1 \to R^1$ be a smooth function satisfying

i) $\psi(t) = 0$, if $t > 1, \psi(t) = 1$ if $t < \frac{1}{2}$, and $\psi(t) \in [0, 1]$;

ii) $\psi'(t) \leq 0$.

Case i) If $\rho(q) = d(q, L) > 0$, then we define the test vector field as

$$v(z) = \psi\left(\frac{d(u(z), q)}{\rho}\right) \cdot \exp_{u(z)}^{-1} q$$

where $0 < \rho < \rho(q)$.

Case ii) If $q \in L$ and $\rho(q) = d(q, L-\{\text{singular points of } L\}) > 0$, then we define the test vector field as

$$v(z) = \psi\left(\frac{d(u(z), q)}{\rho}\right) \cdot \Psi(z)$$

where $\Psi(z) = \exp_{u(z)}^{-1} \pi_L(u(z)) - P_{\pi_L(u(z)),u(z)} \exp_{\pi_L(u(z)),L}^{-1} q$ and $0 < \rho < \rho(q)$. In this case $\exp_{u(z),L}^{-1}$, the inverse exp map in L with the induced metric from M is well-defined near q and $\pi_L(u(z))$ is the projection of $u(z)$ to L along the shortest geodesic near $q, P_{\pi_L(u(z)),u(z)}$ is the parallel transport from $\pi_L(u(z))$ to $u(z)$ along the shortest geodesic between these two points in M near q.

Case iii) If q is a singular point of L, as we have assumed that in small neighborhood $B(q, \rho(q))$ of q in M there are no other singular points of L for some $\rho(q) > 0$. Here $B(q, \rho)$ denotes the geodesic ball centered at q with radius ρ. We assume L is a topological manifold and L is part of the union of two smooth totally-real manifolds, say L_1 and L_2, which intersect at q with angle $\frac{m\pi}{l}$.

In this case we define the test vector field as

$$v(z) = \psi\left(\frac{d(u(z), q)}{\rho}\right) \cdot \Psi(z)$$

where

$$\Psi(z) = -\exp_{u(z)}^{-1} q + \frac{d^2(u(z), L_1)}{d^2(u(z), L_1) + d^2(u(z), L_2)} P_{\pi_2(u(z)),u(z)} [\exp_{\pi_2(u(z))}^{-1} q - \exp_{2,\pi_2(u(z))}^{-1} q]$$

$$+\frac{d^2(u(z),L_2)}{d^2(u(z),L_1)+d^2(u(z),L_2)}P_{\pi_1(u(z)),u(z)}[\exp^{-1}_{\pi_1(u(z))}q-\exp^{-1}_{1,\pi_1(u(z))}q]$$

where $\exp^{-1}_{u(z)}, \exp^{-1}_{i,\pi_i(u(z))}$ denote the inverse exponential maps of M at $u(z)$ and of L_i at $\pi_i(u(z))$ respectively, and $P_{\pi_i(u(z)),u(z)}$ denotes the parallel transport along the shortest geodesic in the geodesic balls and $\pi_i(u(z))$ denotes the projection of $u(z)$ to L_i along the shortest geodesic in the geodesic balls.

In iii) $v(z)$ is not well-defined at those points which are mapped to the singular points of L, but we can assume that $v(z)$ is well-defined almost everywhere on $B(q,\rho)$ along the boundary of the disk near $u(z)\in L_1\cap L_2$, otherwise we can define the test vector field as in the boundaryless case.

We only consider the case iii), and all other cases can be easily done similarly. We follow the argument used by Jost [10] to establish the monotonicity formula for the weakly pseudo-holomorphic disks near the corners. Once we have defined the test vector field along $u(z)$, we just need the estimates from the standard Jacobi field estimates and the conformality of the weakly pseudo-holomorphic to prove above monotonicity formula.

By the standard Jacobi field estimates in Jost [10], we have that

$$\left\|\frac{\partial\exp^{-1}_{u(s,t)}q}{\partial s}+\frac{\partial u}{\partial s}\right\|\le c_2d^2(u(s,t),q)\left\|\frac{\partial u}{\partial s}\right\| \tag{3}$$

$$\left\|\frac{\partial\exp^{-1}_{u(s,t)}q}{\partial t}+\frac{\partial u}{\partial t}\right\|\le c_2d^2(u(s,t),q)\left\|\frac{\partial u}{\partial t}\right\| \tag{4}$$

The above estimates also hold for $\exp_i^{-1}$ for $i=1,2$. And from Jost [10] we have

$$\left\|\frac{\partial P_{\prod_i(u(s,t)),u(s,t)}}{\partial s}\right\|\le c_3d(u(s,t),q)\left\|\frac{\partial u}{\partial s}\right\| \tag{5}$$

$$\left\|\frac{\partial P_{\prod_i(u(s,t)),u(s,t)}}{\partial t}\right\|\le c_3d(u(s,t),q)\left\|\frac{\partial u}{\partial t}\right\| \tag{6}$$

And from the Gauss lemma, we have

$$\frac{\partial}{\partial s}\frac{d^2(u(s,t),L_1)}{d^2(u(s,t),L_1)+d^2(u(s,t),L_1)}\le c_4\frac{1}{d(u(s,t),q)}\cdot\left\|\frac{\partial u}{\partial s}\right\| \tag{7}$$

where c_2,c_3,c_4 are constants depending only on the geometry of M defined by the given metric on M.

By (3) and (4), we have

$$\left\|\frac{\partial\exp^{-1}_{\prod_i(u(s,t))}q}{\partial s}-\frac{\partial\exp^{-1}_{i,\prod_i(u(s,t))}q}{\partial s}\right\|\le c_5d^2(u(s,t),q)\left\|\frac{\partial u}{\partial s}\right\| \tag{8}$$

From (8), we have

$$\left\|\exp^{-1}_{\prod_i(u(s,t))} q - \exp^{-1}_{i,\prod_i(u(s,t))} q\right\| \leq c_6 d^3(u(s,t),q) \tag{9}$$

And by the identity $\|\exp^{-1}_{u(s,t)} q\| = d(u(s,t),q)$, we have

$$\|v(z)\| \leq c_7 d(u(s,t),q) \tag{10}$$

By the conformality of u we have the following inequality

$$\langle \frac{\partial u}{\partial s}, \exp^{-1}_{u(s,t)} q\rangle^2 + \langle \frac{\partial u}{\partial t}, \exp^{-1}_{u(s,t)} q\rangle^2 \leq \frac{1}{2} d^2(u(s,t),q)\|du\|^2 \tag{11}$$

From (2) and (3)–(11), we have

$$\begin{aligned}\iint \psi\|du\|^2 \leq & -\frac{1}{2}\iint \|du\|^2\psi'\left(\frac{d(u(s,t),q)}{\rho}\right)\cdot\frac{1}{\rho}\cdot d(u(s,t),q)\\ & - c_9\iint \|du\|^2\psi'\left(\frac{d(u(s,t),q)}{\rho}\right)\cdot\frac{1}{\rho}\cdot d^3(u(s,t),q)\\ & + c_9\iint \|du\|^2\psi\left(\frac{d(u(s,t),q)}{\rho}\right)\cdot d(u(s,t),q)\end{aligned}$$

Recall the definiton of $I(q,\rho)$ and $I'(q,\rho) = -\frac{1}{\rho^2}\iint \|du\|^2\psi'\cdot d(u(s,t),q)$, here all constants $c_i > 0$ depend only on the geometry of M and L, hence

$$2I(\rho) - \rho I'(\rho) \leq c_9(\rho I(\rho) + \rho^2 I'(\rho)) \tag{12}$$

Now integrating (12) gives

$$\frac{\sigma(\rho_1)}{\rho_1^2}\cdot I(\rho_1) \leq \frac{\sigma(\rho_2)}{\rho_2^2}\cdot I(\rho_2) \tag{13}$$

for any $0 < \rho_1 < \rho_2 \leq \rho(q)$.

Now the proof of Theorem 1 follows from the monotonicity formula and we omit its proof and refer readers to Jost [10] since it is well-known that the continuity of a minimal surface follows from monotonicity formula.

Let $p > 2$ be an integer, let D^+ be the upper part of D^2 and ∂D^+ denote $(-1,1)$.

Lemma 2 *There exists a continuous linear operator*

$$R: L^p(D^+;C^n) \to H^{1,p}(D^+;C^n)$$

such that $\bar\partial \circ R = id$. *Here* $H^{1,p}(D^+;C^n) = \{u \in W^{1,p}(D^+;C^n) : u|_{\partial D^+} \subset R^n \subset R^n + iR^n = C^n\}$. $\bar\partial$ *denotes the standard Cauchy-Riemann operator.*

Proof First we define operator

$$(Pg)(z) = \frac{1}{2\pi i}\iint_{D^+}\left[\frac{g(\xi)}{\xi - z} - \frac{\overline{g(\xi)}}{\bar\xi - z}\right] d\xi d\bar\xi \tag{14}$$

for $g \in L^p(D^+; C^n)$ and $z \in D^+$.

Note that the second term of the right-hand side of (14) is holomorphic in the generalized sense and $\bar{\partial}(Pg)(z) = g(z)$ in $L^p(D^+; C^n)$.

We prove that the second term of (14)

$$(Q_1 g)(z) = \frac{1}{2\pi i} \iint_{D+} \frac{\overline{g(\xi)}}{\bar{\xi} - z} d\xi d\bar{\xi}$$

maps $L^p(D^+; C^n)$ to $W^{1,p}(D^+; C^n), p > 2$.

i)
$$|Q_1 g(z)| \leq \frac{1}{2\pi} \left(i \iint_{D+} |g(\xi)|^p d\xi d\bar{\xi} \right)^{\frac{1}{p}} \left(i \iint_{D+} \frac{1}{|\bar{\xi} - z|^q} d\xi d\bar{\xi} \right)^{\frac{1}{q}} \tag{15}$$

where $\frac{1}{p} + \frac{1}{q} = 1, p > 0, q > 0$ integers.

Because $q < 2$, the second factor in the right-hand side of (15) is bounded above by $\left(\frac{p2\pi}{(p-2)q} \right)^{\frac{1}{q}}$.

Hence $Q_1 g$ is in $L^p(D^+; C^n)$.

ii)
$$\partial_z (Q_1 g)(z) = -\frac{1}{2\pi i} \iint_{D+} \frac{\overline{g(\xi)}}{(\bar{\xi} - z)^2} d\xi d\bar{\xi}$$

We estimate

$$\left(\iint_{D+} \frac{1}{|\bar{\xi} - z|^{2q}} d\xi d\bar{\xi} \right)^{\frac{1}{q}}$$

for $1 < q < 2, z \in D^+$.

Now let G_1, G_0 denote the half-disks centered at Rez of radius $\frac{1}{2}$Imz and 2ρ with $\rho \geq 1$ respectively.

$$i \iint_{G_0 - G_1} \frac{1}{|\bar{\xi} - z|^q} d\xi d\bar{\xi} \leq i \iint_{G_0 - G_1} \frac{1}{|\bar{\xi} - \mathrm{Re}z|^{2q}} d\xi d\bar{\xi}$$
$$= \pi \int_{\frac{1}{2}\mathrm{Im}z}^{2\rho} \frac{1}{r^{2q}} \cdot r dr < \pi 2^{2q-1} (\mathrm{Im}z)^{2-2q} \tag{16}$$

And
$$-\frac{1}{\pi i} \iint_{G_1} \frac{1}{|\bar{\xi} - z|^{2q}} d\xi d\bar{\xi} \leq (\mathrm{Im}z)^{-2q_i} \iint_{G_1} \cdot d\xi d\bar{\xi}$$
$$= (\mathrm{Im}z)^{-2q} \cdot \frac{\pi (\mathrm{Im}z)^2}{2} = \frac{\pi}{2} (\mathrm{Im}z)^{2-2q} \tag{17}$$

Because $1 < q < 2$,

$$|\partial_z Q_1 g(z)| \leq \frac{i}{2\pi} \iint_{D+} |g(\xi)|^{\frac{1}{p}} d\xi d\bar{\xi} \cdot \left(i \iint_{D+} \frac{1}{|\bar{\xi} - z|^{2q}} d\xi d\bar{\xi} \right)^{\frac{1}{q}} \tag{18}$$

where $\frac{1}{p} + \frac{1}{q} = 1$. By (16), (17) and (18) we have

$$\left(i \iint_{D+} |\partial z Q_1 g(z)|^P dz d\bar{z} \right)^{\frac{1}{p}} \leq C(q) \|g\|_{L^p}$$

for some constant $C(q)$ which depends only on q.

For the proof of the first term of the right-hand side of (14) it is in $W^{1,p}(D^+;C^n)$. We refer it to Section 9 in Chapter 1 [11].

It is easy to see that $Pg|_{\partial D^+} \subset R^n$.

Now we assume the angles at the corner points are all rational in the rest of this paper.

Lemma 3 *Let u be a weakly pseudo-holomorphic disk in (M, L). Then the set*

$$S = \{z \in \bar{D}^2 : u(z) \text{ is a singular point of } L\}$$

is finite.

Proof Since it is well-known that there is no cluster point of S in the interior part of D^2 and in the smooth boundary case, this follows from the discussion of Sikorav[6], and since we have proved the continuity of the pseudo-holomorphic disks, we only deal with the corner case.

Let $z \in \partial D^2$ and $u(z)$ be a singular point of L.

Now we choose the canonical coordinate charts near z and $u(z)$ we still denote the map obtained from u under these coordinate transformations by u. Furthermore we assume that the almost complex structure $\tilde{J}(\cdot)$ is obtained from $J(\cdot)$ on M by pushing forward to D^{2n} satisfying $\tilde{J}(0) = i$, the standard complex structure on $R^{2n} = C^n$. With these understood, the corresponding Cauchy-Riemann equation can be rewritten as

$$\bar{\partial} u + q(z)\partial u = 0$$

where $q(z) = (i + \tilde{J}(u(z)))^{-1}(i - \tilde{J}(u(z)))$ and $\partial = \frac{1}{2}\left(\frac{\partial}{\partial s} - i\frac{\partial}{\partial t}\right)$. We note that $q(0) = 0$ and $q \in W^{1,2}(D^+) \cap C^0\left(\overline{D^+}\right)$ since we have proven that u is continuous on $\overline{D^2}$.

Now we define a map $A : W^{1,p}(D^+) \to L^p, p > 2$ simply by setting

$$A(z)w(z) = -q(z)\partial w$$

Then $$\|A(z)w(z)\|_{L^p(D^+_\delta)} \le \|q(z)\|_{L^\infty(D^+_\delta)} \cdot \|w\|_{W^{1,p}(D^+_\delta)} \le C_{10}\|w\|_{W^{1,p}(D^+_\delta)} \tag{19}$$

for $\delta > 0$.

Now let $p > 2$ we define

$$H^{1,p} = \{w \in W^{1,p}(D^+;C^n) : \mathrm{Im}(w|_{\partial D^+}) = 0\}$$

with the norm $\|w\|_{H^{1,p}} = \|w\|_{W^{1,p}(D^+)} + \|w(0)\|$. Then $H^{1,p}$ is a Banach space.

Define

$$\begin{aligned} \Phi = (\bar{\partial}, B) : H^{1,p} &\to L^p \times R^n \\ w &\longmapsto (\bar{\partial} w, w(0)) \end{aligned}$$

Then Φ is a bounded subjective linear map by Lemma 2, now we define a perturbation of Φ by setting

$$\tilde{\Phi} : H^{1,p} \to L^p(D^+; C^n) \times R^n$$
$$\tilde{\Phi} = (\bar{\partial} w + A(z)w, Bw).$$

Since u is continuous, by (19) $\tilde{\Phi}|_{D_\delta^+}$ is onto when $\delta > 0$ is small enough.

Hence we have n complex vectors $\eta_i, i = 1, \cdots, n$, which are the solutions of the following systems

$$\begin{cases} \bar{\partial}\eta(z) + A(z)\eta(z) = 0 \\ \mathrm{Im}\eta(z) = 0, \quad z \in \partial D_\delta^+ \\ \eta(0) = e_i \end{cases}$$

on D_δ^+ and where $e_i = (0, \cdots, 0, 1, 0, \cdots, 0)$ whose components are zero except the i-th one.

Let Θ be the matrix whose columns consist of η_i's. By the construction, Θ has the following properties

i) Θ is continuous on $\overline{D_\delta^+}$;

ii) Θ is an isomorphism if δ is sufficiently small;

iii) $\Theta|_{(-\delta,\delta)}$ is a real matrix;

iv) $\bar{\partial}\Theta + A(z)\Theta = 0$.

Let $u(z) = \Theta(z)\xi(z)$, then we have $\bar{\partial}\xi = 0$ and ξ satisfies the same boundary conditions as $u(z)$ does due to iii) and ξ is continuous, so ξ has only finitely many zeros on $\overline{D_\delta^+}$. Hence u maps only finitely many points on $\overline{D_\delta^+}$ to the same corner point of L. Because we assume that L has only finitely many corners, the lemma is proved.

Lemma 4 *Let $u \in W^{1,2}(D^+, \partial D^+; C^n, R^n) \cap C^0\left(\overline{D^+}\right)$ and $p > 2$. Then*

$$\|u\|_{W^{1,p}(D^+)} \le C_{11}(\|\bar{\partial}u\|_{L^p(D^+)} + \|u\|_{L^p(D^+)}) \tag{20}$$

holds for some constant C_{11} independent of u. Consequently, we have

$$\|u\|_{W^{1,p}(D^+\setminus D_\delta^+)} \le C_{12}\left(\|\bar{\partial}u\|_{L^p(D^+\setminus D_\delta^+)} + \|u\|_{L^p(D^+\setminus D_\delta^+)}\right) \tag{21}$$

holds for some constant C_{12} independent of u and $0 < \delta < 1$.

Proof We assume that $\bar{\partial}u \in L^p$ and $u \in L^p$, otherwise the inequality (20) is trivial. By Lemma 2 we have $w \in W^{1,p}(D^+, \partial D^+; C^n, R^n)$ such that

$$\bar{\partial}w = \bar{\partial}u$$

and
$$\|w\|_{W^{1,p}(D^+)} \le C_{10}\|\bar{\partial}u\|_{L^p(D^+)}$$
for some constant C_{10} independent of u.

Now $\phi = u - w$ is continuous on $\overline{D^+}$ and $\bar{\partial}\phi = 0$, so ϕ is smooth on $\overline{D^+}$ and

$$\|\phi\|_{W^{1,p}(D^+)} \le C_{11}\|\phi\|_{L^p(D^+)}$$

for some constant C_{11} independent of ϕ. Therefore

$$\|u\|_{W^{1,p}(D^+)} \le C_{11}(\|\bar\partial u\|_{L^p(D^+)} + \|u\|_{L^p(D^+)})$$

for some constant C_{11} (we still denote it by C_{11}) independent of u. The proof of (21) easily follows from (20).

Now we give the proof of Theorem 2. Since the regularity of the weakly pseudo-holomorphic disk has been established in the smooth case by Sikarov [6], we only deal with the corner case.

Let u be a weakly pseudo-holomorphic disk in (M, L) and $z_0 \in \partial D^2$ and $u(z_0)$ be a corner point of L and assume the angle at $u(z_0)$ of L is $\frac{m\pi}{l}$. By Lemma 3 we can choose a canonical coordinate charts (D^+, ϕ) and (D^{2n}, Φ) around z_0 and $u(z_0)$ respectively such that u maps no other point in $\overline{D^+}$ to the singular point of L except 0.

Let
$$\tilde u(z) = u(z^l)$$

where $z \in \overline{D^+_{1,l;\delta}} = \left\{z \in C : \|z\| \le \delta, 0 \le \operatorname{Arg}(z) \le \frac{\pi}{l}\right\}$, then $\tilde u$ is J-holomorphic on $D^+_{1,l;\delta}$. Since we can obtain the C^∞-regularity of u away from those points which are mapped to the singular points of L, we assume this fact.

First we extend $\tilde u$ by reflection inductively to $\overline{D^+_{k,l;\delta}} = \left\{z \in C : \|z\| \le \delta, \frac{(k-1)\pi}{l} \le \operatorname{Arg}(z) \le \frac{k\pi}{l}\right\}$ for $k = 1, \cdots, l$.

And we set
$$\tilde u(s,t) = \tilde u(z) = (\tilde u_1 + i\tilde u_{n+1}, \cdots, \tilde u_n + i\tilde u_{2n})$$

where $z = s + it \in \overline{D^+_{1,l;\delta}}$. Now for $s + it \in \overline{D^+_{2,l;\delta}}$ we define

$$\begin{aligned}\tilde u(s,t) =& (\tilde u_1(s',t')\cos\frac{2m\pi}{l} + \tilde u_{n+1}(s',t')\sin\frac{2m\pi}{l} + i(\tilde u_1(s',t')\sin\frac{2m\pi}{l}\\ &- \tilde u_{n+1}(s',t')\cos\frac{2m\pi}{l}\tilde u_n(s',t')\cos\frac{2m\pi}{l} + \tilde u_{2n}(s',t')\sin\frac{2m\pi}{l}\\ &+ i(\tilde u_n(s',t')\sin\frac{2m\pi}{l} - \tilde u_{2n}(s',t')\cos\frac{2m\pi}{l}))\end{aligned}$$

where $s' = s\cos\frac{2\pi}{l} + t\sin\frac{2\pi}{l}, t' = s\cos\frac{2\pi}{l} + t\sin\frac{2\pi}{l}$. It is easy to calculate that

$$\begin{aligned}\frac{\partial\tilde u}{\partial s} + i\frac{\partial\tilde u}{\partial t} =& \Big(\frac{\partial(\tilde u_1 - i\tilde u_{n+1})}{\partial s'}\cos\frac{2(m+1)\pi}{l} - i\frac{\partial(\tilde u_1 - i\tilde u_{n+1})}{\partial t'}\cos\frac{2(m+1)\pi}{l}\\ &+ i\Big(\frac{\partial(\tilde u_1 - i\tilde u_{n+1})}{\partial s'}\sin\frac{2(m+1)\pi}{l} - i\frac{\partial(\tilde u_1 - i\tilde u_{n+1})}{\partial t'}\sin\frac{2(m+1)\pi}{l}\Big),\\ &\cdots, \frac{\partial(\tilde u_n - i\tilde u_{2n})}{\partial s'}\cos\frac{2(m+1)\pi}{l} - i\frac{\partial(\tilde u_n - i\tilde u_{2n})}{\partial t'}\cos\frac{2(m+1)\pi}{l}\\ &+ i\Big(\frac{\partial(\tilde u_n - i\tilde u_{2n})}{\partial s'}\sin\frac{2(m+1)\pi}{l} - i\frac{\partial(\tilde u_n - i\tilde u_{2n})}{\partial t'}\sin\frac{2(m+1)\pi}{l}\Big)\Big)\\ =& \overline{f(s',t')}\cdot\Big(\cos\frac{2(m+1)\pi}{l} + i\sin\frac{2(m+1)\pi}{l}\Big)\end{aligned}$$

where $\bar\partial u(z) = f(z)$ for $z \in D^+_{1,l;\delta}$. Then $\tilde u$ satisfies the following system:

$$\bar\partial\tilde u(s,t) = \begin{cases} f(s,t) & (s,t)\in D^+_{1,l;\delta} \\ \overline{f(s',t')}\cdot\Big(\cos\dfrac{2(m+1)\pi}{l} + i\sin\dfrac{2(m+1)\pi}{l}\Big) & s+it\in D^+_{2,l;\delta}\end{cases}$$

Inductively, we extend $\tilde u$ and f to the maps defined on $\overline{D^+_\delta}$ respectively as above, we still denote them by $\tilde u$ and $\tilde f$. Since we have assumed the smoothness of $\tilde u$ on $\overline{D^+_{1,l;\delta}} - \{0\}$, we have

$$\bar\partial\tilde u(s,t) = \tilde f \quad\text{for}\quad s+it\in D^+_\delta$$

In particular, we have $\tilde u(s,0)\in R^n \subset R^n + iR^n = C^n$. Since $\tilde u$ is continuous on $\overline{D^+_\delta}$, by Lemma 4, we have

$$\|\tilde u\|_{W^{1,p}(D^+_\delta)} \le C_{11}(\|\tilde f\|_{L^p(D^+_\delta)} + \|\tilde u\|_{L^p(D^+_\delta)})$$

for δ small enough and some constant C_{12} independent of $\tilde u$ and $\tilde f$. By the definitions of $\tilde u$ and $\tilde f$, we have

$$\|\tilde u\|_{W^{1,p}(D^+_{1,l;\delta})} \le C_{11}(\|\tilde f\|_{L^p(D^+_{1,l;\delta})} + \|\tilde u\|_{L^p(D^+_{1,l;\delta})}) \tag{22}$$

By Lemma 5, $\tilde u\in W^{k,p}(D^+_{1,l;\delta})$ for any integer $k>0$. Hence $\tilde u$ is smooth.

Lemma 5 *Let $p>2$ be a positive integer and $u\in W^{1,2}(D^+_{1,l;\delta}; C^n)\cap C^0(\overline{D^+_{1,l;\delta}})$ satisfy the following boundary conditions at the corner*

$$u|_{[0,\delta)}\subset R^n_+ \quad\text{and}\quad u|_{(-\delta,0]}\subset R^n_+\cdot e^{i\frac{m\pi}{l}}$$

and $J(\cdot)$ is a smooth almost complex structure on C^n with $J(0)=i$ and u is smooth away from 0. Then

$$\|u\|_{W^{1,p}(D^+_{1,l;\delta_1})} \le C_{13}(\|\bar\partial_J u\|_{L^p} + \|u\|_{L^p})$$

for some $0<\delta_1<\delta$ and some constant C_{13}. where $\bar\partial_J u = \dfrac12\Big(\dfrac{\partial u}{\partial s} + J(u)\dfrac{\partial u}{\partial t}\Big)$.

Proof By (20), we have

$$\|u\|_{W^{1,p}(D^+_{1,l;\delta_1}\backslash D^+_{1,l;\delta_2})} \le C_{14}\Big(\|\bar\partial u\|_{L^p(D^+_{1,l;\delta_1}\backslash D^+_{1,l;\delta_2})} + \|u\|_{L^p(D^+_{1,l;\delta_1}\backslash D^+_{1,l;\delta_2})}\Big)$$

for $0<\delta_2<\delta_1<\delta$. Note that u can be assumed to be smooth in $D^+_{1,l;\delta_1}\backslash D^+_{1,l;\delta_2}$. So we have

$$\begin{aligned}\|u\|_{W^{1,p}(D^+_{1,l;\delta_1}\backslash D^+_{1,l;\delta_2})} \le{}& C_{14}\Big(\|\bar\partial_J u\|_{L^p(D^+_{1,l;\delta_1}\backslash D^+_{1,l;\delta_2})} + \Big\|(i-J(u))\frac{\partial u}{\partial t}\Big\|_{L^p(D^+_{1,l;\delta_1}\backslash D^+_{1,l;\delta_2})} \\ &+ \|u\|_{L^p(D^+_{1,l;\delta_1}\backslash D^+_{1,l;\delta_2})}\Big) \\ \le{}& C_{14}\Big(\|\bar\partial_J u\|_{L^p(D^+_{1,l;\delta_1}\backslash D^+_{1,l;\delta_2})} + \|(i-J(u))\|_{L^\infty(D^+_{1,l;\delta_1}\backslash D^+_{1,l;\delta_2})} \\ &\cdot\Big\|\frac{\partial u}{\partial t}\Big\|_{L^p(D^+_{1,l;\delta_1}\backslash D^+_{1,l;\delta_2})} + \|u\|_{L^p(D^+_{1,l;\delta_1}\backslash D^+_{1,l;\delta_2})}\Big).\end{aligned}$$

where C_{14} is independent of u and since u is continuous on $\overline{D^+_{1,l;\delta}}$ and $J(0)=i$, we can choose δ_1 sufficiently small so that

$$C_{14}\cdot\|i-J(z)\|_{L^\infty\left(\overline{D^+_{1,l;\delta_1}}\right)}\leq\frac{1}{2}$$

Now, let δ_2 tend to zero, so

$$\|u\|_{W^{1,p}\left(D^+_{1,l;\delta_1}\right)}\leq 2C_{14}(\|\bar\partial_J u\|_{L^p}+\|u\|_{L^p})$$

The proof of Theorem 3: Since $\tilde u(z)=u(z^l)$ is smooth by Theorem 2 and $\phi(z)=z^{\frac{1}{l}}$ is Hölder continuous on $\overline{D^+_\delta}$, Theorem 3 follows. Here the Hölder component can be precisely expressed in terms of the corresponding angles.

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