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LOCAL HARDY SPACES AND INHOMOGENEOUS DIRICHLET PROBLEMS IN EXTERIOR REGULAR DOMAINS*

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Abstract In this paper, firstly we give an atomic decomposition of the local Hardy spaces $h_r^p(\Omega)(0 < p \leq 1)$ and their dual spaces, where the domain Ω is exterior regular in $R^n(n \geq 3)$. Then for given data $f \in h_r^p(\Omega)$, we discuss the inhomogeneous Dirichlet problems:

$$\begin{cases} Lu = f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{1}$$

where the operator L is uniformly elliptic. Also we obtain the estimation of Green potential in the local Hardy spaces $h_r^p(\Omega)$.

Key Words Exterior regular domain; local Hardy space; Hölder space; inhomogeneous Dirichlet problem; Green potential.

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0. Introduction

In [1], the authors brought forward the two questions. What are the possible notions of $H^p(\Omega)$ that generalize the usual Hardy spaces $H^p(R^n)$? And in the context of the relevant $H^p(\Omega)$, can one utilize these spaces to partial differential equations. In [1-3], the boundary of the domains Ω in R^n are C^∞ or Lipschitz. In this paper, we only request that the domain is exterior regular. Let's restrict $n \geq 3$.

We say that a domain Ω is exterior regular (brev. $\Omega \in ER(n)$) as [4], if Ω is a bounded domain in R^n, and there is a constant $c > 0, \delta_0 \in (0,1)$, such that for all cube Q centered at $\partial\Omega$ with side-length less than or equal δ_0, then there exists a subcube Q^e with side-length $cl(Q)$ and $Q^e \subset Q \cap (\bar{\Omega})^c$.

We recall the local Hardy spaces $h^p(R^n)$ for $0 < p \leq 1$ in [5], $h^p(R^n) := \{f \in D'(R^n) : \sup_{0<t\leq 1} |\phi_t * f(x)| \in L^p(R^n)\}$, where $\phi \in C_0^\infty(R^n), \int \phi(x)dx = 1, \phi_t = t^{-n}\phi(t^{-1}x)$. In [5], the author gives the atomic decomposition and their dual space in R^n. Let

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$h_r^p(\Omega)$ denote the local Hardy spaces in Ω as [1], i.e., $h_r^p(\Omega) = \{f \in D'(\Omega) : \exists F \in H^p(R^n), s.t. F|_\Omega = f\}$.

Naturally, we may ask how about the atomic decompositions and their dual spaces for $h_r^p(\Omega)$ for the general domains Ω.

Definition 1 *Let the domain Ω be bounded and connected, $0 < p \le 1$, and the function $a(x) \in L^2(\Omega)$, (then there exists a constant $\delta_0 = \delta_0(\Omega) > 0$),*

(1) *there exists a cube $Q \subset \Omega$, such that* $\operatorname{supp} a \subset Q, \|a\|_{L^2(\Omega)} \le |Q|^{1/2-1/p}$;

(2) $\int_\Omega a(x)x^\alpha dx = 0, |\alpha| \le [n(1/p-1)]$, *where $[x]$ denotes the integer part of a real number x;*

(3) *the side length of the cube $l(Q) > \delta_0$;*

(4) *if $l(Q) \le \delta_0$, then $4Q \subset \Omega$;*

(5) $Q \subset \Omega$, *and* $l(Q) \le \operatorname{dist}(Q, \partial\Omega) \le 4l(Q)$.

The function $a(x)$ is called (p, I)-atom if $a(x)$ satisfies (1) (2) (4) (brev. $Q \in I$).
The function $a(x)$ is called (p,II)-atom if $a(x)$ satisfies (1) (5), (brev. $Q \in II$).
The function $a(x)$ is called (p, III)-atom if $a(x)$ satisfies (1) (3), (brev. $Q \in III$).

We have the following atomic decomposition theorem (See [1]):

Theorem 2 *Let $\Omega \in ER(n), 0 < p \le 1$, then $f \in h_r^p(\Omega)$ iff the function f has the atomic decompositon, that is*

$$f(x) = \sum \lambda_I a_I + \sum \lambda_{II} a_{II} + \sum \lambda_{III} a_{III} \quad \text{in } D'(\Omega)$$

where a_I (respectively a_{II}, a_{III}) is (p,I)-atom (respectively (p, II)-atom, (p,III)-atom), and $\sum |\lambda_I|^p + \sum |\lambda_{II}|^p + \sum |\lambda_{III}|^p < \infty$.

For $\alpha \in (0, \infty)$, let $\Lambda_\alpha(R^n)$ denote the inhomogeneous Lipschitz spaces ([5] or see it in the first section), and let

$$C^\alpha(\bar{\Omega}) := \{u \text{ is continuous function} : \exists F \in \Lambda^\alpha(R^n), s.t. u = F|_{\bar{\Omega}}\}$$
$$C_0^\alpha(\Omega) := \{u \in C_\alpha(\bar{\Omega}) : u|_{\partial\Omega} = 0\}$$

We have the dual theorem as follows:

Theorem 3 *Let $\Omega \in ER(n), n/(n+1) < p < 1, \alpha = n(1/p-1)$, we have the dual theorem: $(h_r^p(\Omega))^* = C_0^\alpha(\Omega)$.*

Let $L = -\partial_i(a_{ij}(x)\partial_j)$ be uniformly elliptic operator, i.e. $\exists\lambda > 0, \forall x \in \Omega$, satisfies the following:

(1) $a_{ij}(x) = a_{ji}(x) \in L^\infty(\Omega)$ is real-valued and measurable function;

(2) $\lambda^{-1}|\xi|^2 \le \sum_{i,j} a_{ij}(x)\xi_i\xi_0 \le \lambda|\xi|^2, \forall\xi \in R^n$.

We know that there is a Green function $G(x, y)$ for uniformly elliptic operator in the domain $\Omega \times \Omega \setminus \{(x, y) : x, y \in \Omega, x = y\}$ (See [6]).

Definition 4 *For a function $f \in h_r^p(\Omega)$, we say that $u \in L^1(\Omega)$ is a weak solution of the equation $Lu = f$ vanishing at the boundary Ω if it satisfies:*

$$\int_\Omega uL\Phi dx = \langle f, \Phi\rangle$$

for every $\Phi \in C_0^\infty(\Omega)$, such that $L\Phi \in C(\bar{\Omega})$, where the $\langle\cdot,\cdot\rangle$ is in the dual sense as theorem 3.

For $s > 1$, let $H_0^{1,s}(\Omega)$ denote the usual Sobolev spaces on Ω.

Theorem 5 *Let $\Omega \in ER(n)$, there exists $p_0 = p_0(\lambda, n, \Omega) \in (0,1)$.*

(1) If p satisfies $p_0 < p < 1$, then for every $f \in h_r^p(\Omega)$, the equation $Lu = f$ has a unique solution $u \in H_0^{1,s'}(\Omega)$ (in the sense of the definition 4). Moreover, there is a constant $C = C(n, \lambda, \Omega)$, such that

$$\|u\|_{H_0^{1,s'}(\Omega)} \le C\|f\|_{h_r^p(\Omega)},\ 1/n + 1/s' = 1/p$$

If $f \in C_0^\infty(\Omega)$, the solution has a Green formula

$$u(x) = \int_\Omega G(x,y)f(y)dy \tag{2}$$

(2) If $p = 1$, for every $f \in h_r^1(\Omega)$, the equation $Lu = f$ has a unique solution $u \in H_0^{1,t}(\Omega)(1 \le t < n/(n-1))$ (in the sense of the paper [6]). Moreover, the solution u has the Green formula (2) for a.e. $x \in \Omega$.

The next part of this paper has three sections. In the first section we provide the atomic decomposition of the local Hardy spaces and their dual theorem, in the second section we discuss the inhomogeous Dirichlet problems, and the third section is about the estimations of Green potential.

1. Local Hardy Spaces and Their Dual Spaces on Domains

Before the proof of Theorem 2, we recall the atomic decomposition of the local Hardy space in R^n.

Let $0 < p \le 1$, a bounded and measurable function $a(x)$ supported on a cube $Q \subset R^n$ is called a local p-atom ([5]) if $\|a\|_{L^2(R^n)} \le |Q|^{1/2-1/p}$ and either (1) $l(Q) > 1$ or (2) $l(Q) \le 1$ and $\int_Q a(x)x^\alpha dx = 0$ for all multi-indices α with $|\alpha| \le [n(1/p-1)]$. A distribution f is in $h^p(R^n)$ iff there is a sequence $\{\lambda_j\} \in l^p$ and local p-atoms a_j such that $f = \sum_{j=1}^\infty \lambda_j a_j$. Here the infimum of the norms $\|\lambda_j\|_{l^p}$, taken over all possible atomic decompositions, is comparable to the h^p norm of f.

If $0 < \alpha < 2$, let $\Lambda_\alpha(R^n) := \{f \in L^\infty(R^n) : \|(f(x+t) + f(x-t) - 2f(x))/|t|^\alpha\|_\infty < \infty\}$. If $\alpha > 2, f \in \Lambda_\alpha$ iff $f \in L^\infty$ and $\partial f/\partial x_j \in \Lambda_{\alpha-1}(j = 1, 2, \cdots n)$. The dual of $h^p(R^n)$ is $\Lambda_{n(1/p-1)}$ ([5]).

When a bounded domain Ω in R^n is Lipschitz, E.M. Stein and other authors give the atomic decompositon. Now we prove Theorem 2.

Proof of Theorem 2

"$\Rightarrow$" Given a distribution $f \in h_r^p(\Omega)$, then there exists a distribution $F \in h^p(R^n)$, such that $F|_\Omega = f$, and F has an atomic decomposition:

$$F = \sum_1 \lambda_Q A_Q(x) + \sum_2 \lambda_P B_P(x)$$

where $A_Q(x)$ is local p-atom with $l(Q) \leq 1, B_P(x)$ is local p-atom with $l(Q) > 1$, and $\sum_1 |\lambda_Q|^p + \sum_2 |\lambda_P|^p < \infty$.

For the part of $\sum_1$, we have several cases:

(1) If $Q \subset \Omega, 4Q \cap \partial\Omega = \emptyset$, then we know that $A_Q(x)$ is (p,I)-atom and $Q \in$I in Ω.

(2) If $Q \subset \Omega, 4Q \cap \partial\Omega \neq \emptyset$, then we know that $A_Q(x)$ is (p, III)-atom and $Q \in$III in Ω.

(3) If $Q \cap \bar{\Omega} = \emptyset$, there is nothing to do.

(4) Suppose $Q \cap \Omega \neq \emptyset$ and $Q \cap (\bar{\Omega})^c \neq \emptyset$. In this case, let $\{Q_k\}$ be the dyadic cube decomposition of Whitney type in Ω, i.e., $\Omega = \cup_k Q_k, Q_k^0 \cap Q_j^0 = \emptyset$ (if $k \neq j$), with $l(Q_k) \leq \text{dist}(Q_k, \partial\Omega) \leq 4l(Q_k)$.

Now we take the all $\{Q_k\}$ with $Q \cap Q_k \neq \emptyset$, let $\widetilde{Q_k} = Q_k \cap \Omega$, then we have

$$A_Q(x) = \sum \frac{\|\chi_{Q_k} A\|_2 |Q_k|^{1/2-1/p}}{|Q_k|^{1/2-1/p}\|\chi_{Q_k} A\|_2} \chi_{Q_k} A(x) = \sum \lambda_k a_k(x)$$

where $a_k(x) = \|\chi_{Q_k} A\|_2^{-1} |Q_k|^{1/2-1/p} \chi_{Q_k} A(x), \lambda_k = \|\chi_{Q_k} A\|_2 |Q_k|^{1/p-1/2}$, then $\|a_k\|_2 \leq \|Q_k\|^{1/2-1/p}$, supp$a_k(x) \subset Q_k$, and by Hölder inequality, we have

$$\sum |\lambda_k|^p = \sum \|\chi_{Q_k} A\|_2^p |Q_k|^{(1/p-1/2)p} \leq \left(\sum \|\chi_{Q_k} A\|_2^2\right)^{p/2} \left(\sum |Q_k|\right)^{1-p/2}$$

Claim $\sum_{Q_k \cap Q \neq \emptyset} |Q_k| \leq C|Q|$.

If Claim is true, we can easily prove the formula $\sum |\lambda_k|^p \leq C < \infty$.

Proof of Claim Let x_Q be the center of the cube Q. If $x_Q \in \Omega$, let $x'_Q \in \partial\Omega$ with $\text{dist}(x_Q, x'_Q) = \text{dist}(x_Q, \partial\Omega)$, we know that there exists $y_k \in Q_k \cap Q$ for the cube $\{Q_k\}$ with $Q_k \cap Q \neq \emptyset$, and we have

$$l(Q_k) \leq \text{dist}(Q_k, \partial\Omega) \leq C_1 \text{dist}(y_k, \partial\Omega) \leq C_2 |y_k - x'_Q| \leq c_3 l(Q)$$

So there exists a cube Q' with the center x'_Q and the radius $C'l(Q)$, such that $Q_k \subset Q'$. If $x_Q \notin (\bar{\Omega})^c$, we can assume that $x_Q \notin \partial\Omega$ (otherwise we obtain it with the same as the case $x_Q \in \partial\Omega$), and we have $l(Q_k) \sim |y_k - y'_k|$, where $y'_k \in \partial\Omega$ with $|y_k - y'_k| = \text{dist}(y_k, \partial\Omega)$, and then

$$|y_k - y'_k| \leq |y_k - x_Q| \leq |y_k - x_q| + |x'_Q - x_Q| \leq 2l(Q)$$

As the same previous case, there exists a cube Q' with the center x'_Q and the radius $C'l(Q)$, such that $Q_k \subset Q'$.

The proof of the claim is completed. And we can prove the case B_Q as the case A_Q. The part "$\Rightarrow$" of Theorem 2 is proved.

"$\Leftarrow$" Let

$$f(x) = \sum \lambda_I a_I + \sum \lambda_{II} a_{II} + \sum \lambda_{III} a_{III} \quad \text{in } D'(\Omega)$$

(1) We know that the sum $\sum \lambda_I a_I$ and $\sum \lambda_{III} a_{III}$ belong to $h^p_r(\Omega)$.

(2) For the sum $\sum \lambda_{II} a_{II}$, we can prove it with the methods of the "refelection". The detail is as the following:

Because $Q \in II$, i.e. $l(Q) \leq \operatorname{dist}(Q, \partial\Omega) \leq 4l(Q)$, let $x'_Q \in \partial\Omega$ with $\operatorname{dist}(x_Q, \partial\Omega) = |x_Q - x'_Q|$, then $|x_Q - x'_Q| \approx l(Q)$, we construct a cube Q' with the center x'_Q and the radius $Cl(Q)$ such that $Q \subset Q'$, and then $l(Q) \sim l(Q')$. Because $\Omega \in ER(n)$, there exists a cube Q_e with $Q_e \subset (\bar{\Omega})^c \cap Q'$ and $l(Q_e) = cl(Q')$. Now we construct functions $a'_Q(x)$.

1^0 For every number $N \in \boldsymbol{N}$ and $N \geq 0$, there exists a series of functions $\{\phi_\alpha\} \subset C_0^\infty(B(0,1))$ such that for all α and β with $|\alpha|, |\beta| \leq N$([4]), we have $\int_{R^n} x^\beta \phi_\alpha(x) = \delta_{\alpha\beta}$, where $\delta_{\alpha,\beta} = 0$ if $\alpha \neq \beta$, or $=1$ if $\alpha = \beta$.

Now let $N = [n(1/p - 1)]$, and we define the function

$$a'_Q(x) = a_Q(x) - \sum b_\alpha \phi_\alpha((x - x_{Q_e})/l(Q_e))$$

which satisfies $\int a'_Q(x) x^\beta dx = 0, 0 \leq |\beta| \leq N$. We can take $b_\alpha = \int a_Q(x)(x - x_{Q_e})^2 dx/(l(Q_e))^{\alpha+n}$. Then $\operatorname{Supp} a'_Q \subset Q'$ and

$$\left\|a'_Q\right\|_{L^2} \leq \|a_Q\|_2 + \sum |b_\alpha| \|\phi_\alpha\|_{L^2} \leq C\|a_Q\|_{L^2} \leq C|Q|^{1/2-1/p}$$

i.e., we have $\left\|\widetilde{a'_Q}\right\|_{L^2} \leq C|Q|^{1/2-1/p}$. So, we have an atomic decomposition

$$F(x) = \sum \lambda_I a_I + \sum \lambda_{III} a_{III} + \sum \lambda_{II} \widetilde{a_{II}} \quad \text{in } D'(\Omega)$$

where $\widetilde{a_{II}}$'s are the function a'_Q's with respect to (p, II)-atoms.

The proof of Theorem 2 is completed.

In order to prove Theorem 3, we firstly introduce the spaces $\Lambda_0(\alpha, \Omega)$ in [7].

Let Ω be a bounded Lipschitz domain in $R^n, \alpha > 0$, and $P[\alpha]$ denotes the set of all polynomial with degrees $\leq [\alpha]$, where $[\alpha]$ denotes the integer part of α. Now we define a normed space as follows:

$$\Lambda_0(\alpha, \Omega) := \{f \in L^1_{\text{loc}}(\Omega) : \|f\|_{\Lambda_0(\alpha,\Omega)} < \infty\}$$

where

$$\|f\|_{\Lambda_0(\alpha,\Omega)} := \sup_{Q \in I} \inf_{q \in P[\alpha]} |Q|^{-\alpha/n-1} \int_Q |f - q| dx + \sup_{Q \in II} |Q|^{-\alpha/n-1} \int_Q |f| dx + \|f\|_{L^\infty(\Omega)} < \infty$$

When the domain Ω is Lipschitz and $0 < p \leq 1$, we have the dual theorem $(h^p_r(\Omega))^* = \Lambda_0(\alpha, \Omega), (\alpha/n = 1/p - 1)$([7]). With the same proof of [7], we have the following dual theorem:

Lemma 6 *Let $\Omega \in ER(n), 0 < p \leq 1$, then $(h^p_r(\Omega))^* = \Lambda_0(\alpha, \Omega), (\alpha/n = 1/p - 1)$.*

Lemma 7 *Let $\Omega \in ER(n), n/(n+1) < p < 1$ and $\alpha/n = 1/p - 1$, then $\Lambda_0(\alpha, \Omega) = C^\alpha_0(\Omega)$. Moreover, we have $(h^p_r(\Omega))^* = C^\alpha_0(\Omega)$.*

Proof of Lemma 7

Step 1: $C_0^\alpha(\Omega) \subset \Lambda_0(\alpha, \Omega)$. For any $f \in C_0^\alpha(\Omega), x \in \Omega$, let $x' \in \partial\Omega$, s.t. $\operatorname{dist}(x, \partial\Omega) = |x - x'|$, then if f_Q denotes $|Q|^{-1}\int_Q f(x)dx$, we have

1^0 $|f(x)| \le |f(x) - f(x')| \le C|x|^\alpha$,

2^0 $\|f\|_{C(\bar{\Omega})} < \infty$,

3^0 $\sup_{Q\subset\Omega} |Q|^{-\alpha/n-1}\int_Q |f - f_Q|dx \le C$,

4^0 If $Q \subset \Omega$ is II-cube, we have

$$|Q|^{-\alpha/n}|Q|^{-1}\int_Q |f(x)|dx \le |Q|^{-\alpha/n}|x - x'|^\alpha \le C$$

this can be obtained by $|x - x'| \sim l(Q)$. By following the previous $1^0 - 4^0$, it is easy to prove $\Lambda_0(\alpha, \Omega) \subset C_0^\alpha(\Omega)$.

Step 2: $\Lambda_0(\alpha, \Omega) \subset C_0^\alpha(\Omega)$. For any $f \in \Lambda_0(\alpha, \Omega)$, we should prove the following two claims:

Claim 1 For every $x \in \Omega$, we have $|f(x)| \le C\delta(x)^\alpha$.

Claim 2 For every $x, y \in \Omega$, we have $|f(x) - f(y)| \le C|x - y|^\alpha$.

If Claim 1 and Claim 2 are true, we can obtain $f \in C_0^\alpha(\Omega)$.

Proof of Claim 1 Let Q_j be a Whitney cube of Ω, because $f \in \Lambda_0(\alpha, \Omega)$, we have $|f_{Q_j}| \le Cl(Q_j)^\alpha$. For every $x \in \Omega$, there exists Q_j, such that $x \in Q_j$, if let $Q_j' = Q_j/10$, then

$$|f(x)| \le |f(x) - f_{Q_j}| + |Q_j'|^{-1}\int_{Q_j'} |f|dy \le C\delta(x)^\alpha$$

Proof of Claim 2 It has several cases. For every $x, y \in \Omega$, let $x_0 = (x+y)/2, r = |x - y|$.

Case 1 $\min(\delta(x), \delta(y)) > 12|x - y| = 12r$. In this case, we have $4Q(x, r), 4Q(y, r), 4Q(x_0, 2r) \subset \Omega$, because $|Q|^{-\alpha/n}|Q|^{-1}\int_Q |f - f_Q|dx \le C$, we have $(Q = Q(x, r), Q_j := 2^{-j}Q)$

$$|f(x) - f_Q| \le \sum_{j=1}^{\infty} |f_{Q_j} - f_{Q_{j+1}}| \le \sum_{j=1}^{\infty}(2^{-j}r)^\alpha C \le Cr^\alpha,$$

$$\begin{aligned}|f(x) - f(y)| &\le |f(x) - f_{Q(x,r)}| + |f_{Q(x,r)} - f_{Q(y,r)}| + |f_{Q(y,r)} - f(y)| \\ &\le Cr^\alpha + |f_{Q(x,r)} - f_{Q(x_0,2r)}| + |f_{Q(y,r)} - f_{Q(x_0,2r)}| \le Cr^\alpha\end{aligned}$$

Case 2 $2|x - y| \le \min(\delta(x), \delta(y))$. In this case, the cube Q can be divided by the number of $2^n N$ cubes (here the number N is big enough), and we can prove it as prove Case 1.

Case 3 $\delta(x), \delta(y) \le 2|x - y|$. In this case, using Claim 1, we have

$$|f(x) - f(y)| \le C\delta(x)^\alpha + C\delta(y)^\alpha \le C|x - y|^\alpha$$

Case 4 $\delta(x) \le 2|x-y|, \delta(y) > 2|x-y|$ or $\delta(y) \le 2|x-y|, \delta(x) > 2|x-y|$, we can assume the first case. Let $x^* \in \partial\Omega$, such that $\delta(x) = |x - x^*|$, then

$$|y - x^*| \le |y - x| + |x - x^*| \Rightarrow |y - x^*| \le 3|x - y| \Rightarrow \delta(y) \le 3|x - y|$$

So $2|x-y| \le \delta(y) \le 3|x-y|$. Using Claim 1, we can easily prove this case.

By following Case 1 to Case 4, Claim 2 is proved. So Lemma 7 is proved.

2. Inhomogeneous Dirichlet Problem

In this section, we discuss the inhomogeneous Dirichlet problems, i.e. given data $f \in h_r^p(\Omega)$, we discuss the second elliptic equation as follows:

$$\begin{cases} Lu = f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

where the operator L is uniformly elliptic as the introduction.

In order to prove Theorem 5, we can introduce another equivalent definition easily proved as showed in the reference [6] as follows:

$$\int_\Omega u\phi dx = \langle f, G(\phi)\rangle, \quad \forall \phi \in C(\bar{\Omega})$$

We know when $f \in C_0^\infty(\Omega)$, the previous equation has a unique solution $u \in H_0^{1,2}(\Omega)$ which has a Green formula:

$$u(x) = \int_\Omega G(x,y) f(y) dy$$

If $f \in h_r^p(\Omega)$, do we have the Green formulae? If it has, what's the meaning?

Proof of Theorem 5

When $p_0 < p < 1$, we prove Theorem 5 as follows:

1^0 The space $C_0^\infty(\Omega)$ is dense in the spaces $h_r^p(\Omega)$. The detail belongs to [7].

2^0 For every $f \in C_0^\infty(\Omega)$, the equation (1) has a unique solution $u \in H_0^{1,2}(\Omega)$. Moreover, we should show that it has a Green representansion and the inequalities as follows:

$$u(x) = \int_\Omega G(x,y) f(y) dy \text{ and } \|u\|_{H^{1,s'}(\Omega)} \le C\|f\|_{h_r^p(\Omega)}$$

In fact, because $\Omega \in ER(n) \subset S = \{\Omega$ is open set in R^n: if exist two numbers $\alpha(0 < \alpha \le 1)$ and $r_0 > 0$ such that $|B_r(x_0) \cap \Omega^c| \ge \alpha|B_r(x_0)|$ for all $x_0 \in \partial\Omega, 0 < r \le 1\}$([8]). For every $\phi \in C(\bar{\Omega})$, by [6] [8], we know $G(\phi) \in C_0^\alpha(\Omega)$ and $\|G(\phi)\|_{C_0^\alpha(\Omega)} \le C\|\phi\|_{H^{-1,s}(\Omega)}$, then

$$\left|\int_\Omega u\phi dx\right| = |\langle f, G(\phi)\rangle| \le \|f\|_{h_r^p(\Omega)} \|G(\phi)\|_{C_0^\alpha(\Omega)} \le C\|f\|_{h_r^p(\Omega)} \|\phi\|_{H^{-1,s}(\Omega)}$$

So by the dual theorem, we obtain $\|u\|_{H_0^{1,s'}(\Omega)} \le C\|f\|_{h_r^p(\Omega)}$.

3^0 We should show that for every $f \in h_r^p(\Omega)$, the equation (1) has a solution $u \in H_0^{1,s'}(\Omega)$ and

$$u(x) = \langle G(x,\cdot), f\rangle,$$
$$\|u\|_{H_0^{1,s'}(\Omega)} \leq C\|f\|_{h_r^p(\Omega)}$$

By 1^0, we know that for every $f \in h_r^p(\Omega), \exists\{f_k\} \subset C_0^\infty(\Omega)$, such that $f_k \to f$ in $h_r^p(\Omega)$, and the inhomogeneous Dirichlet equation $Lu = f_k$ has a unique weak solution $u_k \subset H_0^{1,s'}(\Omega)$ which satisfies the equation $Lu_k = f_k$, moreover, $u_k(x) = \int_\Omega G(x,y)f_k(y)dy$. If the number k is large enough, we have

$$\|u_k\|_{H_0^{1,s'}(\Omega)} \leq C\|f_k\|_{h_r^p(\Omega)} < 2C\|f\|_{h_r^p(\Omega)}$$
$$\int_\Omega u_k\phi dx = \langle f_k, G(\phi)\rangle, \quad \forall\phi \in C(\bar{\Omega})$$

So there exists a function $u \in H^{1,s'}(\Omega)$ and a subsequence of $\{u_{k_j}\}$, such that $u_{k_j} \to u$ in $H_0^{1,s'}(\Omega)$ in the weak convergent sense ($j \to \infty$). Letting $j \to \infty$, we have

$$\int_\Omega u\phi dx = \langle f, G(\phi)\rangle, \quad \forall\phi \in C(\bar{\Omega})$$
$$\|u\|_{H_0^{1,s'}(\Omega)} \leq 2C\|f\|_{h_r^p(\Omega)}$$

If $f = \sum_{j=1}^{\infty} \lambda_j a_j$, we can easily prove $u = G(f)$, this means

$$u(x) = \lim_{k\to\infty} \sum_{j=1}^{k} G(\lambda_j a_j)$$

in the dual sense. Moreover, it is easy to prove that this solution is unique.

When $p = 1$, we show that $h_r^1(\Omega) \subset L^1(\Omega)$. In fact, we can easily prove this assertion by using the atomic decomposition. The rest part can be proved as [6].

Theorem 5 is completed.

3. Some Other Estimations

In [6] and [8], given $f \in L^s(\Omega)(s > n)$, the inhomogeneous Dirichlet problem has a unique solution $u \in H_0^{1,2}(\Omega)$, moreover, the solution u is Hölder continuous in Ω. In this section, we deduce the number s which satisfies $n/2 < s < n/(2-\alpha_0)$ here $\alpha_0 \in (0,1]$ as Theorem 5, then we have the following theorems about the normed estimations:

Theorem 8 *Let $\Omega \in S, 0 < \alpha < \alpha_0, 1/s = 2/n - \alpha/n$. Given data $f \in L^s(\Omega)$, then the inhomogeneous Dirichlet problem (1) has a unique solution $u \in H_0^{1,2}(\Omega)$, moreover, $u \in C_0^\alpha(\Omega)$, and*

$$u(x) = \int_\Omega G(x,y)f(y)dy, \text{ a.e. } x \in \Omega$$
$$\|u\|_{C_0^\alpha(\Omega)} \leq C(\lambda, n, s)\|f\|_{L^s(\Omega)}$$

Corollary 9 *Let $\Omega \in ER(n), n/(n+\alpha_0) < p < 1, 1/s' = 1/p - 2/n$, then the operator G can be extended from the spaces $C_0^\infty(\Omega)$ to the spaces $h_r^p(\Omega)$, moreover, there exists a constant $C = C(\lambda, \alpha, n, s, d)(d = \mathrm{diam}(\Omega))$, such that*

$$\|G(f)\|_{L^{s'}(\Omega)} \le C\|f\|_{h_r^p(\Omega)}$$

Proof of Theorem 8 By Theorem 3, there are three cases that should be considered.

Case 1: $Q \in I$, i.e. $4Q \subset \Omega$, we should show that for every $f \in L^s(\Omega)$, the following is true

$$|Q|^{-1}\int_Q |(Gf)(x) - (Gf)_Q|dx \le Cl(Q)^\alpha$$

Let $f = f_1 + f_2$, where $f_1 = f\chi_{Q^*}, f_2 = f\chi_{Q^{*c}}(Q^* = 2Q)$, then

1^0 For the function f_1, we have

$$\begin{aligned}|Q|^{-1}\int_Q |(Gf_1)(x)|dx &\le |Q|^{-1}\int_Q\int_{Q^*}|G(x,y)||f(y)|dydx\\ &= |Q|^{-1}\int_{Q^*}|f(y)|dy\int_Q |G(x,y)|dx\\ &= C(\lambda,n)(|Q|^{-1}\int_{Q^*}|f(y)|^s dy)^{1/s}\int_0^{cl(Q)}|x-y|^{2-n}dx\\ &\le C\|f\|_{L^s(\Omega)}|Q|^{-1/s}l(Q)^2 = C|Q|^{-1/s+2/n}\|f\|_{L^s(\Omega)}\end{aligned}$$

2^0 For the function f_2, any $x, y \in Q$, if set $E_j = (2^{j+1}Q\backslash 2^jQ)\cap\Omega)$ for $j \ge 1$, by [9], we have

$$\begin{aligned}|Gf_2(x) - Gf_2(y)| &\le \int_{(Q^*)^c}|G(x,z) - G(y,z)||f(z)|dz\\ &\le C(\lambda,n)\int_{(Q^*)^c}|x-z|^{\alpha_0}(|x-z|^{2-n-\alpha_0} + |y-z|^{2-n-\alpha_0})|f(z)|dz\\ &\le Cl(Q)^{\alpha_0}\int_{(Q^*)^c}|x_Q - z|^{2-n-\alpha_0}|f(z)|dz\\ &\le Cl(Q)^{\alpha_0}\sum_{j=0}^\infty\int_{E_j}|x_Q - z|^{2-n-\alpha_0}|f(z)|dz\\ &\le Cl(Q)^{\alpha_0}\sum_{j=0}^\infty(2^jl(Q))^{2-n-\alpha_0}\int_{E_j}|f(z)|dz\\ &\le Cl(Q)^{\alpha_0}\sum_{j=0}^\infty(2^jl(Q))^{2-n-\alpha_0}|E_j|^{1-1/s}\left(\int_{E_j}|f(z)|^s dz\right)^{1/s}\\ &\le Cl(Q)^{2-n/s}\sum_{j=0}^\infty(2^j)^{2-\alpha_0-n/s}\left(\int_{E_j}|f(z)|^s dz\right)^{1/s}\\ &\le Cl(Q)^{2-n/s}\|f\|_{L^s(\Omega)}\end{aligned}$$

Case 2: $Q \in II$, i.e. $l(Q) \le \text{dist}(Q, \partial\Omega) \le 4l(Q)$.

1^0 As the proof of Case 1.1^0, we can obtain

$$|Q|^{-1}\int_Q |Gf_1(x)|dx \le C|Q|^{2/n-1/s}\|f\|_{L^s(\Omega)}$$

2^0 Because $\Omega \in S$, by [9], we have

$$G(x,y) \le C\text{dist}(x,\partial\Omega)^{\alpha_0}|x-y|^{2-n-\alpha_0}$$

So, if $x \in Q, y \in \Omega\backslash(2Q\cap\Omega)$, we have

$$G(x,y) \le Cl(Q)^{\alpha_0}|x-y|^{2-n-\alpha_0}$$

It is easy to prove

$$|Q|^{-1}\int_Q |Gf_2(x)|dx \le C|Q|^{2/n-1/s}\|f\|_{L^s(\Omega)}$$

Case 3: $Q \in III$, i.e. $Q \subset \Omega$, and $l(Q) \ge \delta_0$, we have

$$\begin{aligned}|Q|^{-1}\int_Q |Gf(x)|dx &\le C|Q|^{-1}\int_Q\int_\Omega |G(x,y)||f(y)|dydx\\ &= C(\lambda,n)|Q|^{-1}\int_\Omega |f(x)|dx\int_\Omega |x-y|^{2-n}dy\\ &\le C|\Omega|^{2/n}|Q|^{-1}|\Omega|^{1-1/s}\|f\|_{L^s(\Omega)}\\ &\le C(|\Omega|/|Q|)^{2/n-1/s+1}\|f\|_{L^s(\Omega)}|Q|^{2/n-1/s}\end{aligned}$$

So, by $l(Q) \ge \delta_0$, we can obtain

$$|Q|^{-1}\int_Q |Gf(x)|dx \le C(\lambda,s,n,\Omega)\|f\|_{L^s(\Omega)}l(Q)^{n(2/n-1/s)}$$

Because $1/s = 2/n - \alpha/n$, we obtain $s > n/2$ and $G(f) \in L^\infty(\Omega)$ with the Hölder inequality.

Combining with the three cases, Theorem 6 is completed.

Proof of Corollary 9 For every $f \in C_0^\infty(\Omega)$, and $\phi \in C_0^\infty(\Omega)$, because $G(x,y) = G(y,x)$, by Theorem 8, we have

$$\left|\int_\Omega G(f)\phi dx\right| = \left|\int_\Omega fG(\phi)dx\right| \le \|f\|_{h_r^p(\Omega)}\|G(\phi)\|_{C_0^\alpha(\Omega)} \le C\|f\|_{h_r^p(\Omega)}\|\phi\|_{L^s(\Omega)}$$

So, we get

$$\|G(f)\|_{L^{s'}(\Omega)} \le C\|f\|_{h_r^p(\Omega)}$$

For general $f \in h_r^p(\Omega)$, we can also obtain it by using dense properties.

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