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ANISOTROPIC PARABOLIC EQUATIONS WITH MEASURE DATA*

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Abstract In this paper, we prove the existence of solutions to anisotropic parabolic equations with right hand side term in the bounded Radon measure $M(Q)$ and the initial condition in $M(\Omega)$ or in L^m space (with m "small").

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1. Introduction and Statement of Results

The existence of solutions to nonlinear elliptic equations and parabolic equations with measure data has been discussed in [1]–[4]. For the case of anisotropic elliptic equations, L.Boccardo, T.Gallouët and P.Marcellini studied it in [5]. In this paper, we will extend the analogous results of [5] for anisotropic elliptic equations to anisotropic parabolic equations and obtain the appropriate function space for solutions. We will consider the following anisotropic parabolic equations:

$$(P)\begin{cases} \dfrac{\partial u}{\partial t} - \operatorname{div}\left(a(x,t,u,Du)\right) = f & \text{in } Q \\ u = 0 & \text{on } \Sigma \\ u(x,0) = u_0 & \text{in } \Omega \end{cases}$$

Here Ω is a bounded open set in $R^N, N \geq 2$, with smooth boundary $\partial\Omega, Q$ is the cylinder $\Omega \times (0,T)$, where T is a real positive number, and Σ is the "latreal surface" $\partial\Omega \times (0,T), p_i > 1, i = 1, 2, \cdots, N$.

Let a be a Carathéodory function in $Q \times R \times R^N$. We assume there exist two real positive constants α, β and a nonnegative function $h \in L^1(Q)$, such that for every component a_i of a, almost every $(x,t) \in Q$, and for any $s \in R, \xi \in R^N, \eta \in R^N$,

$$\mathbf{a}(x,t,s,\xi)\xi \geq \alpha \sum_{i=1}^{N} |\xi_i|^{p_i} \tag{1.1}$$

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$$|a_i(x,t,s,\xi)| \le \beta\left(h(x,t) + |s|^{\bar{p}} + \sum_{j=1}^{N} |\xi_j|^{p_j}\right)^{1-\frac{1}{p_i}}, \quad i = 1,2,\cdots,N \tag{1.2}$$

where $\bar{p}$ satisfies $\frac{1}{\bar{p}} = \frac{1}{N}\sum_{i=1}^{N}\frac{1}{p_i}$.

$$[\mathbf{a}(x,t,s,\xi) - \mathbf{a}(x,t,s,\eta)][\xi - \eta] > 0, \quad \xi \ne \eta \tag{1.3}$$

In particular, if $\mathbf{a}$ doesn't depend on x, t and s, namely $\mathbf{a}(x,t,s,\xi) \equiv \mathbf{a}(\xi)$, $\mathbf{a}(\xi)$ is the vector field whose components are $a_i(\xi) = |\xi_i|^{p_i-2}\xi_i, i = 1,2,\cdots,N, p_i > 1$.

We will specify in the statement of the theorems the different hypotheses on f and u_0. The general case is when f and u_0 are the bounded Radon measures on Q and Ω respectively, we will also consider the more regular case when f and u_0 belong to some Lebesgue or Orlicz space.

Definition 1.1 We will say that u is a solution of (P) if $u \in L^1(0,T;W_0^{1,1}(\Omega))$, $a(x,t,u,Du) \in L^1(Q)$ and u satisfies the equation (P) in the following weak sense:

$$-\int_Q u\phi' dxdt + \int_Q \mathbf{a}(x,t,u,Du)D\phi dxdt = \int_Q \phi df + \int_\Omega \phi(x,0)du_0 \tag{1.4}$$

for every $\phi \in C^\infty(\bar{Q})$ which is zero in a neighborhood of $\Sigma \cup (\Omega \times \{T\})$

Set

$$W^{1,(p_i)}(\Omega) = \{u | u \in L^{p_i}(\Omega), D_i u \in L^{p_i}(\Omega)\}, \quad i = 1,2,\cdots,N \tag{1.5}$$

Define

$$\|u\|_{W^{1,(p_i)}(\Omega)} = \|D_i u\|_{L^{p_i}(\Omega)} + \|u\|_{L^{p_i}(\Omega)}, \quad \forall u \in W^{1,(p_i)}(\Omega) \tag{1.6}$$

$W^{1,(p_i)}(\Omega)$ becomes reflexive Banach space. We will denote by $W_0^{1,(p_i)}(\Omega)$ the closure of $C_0^\infty(\Omega)$ relative to the norm (1.6) in $W^{1,(p_i)}(\Omega)$. Suppose

$$2 - \frac{1}{N+1} < p_i < \frac{\bar{p}(N+1)}{N}, \quad i = 1,2,\cdots,N \tag{1.7}$$

We now state the main results of this paper.

Theorem 1.1 *Assume* (1.1)–(1.3) *and* (1.7) *hold, let* $\bar{p} \le N + \frac{N}{N+1}$,

$$f \in M(Q), \quad u_0 \in M(\Omega) \tag{1.8}$$

where $M(Q)$ and $M(\Omega)$ denote the space of bounded (finite) Radon measure on Q and Ω respectively.

Then there exists a solution u of the problem (P) such that

$$u \in \bigcap_{i=1}^{N} L^{q_i}(0,T;W_0^{1,(q_i)}(\Omega)), \quad \forall q_i \in \left[1, \frac{p_i}{\bar{p}}\left(\bar{p} - \frac{N}{N+1}\right)\right), \quad i = 1,2,\cdots,N \tag{1.9}$$

In order to obtain $q_i = \frac{p_i}{\bar{p}}\left(\bar{p} - \frac{N}{N+1}\right)$, $(i = 1, 2, \cdots, N)$ in (1.9), we have to make stronger assumptions on f and u_0. This is what is stated in the following.

Theorem 1.2 *Assume* (1.1)–(1.3) *and* (1.7) *hold, let* $\bar{p} \le N + \frac{N}{N+1}$.

$$f \in L^1(0,T; L_\gamma(\Omega)), \quad u_0 \in L_\gamma(\Omega) \tag{1.10}$$

where $L_\gamma(\Omega)$ *is the Orlicz space generated by the function* $\gamma(s) = s\log(1+s)$. *Then there exists a solution* u *of the problem* (P) *such that*

$$u \in \bigcap_{i=1}^{N} L^{q_i}(0,T; W_0^{1,(q_i)}(\Omega)), \quad q_i = \frac{p_i}{\bar{p}}\left(\bar{p} - \frac{N}{N+1}\right), \quad i = 1, 2, \cdots, N \tag{1.11}$$

Now, we will improve the regularity of f and u_0 to obtain more summability of the gradients of solutions of (P).

Theorem 1.3 *Assume* (1.1)–(1.3) *and* (1.7) *hold, let* $\bar{p} < N$ *and* ρ *satisfy*

$$1 < \rho < \bar{\rho} = \frac{N\bar{p}}{N\bar{p} - N + \bar{p}} \tag{1.12}$$

Then there exists a constant $\bar{\sigma} = \bar{\sigma}(\rho)$ *satisfying*

$$\rho < \bar{\sigma} < \frac{\bar{p}(\rho+N) - \rho N}{\bar{p}(\rho+N) - 2N} \tag{1.13}$$

such that the following holds: if

$$1 < \sigma < \bar{\sigma}, \quad \kappa = \rho + (\sigma - 1)\left(\frac{N+\rho}{N}\bar{p} - 2\right) \tag{1.14}$$

and

$$f \in L^\sigma(0,T; L^\rho(\Omega)), \quad u_0 \in L^\kappa(\Omega) \tag{1.15}$$

Then there exists a solution u *of the problem* (P) *such that*

$$u \in \bigcap_{i=1}^{N} L^{q_i}(0,T; W_0^{1,(q_i)}(\Omega)), \quad q_i = \frac{p_i}{\bar{p}}\left(\sigma\bar{p} + \frac{(\rho - 2\sigma)N}{N+\rho}\right), \quad i = 1, 2, \cdots, N \tag{1.16}$$

The content of this paper is in close relation with [1], [3], [5]. Our results extend those contained in [1] and [3]. This is obtained by means of a technique which is inspired by that used in [3] for anisotropic elliptic equations. Moreover, our theorems extend the anologous results of [5] to anisotropic parabolic equations.

This paper is organized as follows: In Section 2, *a priori* estimates for solutions to the problem (P) are given; Section 3 is devoted to the proof of Theorem 1.1–1.3.

2. *A Priori* Estimates

In this section, we state and prove *a priori* estimates for the solutions u of the problem (P). Throughout the paper, we will denote by c_n the positive constants depending only on the data of the problem, but not on u.

Lemma 2.1 *Assume (1.1)–(1.3) and (1.7) hold, let $\bar{p} \le N + \frac{N}{N+1}$,*

$$f \in L^\infty(Q), \quad u_0 \in L^\infty(\Omega) \tag{2.1}$$

Then, for every $q_i \in \left[1, \frac{p_i}{\bar{p}}\left(\bar{p} - \frac{N}{N+1}\right)\right)$, $i = 1, 2, \cdots, N$, there exists a positive constant c depending on $Q, \alpha, N, p_i, q_i, \|f\|_{L^1(Q)}, \|u_0\|_{L^1(Q)}$, such that

$$\|D_i u\|_{L^{q_i}(Q)} \le c \tag{2.2}$$

and

$$\|u\|_{L^{\bar{q}}(Q)} \le c \tag{2.3}$$

where $\bar{q}$ satisfies $\frac{1}{\bar{q}} = \frac{1}{N}\sum_{i=1}^{N}\frac{1}{q_i}$.

Proof It is simply modifies the classical J.L.Lions ([6]) method, the problem (P) has a solution $u \in \bigcap_{i=1}^{N} L^{p_i}(0,T; W_0^{1,(p_i)}(\Omega)) \bigcap C([0,T]; L^2(\Omega))$ such that $u' \in \sum_{i=1}^{N} L^{p_i'}(0,T; (W_0^{1,(p_i)}(\Omega))')$, where $\sum_{i=1}^{N} L^{p_i'}(0,T; (W_0^{1,(p_i)}(\Omega))')$ denotes the dual space of $\bigcap_{i=1}^{N} L^{p_i}(0,T; W_0^{1,(p_i)}(\Omega))$ with $p_i' = \frac{p_i}{p_i - 1}$ and

$$\int_0^T < u'(t), v(t) > dt + \int_Q \mathrm{a}(x,t,u,Du)Dv\,dxdt = \int_0^T < f(t), v(t) > dt$$
$$\forall v \in \bigcap_{i=1}^{N} L^{p_i}(0,T; W_0^{1,(p_i)}(\Omega)) \tag{2.4}$$

and

$$u(0) = u_0 \tag{2.5}$$

Define the function $\Psi_1 : R \to R$ by

$$\Psi_1(s) = \begin{cases} 1 & \text{if } s > 1 \\ s & \text{if } |s| \le 1 \\ -1 & \text{if } s < -1 \end{cases} \tag{2.6}$$

For any given $\tau \in (0,T)$, taking $v(x,t) = \Psi_1(u(x,t))\chi(0,\tau)(t)$ in (2.4) (here $\chi(0,\tau)$ is the characteristic function of the interval $(0,\tau)$), we obtain

$$\int_\Omega \psi_1(u(x,\tau))dx - \int_\Omega \psi_1(u_0(x))dx \le \int_Q |f||\Psi_1(u)|dxdt \tag{2.7}$$

where ψ_1 is a primitive of Ψ_1; that is

$$\psi_1(y) = \int_0^y \Psi_1(s)ds, \quad \forall y \in R \tag{2.8}$$

Since we have $|s| - \frac{1}{2} \le \psi_1(s) \le |s|$ for every $s \in R$, we obtain for every $\tau \in (0, T)$,

$$\int_\Omega |u(x,\tau)|dx \le \|f\|_{L^1(Q)} + \frac{1}{2} meas\Omega + \int_\Omega |u_0(x)|dx \tag{2.9}$$

This implies that there exists a positive constant c_1 such that

$$\|u\|_{L^\infty(0,T;L^1(\Omega))} \le c_1 \tag{2.10}$$

For given $\lambda > 1$, let us define the function

$$\Psi_2(s) = \int_0^s (1 + |y|)^{-\lambda} dy \tag{2.11}$$

Suppose ψ_2 is a primitive of $\Psi_2(s)$; that is

$$\psi_2(z) = \int_0^z \Psi_2(s)ds \tag{2.12}$$

For any given $\tau \in (0, T)$, taking $v(x,t) = \Psi_2(u(x,t))\chi(0,\tau)(t)$ in (2.4), we obtain

$$\int_\Omega \psi_2(u(x,\tau))dx - \int_\Omega \psi_2(u_0(x))dx + \int_0^\tau \int_\Omega \mathbf{a}(x,t,u,Du)D(\Psi_2(u))dxdt = \int_0^\tau \int_\Omega f\Psi_2(u)dxdt \tag{2.13}$$

Using (1.1), (2.11) and (2.12), we obtain

$$\sum_{i=1}^N \int_Q \frac{|D_iu|^{p_i}}{(1+|u|)^\lambda} dxdt \le \frac{1}{\alpha(\lambda - 1)} \left(\int_Q |f|dxdt + \|u_0\|_{L^1(\Omega)} \right) \tag{2.14}$$

(2.14) implies that there exists a positive constant c_2 such that

$$\sum_{i=1}^N \int_Q \frac{|D_iu|^{p_i}}{(1+|u|)^\lambda} dxdt \le c_2 \tag{2.15}$$

By the Hölder's inequality, this implies that

$$\begin{aligned} \int_Q |D_iu|^{q_i} dxdt &= \int_Q \frac{|D_iu|^{q_i}}{(1+|u|)^{\lambda q_i/p_i}} (1+|u|)^{\lambda q_i/p_i} dxdt \\ &\le \left(\int_Q \frac{|D_iu|^{p_i}}{(1+|u|)^\lambda} dxdt \right)^{q_i/p_i} \left(\int_Q (1+|u|)^{\lambda q_i/(p_i - q_i)} dxdt \right)^{1-q_i/p_i} \\ &\le c_2^{q_i/p_i} \left(\int_Q (1+|u|)^{\lambda q_i/(p_i-q_i)} dxdt \right)^{1-q_i/p_i} \end{aligned} \tag{2.16}$$

The assumption of q_i implies that $\bar{q} < N$ and $1 < \bar{q}^*$ with $\bar{q}^* = \dfrac{N_{\bar{q}}}{N-\bar{q}}$. Use the following interpolation argument

$$\|u\|_{L^r(\Omega)} \le \|u\|^{\tau}_{L^{\bar{q}^*}(\Omega)} \|u\|^{1-\tau}_{L^1(\Omega)} \tag{2.17}$$

with

$$\frac{1}{r} = \frac{\tau}{\bar{q}^*} + \frac{1-\tau}{1}, \quad 0 < \tau < 1 \tag{2.18}$$

If r satisfies $\bar{q}^*(r-1)/(\bar{q}^*-1) = \bar{q}$, then

$$r = \frac{N+1}{N}\bar{q} \tag{2.19}$$

By the nonisotropic Sobolev inequality (cf. [7]), we have

$$\|u(t)\|_{L^{\bar{q}^*}(\Omega)} \le c_3 \prod_{j=1}^{N} \|D_j u(t)\|^{\frac{1}{N}}_{L^{q_j}(\Omega)} \tag{2.20}$$

(2.10), (2.17), (2.19) and (2.20) yield

$$\begin{aligned} \int_0^T \|u(t)\|^r_{L^r(\Omega)} dt &\le \int_0^T \|u(t)\|^{\tau r}_{L^{\bar{q}^*}(\Omega)} \|u(t)\|^{(1-\tau)r}_{L^1(\Omega)} dt \\ &\le c_4 \int_0^T \|u(t)\|^{\bar{q}}_{L^{\bar{q}^*}(\Omega)} dt \\ &\le c_3 \cdot c_4 \int_0^T \prod_{j=1}^{N} \|D_j u(t)\|^{\frac{\bar{q}}{N}}_{L^{q_j}(\Omega)} dt \end{aligned} \tag{2.21}$$

Since

$$\sum_{j=1}^{N} \frac{\bar{q}}{N q_j} = 1, \text{ and } \frac{\bar{q}}{N q_j} = \frac{1}{q_j \sum_{i=1}^{N} \frac{1}{q_i}} \le 1 \tag{2.22}$$

Hölder's inequality implies that

$$\|u\|^r_{L^r(Q)} \le c_5 \prod_{j=1}^{N} \|D_j u\|^{\frac{\bar{q}}{N}}_{L^{q_j}(Q)} \tag{2.23}$$

Let

$$\frac{\lambda q_i}{p_i - q_i} = r \tag{2.24}$$

(2.19) and (2.24) imply that

$$\lambda = \frac{(p_i - q_i)\bar{q}(N+1)}{q_i N} > 1 \tag{2.25}$$

For every i, with $1 \le i \le N$, (2.16) and (2.23) imply that

$$\|D_i u\|_{L^{q_i}(Q)} \le c_6 + c_7 \prod_{j=1}^{N} \|D_j u\|_{L^{q_j}(Q)}^{\bar{q}(1/q_i - 1/p_i)/N} \tag{2.26}$$

Set

$$d = \prod_{j=1}^{N} \|D_j u\|_{L^{q_j}(Q)} \tag{2.27}$$

Then there exist two positive constants such that

$$d \le c_8 + c_9 d^{\frac{\bar{q}}{N}\sum_{i=1}^{N}\left(\frac{1}{q_i} - \frac{1}{p_i}\right)} \tag{2.28}$$

Since

$$\frac{\bar{q}}{N}\sum_{i=1}^{N}\left(\frac{1}{q_i} - \frac{1}{p_i}\right) = 1 - \frac{\bar{q}}{\bar{p}} < 1 \tag{2.29}$$

(2.28) implies that there exists a positive constant c_{10} such that

$$d \le c_{10} \tag{2.30}$$

Hence it follows from (2.30), (2.27) and (2.26) that (2.2) holds. (2.2) and (2.20) yield (2.3). This finishes the proof of Lemma 2.1.

Lemma 2.2 *Assume* (1.1)–(1.3), (1.7) *and* (2.1) *hold. Let* $\bar{p} \le N + \dfrac{N}{N+1}$, *then for* $q_i = \dfrac{p_i}{\bar{p}}\left(\bar{p} - \dfrac{N}{N+1}\right), i = 1, 2, \cdots, N$, *there exists a positive constant* c *depending on* $Q, \alpha, N, p_i, q_i, \|f\|_{L^1(0,T;L_\gamma(\Omega))}$ *and* $\|u_0\|_{L_\gamma(\Omega)}$, *such that*

$$\|D_i u\|_{L^{q_i}(Q)} \le c \tag{2.31}$$

and

$$\|u\|_{L^{\bar{q}}(Q)} \le c \tag{2.32}$$

Proof We work in exactly the same way as that of Lemma 2.2 in [3], we can prove that there exists a positive constant c_{11} such that

$$\|u\|_{L^\infty(0,T;L^1(\Omega))} \le c_{11} \tag{2.33}$$

and

$$\sum_{i=1}^{N} \int_Q \frac{|D_i u|^{p_i}}{1+|u|} dxdt \le c_{11} \tag{2.34}$$

The remainder proof is similar to the latter half part of the proof in Lemma 2.1. We only take $\lambda = 1$ here. So Lemma 2.2 is proved.

Lemma 2.3 *Assume* (1.1)–(1.3), (1.7) *and* (2.1) *hold. Let* $\bar{p} < N$. *Then for* $q_i = \frac{p_i}{\bar{p}}\left(\sigma\bar{p} - \frac{(\rho - 2\sigma)N}{N+\rho}\right), i = 1, 2, \cdots, N$, *there exists a positive constant* c *depending on* $Q, \alpha, N, p_i, \rho, \|f\|_{L^\sigma(0,T;L^\rho(\Omega))}, \|u_0\|_{L^k(\Omega)}$, *such that*

$$\|D_i u\|_{L^{q_i}(Q)} \leq c \tag{2.35}$$

and

$$\|u\|_{L^{\bar{q}}(Q)} \leq c \tag{2.36}$$

Proof In order to prove (2.35) and (2.36), we modify the proof of Lemma 2.3 in [3]. We replace q, r and q^* with $\bar{q}, \frac{N+\rho}{N}(\bar{p} - \bar{q})$ and $\bar{q}^*$ respectively, and combine it with nonisotropic Sobolev inequality (cf. [7]). Thus Lemma 2.3 can be proved.

3. The Proof of Theorems 1.1–1.3

Proof of Theorem 1.1

Let us consider the following approximate problems:

$$(P_n)\begin{cases} \frac{\partial u_n}{\partial t} - \operatorname{div}(\mathrm{a}(x, t, u_n, Du_n)) = f_n & \text{in } Q \\ u_n(x, t) = 0 & \text{on } \Sigma \\ u_n(x, 0) = u_{0n}(x) & \text{for a.e. } x \in \Omega \end{cases}$$

Since f and u_0 are the bounded Radon measures, we may choose two sequences $\{f_n\} \subset L^\infty(Q), \{u_{0n}\} \subset L^\infty(\Omega)$, such that

$$f_n \to f \text{ in the weak* topology of measures} \tag{3.1}$$

$$u_{0n} \to u_0 \text{ in the weak* topology of measures} \tag{3.2}$$

and

$$\|f_n\|_{L^1(Q)} \leq B = \|f\|_{M(Q)} \text{ and } \|u_{0_n}\|_{L^1(\Omega)} \leq C = \|u_0\|_{M(\Omega)} \tag{3.3}$$

By Lemma 2.1 and (3.3), there exists a positive constant c independent of n such that

$$\|D_i u_n\|_{L^{q_i}(Q)} \leq c, \ \forall q_i \in \left[1, \frac{p_i}{\bar{p}}\left(\bar{p} - \frac{N}{N+1}\right)\right), \ i = 1, 2, \cdots, N \tag{3.4}$$

and

$$\|u_n\|_{L^{\bar{q}}(Q)} \leq c \tag{3.5}$$

By (3.4) and (3.5), there exists a subsequence of $\{u_n\}$ (still denoted by $\{u_n\}$) such that

$$D_i u_n \rightharpoonup D_i u \text{ weakly in } L^{q_i}(Q), \ i = 1, 2, \cdots, N \tag{3.6}$$

and

$$u_n \rightharpoonup u \text{ weakly in } L^{\bar{q}}(Q) \tag{3.7}$$

By (1.2), (3.4) and (3.5), we have that div $(\mathbf{a}(x,t,u_n,Du_n))$ is bounded in $\sum_{i=1}^{N} L^{r_i}(0,T;$ $W^{-1,r_i}(\Omega))$,

$$r_i < \frac{p_i}{p_i-1}\left(1-\frac{N}{\bar{p}(N+1)}\right), \quad i=1,2,\cdots,N \tag{3.8}$$

Let $r_0 = \min_{1\le i\le N} r_i$, then we have that $\{u_n'\}$ is bounded in $L^1(Q)+L^{r_0}(0,T;W^{-1,r_0}(\Omega))$. So $\{u_n'\}$ is bounded in $L^1(0,T;W^{-1,s}(\Omega))$, for all $s < \min\left\{\frac{N}{N+1}, r_0\right\}$. Let $q_0 = \min_{1\le i\le N}\{q_i\}$, then $\{u_n\}$ is bounded in $L^{q_0}(0,T;W_0^{1,q_0}(\Omega))$, $\{u_n'\}$ is bounded in $L^1(0,T;$ $W^{-1,s}(\Omega))$. Using Corollary 4 in [8], we obtain that

$$u_n \to u \text{ strongly in } L^1(Q) \tag{3.9}$$

This implies that

$$u_n \to u \text{ a. e. in } Q \tag{3.10}$$

We work in exactly the same way as that in [4], we obtain that

$$Du_n \to Du \text{ in measure} \tag{3.11}$$

By (3.11), there exists a subsequence of $\{Du_n\}$ (still denoted by $\{Du_n\}$) such that

$$Du_n \to Du \text{ a.e. in } Q \tag{3.12}$$

By the assumptions of $\mathbf{a}$, from (3.4), (3.5), (3.10), (3.12) and the Vitali's theorem, we obtain that for every i with $1\le i\le N$,

$$a_i(x,t,u_n,Du_n) \to a_i(x,t,u,Du) \text{ strongly in } L^{r_i}(Q), \forall r_i \in \left[1, \frac{p_i}{p_i-1}\left(1-\frac{N}{\bar{p}(N+1)}\right)\right). \tag{3.13}$$

So we can pass to the limit in the approximate problem (P_n). Therefore we get that u is a solution of the problem (P).

Remark 3.1 We can choose f_n and u_{0n} as follows. Define the C_0^∞-function η : $R^N \to R$ as follows

$$\eta(x) = \begin{cases} c\exp\dfrac{1}{|x|^2-1} & \text{if } |x|<1 \\ 0 & \text{if } |x|\ge 1 \end{cases}$$

Here, c is so chosen that $\int_{R^N}\eta(x)dx = 1$. Next, set

$$\eta_n(x) = n^N\eta(nx) \quad (n=1,2,\cdots, x\in R^N)$$

Let

$$u_{0n} = \int_{R^N}\eta_n(x-y)du_0(y)$$

similar to the definition of u_{0n}, we can define f_n.

Proof of Theorem 1.2 We only replace (3.1) and (3.2) with the following (3.14) and (3.15),

$$f_n \to f \text{ strongly in } L^1(0,T;L_\gamma(\Omega)) \tag{3.14}$$

and

$$u_{0n} \to u_0 \text{ strongly in } L_\gamma(\Omega) \tag{3.15}$$

It is similar to (3.13), we have that for every i with $1 \le i \le N$,

$$a_i(x,t,u_n,Du_n) \rightharpoonup a_i(x,t,u,Du) \text{ weakly in } L^{r_i}(\Omega), r_i = \frac{p_i}{p_i-1}\left(1 - \frac{N}{\bar{p}(N+1)}\right) \tag{3.16}$$

Proof of Theorem 1.3

Suppose that

$$f_n \to f \text{ strongly in } L^\sigma(0,T;L^\rho(\Omega)) \tag{3.17}$$

and

$$u_{0n} \to u_0 \text{ strongly in } L^\kappa(\Omega) \tag{3.18}$$

We also obtain that for every i with $1 \le i \le N$,

$$a_i(x,t,u_n,Du_n) \rightharpoonup a_i(x,t,u,Du) \text{ weakly in } L^{r_i}(\Omega), r_i = \frac{p_i}{p_i-1}\left(\sigma + \frac{(\rho-2\sigma)N}{\bar{p}(N+\rho)}\right) \tag{3.19}$$

Remark 3.2 We note $2 - \frac{1}{N+1} < p_i (i = 1,2,\cdots,N)$ implies that $\bar{p} - \frac{N}{N+1} > 1$ and the condition $\bar{p} \le N + \frac{N}{N+1}$ implies that $\bar{q} < N$. The condition $p_i < \frac{\bar{p}(N+1)}{N} (i = 1,2,\cdots,N)$ implies that $r_i > 1 (i = 1,2,\cdots,N)$ in (3.13), (3.16) and (3.19). These bounds are due to technical reason. (See the proofs of Lemmas 2.1–2.3 and Theorems 1.1–1.3).

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