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# GLOBAL APPROXIMATELY CONTROLLABILITY AND FINITE DIMENSIONAL EXACT CONTROLLABILITY FOR PARABOLIC EQUATION*

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**Abstract** We study the globally approximate controllability and finite-dimensional exact controllability of parabolic equation where the control acts on a mobile subset of $\Omega$ ,or, a curve in $Q = \Omega \times (0, T)$.
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## 1. Introduction

Let $\Omega$ be a bounded, open, connected set in $R^n$ with boundary $\partial\Omega$. Consider the following homogeneous Dirichlet problem for the parabolic equation:

$$\begin{aligned}&\frac{\partial u}{\partial t} = \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}(a_{ij}(x,t)\frac{\partial u}{\partial x_j}) - \sum_{i=1}^{n} b_i(x,t)\frac{\partial u}{\partial x_i} - a(x,t)u \\ &\qquad \text{in} \quad Q = (0,T) \times \Omega, \qquad (1.1a) \\ &u|_{\Sigma} = 0 \qquad \text{in} \quad \Sigma = \partial\Omega \times (0,T), \ \ u|_{t=0} = u_0 \qquad \text{in} \quad \Omega,\end{aligned}$$

under the condition of uniform ellipticity, namely,

$$\mu \sum_{i=1}^{n} \xi_i^2 \le \sum_{i,j=1}^{n} a_{ij}(x,t)\xi_i\xi_j \ \ \forall \xi_i \in R \quad \text{a.e.} \quad \text{in} \quad Q, \ \mu > 0, \qquad (1.1b)$$

where $a_{ij} = a_{ji}$, $a_{ij} \in L^\infty(Q)$, $i, j = 1, ..., n$. To guarantee the solvability and unique continuation, some other assumptions are needed [1-2]:

$$u_0 \in L^2(\Omega), \ \| \sum_{i=1}^{n} b_i^2, a\|_{q,r,Q} \le \mu, \quad \frac{1}{r} + \frac{n}{2q} = 1 - k, \qquad (1.2a)$$

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$$\begin{cases} q \in [\frac{n}{2(1-k)}, \infty],\ r \in [\frac{1}{1-k}, \infty],\ 0 < k < 1, \quad \text{for} \quad n \geq 2, \\ q \in [1, \infty],\ r \in [\frac{1}{1-k}, \frac{2}{1-2k}],\ 0 < k < \frac{1}{2}, \quad \text{for} \quad n = 1, \end{cases} \tag{1.2b}$$

where $\|z\|_{q,r,Q} = (\int_0^T (\int_\Omega |z|^q dx)^{\frac{r}{q}} dt)^{\frac{1}{r}}$;

$$\frac{\partial a_{ij}}{\partial t} \in L^1(0, T; L^\infty(\Omega)),\ b_i, a \in L^\infty(Q), \tag{1.3}$$

$$u_0 \in H_0^1(\Omega),\ \partial\Omega \in C^2,\ \frac{\partial a_{ij}}{\partial x_k}, b_i, a \in L^\infty(Q). \tag{1.4}$$

The conditions (1.2) ensure the existence and uniqueness of a solution to (1.1) from the space $C([0,T]; L^2(\Omega)) \cap H_0^{1,0}(Q)$(see Ladyzenskaja [1]), which satisfies the energy estimate:

$$\|u\|_{C([0,T];L^2(\Omega))} + \|u\|_{H^{1,0}(Q)} \leq c\|u_0\|_{L^2(\Omega)}. \tag{1.5}$$

Here $c$ depends on $T$ and the parameters in (1.1b), (1.2). Under the assumptions (1.4) this solution lies in $H_0^{2,1}(Q)$. The assumptions (1.3) allow one to use the backward uniqueness result.

The reference [2] gives the following unique continuation results:

**Proposition 1.1** *Let $n \leq 3$. Given $T > \epsilon > 0$, there exists a measurable curve $(\epsilon, T) \ni t \to \hat{x}(t) \in \bar{\Omega}$ such that every solution $u \in H_0^{2,1}(Q)$ to (1.1), (1.3), (1.4) which vanishes along $\hat{x}(\cdot)$ and vanishes in $Q$.*

**Proposition 1.2** *Given $T > \epsilon > 0$, there exists a set-valued map $(\epsilon, T) \ni t \to S(t) \subset \Omega$ , $\text{mes}\{S(t)\} > 0$ such that every solution $u \in C([0,T]; L^2(\Omega)) \cap H_0^{1,0}(Q)$ to (1.1), (1.2), (1.3) which satisfies the equality $\int_{S(t)} u dx = 0$ on $(\epsilon, T)$ vanishes in $Q$.*

Furthermore, [2] studies the approximate controllability of the following control system:

$$\begin{aligned} \frac{\partial \varphi}{\partial t} =& \sum_{i,j=1}^n \frac{\partial}{\partial x_i}(a_{ij}(x, T-t)\frac{\partial \varphi}{\partial x_j}) \\ &+ \sum_{i=1}^n \frac{\partial}{\partial x_i}(b_i(x, T-t)\varphi) - a(x, T-t)\varphi + B(T-t)v(t) \quad \text{in} \quad Q, \\ \varphi =& 0 \quad \text{in}\, \Sigma, \quad \varphi|_{t=0} = 0, \end{aligned} \tag{1.6}$$

where $B(\cdot)$ is a linear operator defined on a linear manifold $V \subseteq L^2(0,T)$ by one of the following formulas:

$$B(T-t)v(t) = v(t) \times \begin{cases} 1, \text{ if } & x \in S(T-t), \\ 0, \text{ if } & x \notin S(T-t), \end{cases} \quad S(t) \subset \Omega \quad \text{a.e. in} \quad [0,T], \tag{1.7}$$

or

$$B(T-t)v(t) = v(t)\delta(x - \hat{x}(T-t)), \quad \hat{x}(t) \in \bar{\Omega} \quad \text{a.e. in} \quad [0,T], \tag{1.8}$$

where $\delta(x)$ is Dirac's function, and $v \in V = \{v \in L^2(0,T;L^2(\Omega)|v(x,t)=0, \text{a.e. in}(T-\epsilon,T)\}$ is a control function. The reference [1] just gives the approximate controllability results for linear parabolic system. We will show the finite-dimensional exact controllability of linear parabolic system under the same assumptions as in [2], and morover, extend the results to nonlinear systems.

## 2. Global Approximately Controllability and Finite Dimensional Exact Controllability for Linear Parabolic Equation

In this section, we intend to study the global approximately controllability and finite dimensional exact controllability of (1.6), (1.7). First, we deal with the case that control acts on a positive measure subset of $Q$, and then, the case that control acts on a curve.

### 2.1 The case that control acts on a positive measure subset of $Q$

Given $\epsilon > 0$, $\alpha > 0$, $\varphi^1 \in L^2(\Omega)$, let $E$ be any finite-dimensional subspace of $L^2(\Omega)$ and $\Pi_E$ the orthogonal projection from $L^2(\Omega)$ into $E$, and $S(t)$ a subset constructed as in Proposition 1.2. Define functional $J : L^2(\Omega) \to \mathrm{R}$ as follows:

$$J(u_0) = \frac{1}{2}\int_\epsilon^T \int_{S(t)} u^2 dxdt + \alpha\|(I-\Pi_E)u_0\| - \int_\Omega \varphi^1 u_0 dx \tag{2.1}$$

where $u$ is the solution to (1.1) with initial datum $u_0$. We have the following results:

**Theorem 2.1** *The functional $J : L^2(\Omega) \to \mathrm{R}$ is continuous and convex. Furthermore, it is coercive. More precisely,*

$$\liminf_{\|u_0\|_{L^2(\Omega)}\to\infty} \frac{J(u_0)}{\|u_0\|_{L^2(\Omega)}} \geq \alpha \tag{2.2}$$

**Proof** The continuity can be easily deduced from the energy estimate (1.5), and convexity, from unique continuation Proposition 1.2. To prove (2.2), we proceed as in Zuazua[3]. Given a sequence $\{u_0^j\}$ in $L^2(\Omega)$ with $\|u_0^j\|_{L^2(\Omega)} \to \infty$, we normalize it:

$$\hat{u}_0^j = u_0^j/\|u_0^j\|_{L^2(\Omega)}.$$

We have

$$J(u_0^j)/\|u_0^j\|_{L^2(\Omega)} = \frac{\|u_0^j\|_{L^2(\Omega)}}{2}\int_\epsilon^T \int_{S(t)} |\hat{u}^j|^2 dxdt$$
$$+\alpha\|(I-\Pi_E)\hat{u}_0^j\|_{L^2(\Omega)} - \int_\Omega \varphi^1 \hat{u}_0^j dx$$

where $\hat{u}^j$ is the solution of (1.1) with initial data $\hat{u}_0^j$.

We distinguish the following two cases.

**Case 1:**

$$\liminf_{j\to\infty}\int_\epsilon^T\int_{S(t)}|\hat{u}^j|^2dxdt>0.$$

When this holds, we clearly have

$$\liminf_{j\to\infty}\frac{J(u_0^j)}{\|u_0^j\|_{L^2(\Omega)}}=\infty.$$

**Case 2:**

$$\liminf_{j\to\infty}\int_\epsilon^T\int_{S(t)}|\hat{u}^j|^2dxdt=0.$$

In this case, by extracting subsequences (that we denote by the index $j$ to simplify the notation) we have that

$$\int_\epsilon^T\int_{S(t)}|\hat{u}^j|^2dxdt\to 0 \tag{2.3}$$

and

$$\hat{u}_0^j\rightharpoonup u_0 \quad \text{weakly} \quad \text{in} \quad L^2(\Omega). \tag{2.4}$$

In view of (2.3) and (2.4) the solution of (1.1), (1.2) with data $u_0$ satisfies

$$u=0 \quad \text{in} \quad S(\cdot).$$

But then, by Proposition 1.2, $u\equiv 0$. In particular $u(0)=u_0=0$ in $\Omega$ and therefore

$$\hat{u}_0^j\rightharpoonup 0 \quad \text{weakly} \quad \text{in} \quad L^2(\Omega).$$

Since $E$ is finite-dimensional (and $\Pi_E$ compact),

$$\|(I-\Pi_E)\hat{u}_0^j\|_{L^2(\Omega)}\to 1.$$

Therefore,

$$\liminf_{j\to\infty}\frac{J(u_0^j)}{\|u_0^j\|_{L^2(\Omega)}}\geq\alpha.$$

This proves the claim (2.2).

**Theorem 2.2** *System (1.6),(1.7) is globally approximately controllable and finite-dimensional exact controllable in the sense that, given $T>\epsilon>0$, $\varphi^1\in L^2(\Omega)$, finite-dimensional subspace $E$ and $\alpha>0$, there exists a set-valued map $(\epsilon,T)\ni t\to S(t)\subset\Omega$ and a control $v\in V$ such that*

$$\|\varphi(x,T)-\varphi^1\|\leq\alpha$$

*and*

$$\Pi_E\varphi(x,T) = \Pi_E\varphi^1$$

**Proof** By Proposition 1.2 it is also easy to deduce that $J$ is strictly convex. Then $J$ has a unique critical point which is its minimizer:

$$\hat{u}_0 \in L^2(\Omega): \quad J(\hat{u}_0) = \min_{u_0\in L^2(\Omega)} J(u_0).$$

Given any $u_0 \in L^2(\Omega)$ and $\lambda \in R$ we have

$$J(\hat{u}_0) \le J(\hat{u}_0 + \lambda u_0)$$

or, in other words,

$$\begin{aligned}\alpha\|(I-\Pi_E)\hat{u}_0\|_{L^2(\Omega)} \le & \frac{\lambda^2}{2}\int_\epsilon^T\int_{S(t)} u^2 dxdt + \lambda\int_\epsilon^T\int_{S(t)} \hat{u}udxdt \\ & + \alpha\|(I-\Pi_E)(\hat{u}_0+\lambda u_0)\|_{L^2(\Omega)} - \lambda\int_\Omega \varphi^1 u_0 dx.\end{aligned}$$

where $u$ is the solution of (1.1), (1.2) with data $u_0$.

Dividing this inequality by $\lambda > 0$ and letting $\lambda \to 0^+$, we obtain that

$$\begin{aligned}\int_\Omega \varphi^1 u_0 \le & \int_\epsilon^T\int_{S(t)} \hat{u}udxdt \\ & + \alpha\liminf_{\lambda\to 0^+}\frac{\|(I-\Pi_E)(\hat{u}_0+\lambda u_0)\|_{L^2(\Omega)} - \|(I-\Pi_E)\hat{u}_0\|_{L^2(\Omega)}}{\lambda} \\ \le & \int_\epsilon^T\int_{S(t)} \hat{u}udxdt + \alpha\|(I-\Pi_E)u_0\|_{L^2(\Omega)}.\end{aligned}$$

Reproducing this argument with $\lambda < 0$, we obtain finally that

$$|\int_\epsilon^T\int_{S(t)} \hat{u}udxdt - \int_\Omega \varphi^1 u_0| \le \alpha\|(I-\Pi_E)u_0\|_{L^2(\Omega)} \tag{2.5}$$

On the other hand, the reference [2] ((6.6) pp.461) gives

$$\int_\epsilon^T\int_{S(t)} u(x,t)v(T-t)dxdt = \int_\Omega u_0\varphi(x,T)dx. \tag{2.6}$$

Let

$$v(t) = \begin{cases} \hat{u}(x,T-t), \ \text{in}(0,T-\epsilon), \\ 0, \ \text{in}(T-\epsilon,T). \end{cases}$$

Then, combining (2.5) with (2.6) we obtain that

$$|\int_\Omega u_0(\varphi(x,T)-\varphi^1)dx| \le \alpha\|(I-\Pi_E)u_0\|_{L^2(\Omega)}$$

holds for any $u_0 \in L^2(\Omega)$. Thus

$$\|\varphi(x,T) - \varphi^1\| \le \alpha$$

and

$$\Pi_E(\varphi(x,T) - \varphi^1) = 0.$$

As a consequence of Theorem 2.2, we have more general controllability result as follows:

**Theorem 2.3** *The parabolic system*

$$\begin{aligned} \frac{\partial \phi}{\partial t} &= \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}(a_{ij}(x,T-t)\frac{\partial \phi}{\partial x_j}) \\ &\quad + \sum_{i=1}^{n} \frac{\partial}{\partial x_i}(b_i(x,T-t)\phi) - a(x,T-t)\phi + B(T-t)v(t) \quad \text{in } Q, \\ \phi &= 0 \quad \text{in } \Sigma, \ \ \phi|_{t=0} = \phi_0 \in L^2(\Omega) \end{aligned} \tag{2.7}$$

*is also globally approximate controllable and finite-dimensional exact controllable. i.e., that, for any $\phi_0$, $\phi^1 \in L^2(\Omega)$, finite-dimensional subspace $E$, $\epsilon > 0$, $\alpha > 0$, there exist a set-valued map $(\epsilon, T) \ni t \to S(t) \subset \Omega$ and a control $v \in V$ such that the solution satisfies*

$$\|\phi(x,T) - \phi^1\|_{L^2(\Omega)} \le \alpha$$

*and*

$$\Pi_E \phi(x,T) = \Pi_E \phi^1.$$

**Proof** In fact, system (2.7) can be decomposed as $\phi = \varphi + \tilde{\varphi}$ where $\varphi$ is the solution of (1.6) and $\tilde{\varphi}$ satisfies

$$\begin{aligned} \frac{\partial \tilde{\varphi}}{\partial t} &= \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}(a_{ij}(x,T-t)\frac{\partial \tilde{\varphi}}{\partial x_j}) \\ &\quad + \sum_{i=1}^{n} \frac{\partial}{\partial x_i}(b_i(x,T-t)\tilde{\varphi}) - a(x,T-t)\tilde{\varphi} \quad \text{in } \ Q, \\ \tilde{\varphi} &= 0 \quad \text{in } \ \Sigma, \ \ \tilde{\varphi}|_{t=0} = \phi_0 \quad \text{in } \ \Omega. \end{aligned}$$

Of course, $\tilde{\varphi}(T)$ is determined by $\phi_0$. Then, for $\varphi^1 = \phi^1 - \tilde{\varphi}(T)$. From Theorem 2.2, there exists a control $v \in V$ such that

$$\|\varphi(T) - (\phi^1 - \tilde{\varphi}(T))\|_{L^2(\Omega)} \le \alpha$$

and

$$\Pi_E \varphi(T) = \Pi_E(\phi^1 - \tilde{\varphi}(T)).$$

i.e.,

$$\|\phi(T) - \phi^1\|_{L^2(\Omega)} \le \alpha$$

and

$$\Pi_E\phi(T) = \Pi_E\phi^1.$$

This concludes the proof.

**Remark** Our control acts only during a subperiod $(0, T-\epsilon)$, not the whole interval $[0, T]$. This is the difference from classical control theory.

### 2.2 The case that control acts on a curve in $Q$

Let $n \le 3$. Given $T > \epsilon > 0$, let $\hat{x}(t)$, $t \in (0, T)$ be an arbitrary curve satisfying Proposition 1.1. Consider the system

$$\begin{aligned}\frac{\partial\phi}{\partial t} =& \sum_{i,j=1}^{n}\frac{\partial}{\partial x_i}(a_{ij}(x, T-t)\frac{\partial\phi}{\partial x_j})\\ &+\sum_{i=1}^{n}\frac{\partial}{\partial x_i}(b_i(x, T-t)\phi) - a(x, T-t)\phi + v(t)\delta(x - \hat{x}(T-t)) \quad \text{in } Q, \quad (2.8)\\ \phi =& 0 \quad \text{in } \Sigma, \quad \phi|_{t=0} = \phi_0 \in L^2(\Omega).\end{aligned}$$

We have the following result:

**Theorem 2.4** *The system (2.8) is globally approximately controllable and finite-dimensional exactly controllable in the sense that, for any $\phi_0$, $\phi^1 \in H^{-1}(\Omega)$, let $E$ be finite-dimensional subspace in $H_0^1(\Omega)$. Then there exists a measurable curve $(\epsilon, T) \ni t \to \hat{x}(t) \in \bar{\Omega}$ and a control $v \in V$ such that the solution satisfies*

$$\|\phi(T) - \phi^1\|_{H^{-1}(\Omega)} \le \alpha \quad (2.9)$$

*and*

$$\Pi_E\phi(T) = \Pi_E\phi^1 \quad (2.10)$$

*where $\Pi_E$ is the orthogonal projection from $H^{-1}(\Omega)$ into $E$.*

This theorem can be proved in the same way as that of Theorem 2.2 and 2.3. We just give a sketch: Define functional $J : H_0^1(\Omega) \to \mathrm{R}$ as

$$J(u_0) = \frac{1}{2}\int_{\epsilon}^{T}|u(\hat{x}(t), t)|^2 dxdt + \alpha\|(I - \Pi_E)u_0\|_{H_0^1(\Omega)} - \int_{\Omega}\varphi^1 u_0 dx,$$

where $u$ is the solution to (1.1) with initial datum $u_0 \in H_0^1(\Omega)$. It is easy to verify that the functional $J$ is continuous and strictly convex, and moreover, the coercive in $H_0^1(\Omega)$. So it has a unique critical point which is its minimizer:

$$\hat{u}_0 \in H_0^1(\Omega) : \ J(\hat{u}_0) = \min_{u_0 \in H_0^1(\Omega)} J(u_0).$$

Let $\hat{u}$ be the solution to (1.1) with initial data $\hat{u}_0$. Then

$$v(t) = \begin{cases} \hat{u}(x, T-t), \text{ in } \ (0, T-\epsilon), \\ 0, \text{ in } \ (T-\epsilon, T) \end{cases}$$

is the control such that (2.9) and (2.10) hold.

## 3. The Nonlinear Case

In this section, we intend to deal with the semilinear control system:

$$\begin{cases} \dfrac{\partial \phi}{\partial t} = \triangle \phi + g(\phi) + B(T-t)v(t) \\ \phi = 0 \quad \text{in} \quad \Sigma, \quad \phi|_{t=0} = \phi_0, \end{cases} \tag{3.1}$$

with $v$ and $B(\cdot)$ as in (1.7) or (1.8) and $\hat{x}(\cdot)$, $S(t)$ as in Propositions 1.1 and 1.2 respectively. We have the following result:

**Theorem 3.1** *Assume that $g$ is of $C^1$, globally Lipschitz and $g(0) = 0$, then the system (3.1) is globally approximately controllable and finite-dimensional exactly controllable in $L^2(\Omega)$ with control $v \in V$.*

**Sketch of the proof** We introduce the nonlinearity

$$h(s) = \begin{cases} g(s)/s, & \text{if} \quad s \neq 0, \\ g'(0), & \text{if} \quad s = 0. \end{cases}$$

Given any $z \in L^2(\Omega)$ we consider the "linearized " system:

$$\begin{cases} \dfrac{\partial \phi}{\partial t} = \triangle \phi + h(z)\phi + B(T-t)v(t) \\ \phi = 0 \quad \text{in} \quad \Sigma, \quad \phi|_{t=0} = \phi_0, \end{cases} \tag{3.2}$$

We observe that the potential $h(z)$ belongs to $L^\infty(\Omega \times (0,T))$. Moreover,

$$\|h(z)\|_{L^\infty(\Omega\times(0,T)} \leq \|g'\|_{L^\infty(R)}.$$

By Theorem 2.3 we may construct a control $v_z \in V$, depending on $z$, such that

$$\|\varphi(x,T) - \varphi^1\| \leq \alpha$$

and

$$\Pi_E \varphi(x,T) = \Pi_E \varphi^1$$

Using the techniques in [3], one can deduce that the controls $v_z$ obtained by the method in Theorem 2.2 and 2.3 or (2.4) are uniformly bounded. More precisely, there exists $C > 0$ such that

$$\|v_z\|_{L^2(\Omega\times(0,T))} \leq C, \quad \forall z \in L^2(\Omega \times (0,T)).$$

and therefore the solutions $\phi_z$ of (3.1) corresponding to $v_z$ are also uniformly bounded in $H_0^{1,0}(Q)$ and $C([0,T]; L^2(\Omega))$, and thus belong to a compact subset in $L^2(Q)$(see [4]). In fact, we have constructed a nonlinear map:

$$N: \quad L^2(Q) \to L^2(Q)$$

such that $N(z) = \phi_z$ which is continuous and compact. Then, by Schauder's fixed point theorem, the fixed point exists and the proof is concluded.

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