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CAUCHY PROBLEM FOR GENERAL FIRST ORDER INHOMOGENEOUS QUASILINEAR HYPERBOLIC SYSTEMS

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Abstract In this paper, we consider Cauchy problem for general first order inhomogeneous quasilinear strictly hyperbolic systems. Under the matching condition, we first give an estimate on inhomogeneous terms. By this estimate, we obtain the asymptotic behaviour for the life-span of C^1 solutions with "slowly" decaying and small initial data and prove that the formation of singularity is due to the envelope of characteristics of the same family.
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1. Introduction and Main Results

Consider the following first order inhomogeneous quasilinear system

$$\frac{\partial u}{\partial t} + A(u)\frac{\partial u}{\partial x} = F(u), \tag{1.1}$$

where $u = (u_1, \cdots, u_n)^T$ is the unknown vector function of $(t, x), A(u) = (a_{ij}(u))$ is an $n \times n$ matrix with suitably smooth elements $a_{ij}(u)(i, j = 1, \cdots, n)$ and $F(u) = (f_1(u), \cdots, f_n(u))^T$ is a vector function of u with suitably smooth elements $f_i(u)(i = 1, \cdots, n)$.

Suppose that the system (1.1) is strictly hperbolic in a neighbourhood of $u = 0$, namely, for any given u in this domain, $A(u)$ has n distinct real eigenvalues

$$\lambda_1(u) < \lambda_2(u) < \cdots < \lambda_n(u). \tag{1.2}$$

For $i = 1, \cdots, n$, let $l_i(u) = (l_{i1}(u), \cdots, l_{in}(u))$ (resp. $r_i(u) = (r_{r1}(u), \cdots, r_{in}(u))^T$) be a left (resp. right) eigenvecdtor corresponding to $\lambda_i(u)$:

$$l_i(u)A(u) = \lambda_i(u)l_i(u) \text{ (resp. } A(u)r_i(u) = \lambda_i(u)r_i(u)) \tag{1.3}$$

We have

$$\det|l_{ij}(u)| \neq 0 \text{ (resp. } \det|r_{ij}(u)| \neq 0). \tag{1.4}$$

All $\lambda_i(u), l_{ij}(u)$ and $r_{ij}(u)(i,j = 1,\cdots,n)$ have the same regularity as $a_{ij}(u)(i,j = 1,\cdots,n)$. Without loss of generality, we may suppose that

$$l_i(u)r_j(u) \equiv \delta_{ij}, \quad i,j = 1,\cdots,n \tag{1.5}$$

and

$$r_i^T(u)r_i(u) \equiv 1, \quad i = 1,\cdots,n, \tag{1.6}$$

where δ_{ij} stands for Kronecker's symbol.

For the following initial data

$$t = 0: \quad u = \varphi(x), \tag{1.7}$$

where $\varphi(x)$ is a "small" C^1 vector function of x with certain decay properties as $|x| \to \infty$, Li et al.[1,2] presented a complete result on the global existence and the blow-up phenomenon of C^1 solution $u = u(t,x)$ to Cauchy problem (1.1) and (1.7) in the case $F(u) \equiv 0$. In the case that $F(u)$ satisfies the so-called matching condition, Koing [3] gave a quite complete result for the global existence and the breakdown of C^1 solution $u = u(t,x)$ to Cauchy problem (1.1) and (1.7). Kiong [4] also proved that the results given in [2] on the breakdown of C^1 solution are still valid for "slow" decaying initial data. In this paper, in the case that the inhomogeneous term satisfies the matching condition, we will prove that the system (1.1) has the same result as in the homogeneous case. We will first give an estimate on the inhomogeneous term under the matching condition. By this estimate, the corresponding proof given in [3] can be simplified. On the other hand, we generalize the results in [3] on the breakdown of C^1 solution for "slow" decaying initial data.

For the completeness of statement, we first recall the concepts of the weak linear degeneracy (see [5] or [1]) and the matching condition (see [6] or [3]).

Definition 1.1 *The i-th characteristic $\lambda_i(u)$ is weakly linearly degenerate if along the i-th characteristic trajectory $u = u^{(i)}(s)$ passing through $u = 0$, defined by*

$$\begin{cases} \dfrac{du}{ds} = r_i(u), \\ s = 0: \quad u = 0, \end{cases} \tag{1.8}$$

we have

$$\nabla\lambda_i(u)r_i(u) \equiv 0, \ \forall|u| \text{ small}, \quad \text{namely}, \quad \lambda_i(u^{(i)}(s) \equiv \lambda_i(0), \quad \forall|s| \text{ small} \tag{1.9}$$

If all characteristics are wakly linearly degenerate, the system (1.1) is said to be weakly linearly degenerate.

Definition 1.2 *$F(u)$ is called to satisfy the matching condition if along all characteristic trajectories passing through $u=0$, we have*

$$F(u) \equiv 0, \quad \forall |u| \text{ small}, \quad \text{namely}, \quad F(u^{(i)}(s)) = 0, \quad \forall |s| \text{ small}, \quad i = 1, \cdots, n \tag{1.10}$$

In this case, it is easy to see that

$$F(0) = 0, \quad \nabla F(0) = 0. \tag{1.11}$$

Suppose that $A(u) \in C^k$, where k is an integer ≥ 1. By Lemma 2.5 in [5], there exists an invertible C^{k+1} transformation $u = u(\tilde{u})(u(0) = 0)$ such that in $\tilde{u}$-space, for each $i = 1, \cdots, n$, the i-th characteristic trajectory passing through $\tilde{u} = 0$ coincides with the $\tilde{u}_i$-axis at least for $|\tilde{u}_i|$ small, namely,

$$\tilde{r}_i(\tilde{u}_i e_i) \| e_i, \quad \forall |\tilde{u}| \text{ small}, \quad i = 1, \cdots, n, \tag{1.12}$$

where $e_i = (0, \cdots, 0, 1^{(i)}, 0, \cdots, 0)^T$ and $\tilde{r}_i$ denotes the i-th right eigenvector in $\tilde{u}$-space. Such a transformation is called the normalized transformation and the corresponding unknown variables $\tilde{u} = (\tilde{u}_1, \cdots, \tilde{u}_n)$ are called the normalized variables or normalized coordinates. Noting (1.15)–(1.16) in [2], we can always find suitable normalized coordinates $\tilde{u}$ such that

$$\frac{\partial \tilde{u}_i}{\partial u}(0) = l_i(0), \quad i = 1, \cdots, n, \tag{1.13}$$

i.e.,

$$\frac{\partial \tilde{u}}{\partial u}(0) = L(0), \tag{1.14}$$

where $L(u)$ is the matrix composed by the left eigenvectors $l_i(u)(i = 1, \cdots, n)$.

The following theorem is proved in [3] (see Theoroem 3.1 in [3]).

Theorem A *Suppose that in a neighbourhood of $u = 0, A(u) \in C^2$, the system (1.1) is strictly hyperbolic and weakly linearly degenerate, and $F(u) \in C^2$ satisfy the matching condition. Suppose furthermore that $\varphi(x)$ is a C^1 vector function satisfying that there exists a constant $\mu > 0$ such that*

$$\theta \overset{\triangle}{=} \sup_{x \in \mathbf{R}} \{(1 + |x|)^{1+\mu} (|\varphi(x)| + |\varphi'(x)|)\} < \infty. \tag{1.15}$$

Then there exists $\theta_0 > 0$ so small that for any given $\theta \in [0, \theta_0]$, the Cauchy problem (1.1) and (1.7) admits a unique global C^1 solution $u = u(t, x)$ on all $t \in \mathbf{R}$.

When the system (1.1) is not weakly linearly degenerate, there exists a nonempty set $J \subseteq \{1, 2, \cdots, n\}$ such that $\lambda_i(u)$ is not weakly linearly degenerate if and only if $i \in J$. Note (1.9), for any fixed $i \in J$, either there exists an integer $\alpha_i \geq 0$ such that

$$\left.\frac{d^l \lambda_i(u^{(i)}(s))}{ds^l}\right|_{s=0} = 0, \ l = 1, \cdots, \alpha_i, \quad \text{but} \quad \left.\frac{d^{\alpha_i+1} \lambda_i(u^{(i)}(s))}{ds^{\alpha_i+1}}\right|_{s=0} \neq 0, \tag{1.16}$$

or

$$\left.\frac{d^l\lambda_i(u^{(i)}(s))}{ds^l}\right|_{s=0} = 0,\ \ l = 1, 2, \cdots, \tag{1.17}$$

where $u = u^{(i)}(s)$ is defined by (1.8). In the case that (1.17) holds, we define $\alpha_i = +\infty$.

The following theorems are the main results in this paper.

Theorem 1.1 *Suppose that $A(u)$ is suitably smooth, the system* (1.1) *is strictly hperbolic and $F(u) \in C^2$ satisfies the matching condition in a neighbourhood of $u = 0$. Suppose furthermore that $\varphi(x) = \varepsilon\psi(x)$, where $\varepsilon > 0$ is a small parameter and $\psi(x)$ is a C^1 vector function satisfying*

$$\sup_{x\in\mathbf{R}}\{(1 + |x|)(|\psi(x)| + |\psi'(x)|)\} < \infty. \tag{1.18}$$

Suppose finally that the system (1.1) is not weakly linearly degenerate and

$$\alpha = \min\{\alpha_i | i \in J\} < +\infty, \tag{1.19}$$

where α_i is defined by (1.16)–(1.17). Let

$$J_1 = \{i | i \in J, \alpha_i = \alpha\}. \tag{1.20}$$

If there exists $i_0 \in J_1$ such that

$$l_{i_0}(0)\psi(x) \not\equiv 0 \tag{1.21}$$

where $l_{i_0}(u)$ stands for the i_0-th left eigenvector, then there exists $\varepsilon_0 > 0$ so small that for any given $\varepsilon \in (0, \varepsilon_0]$, the first order derivative u_x of the C^1 solution $u = u(t, x)$ to the Cauchy problem (1.1) and (1.7) must blow up in a finite time and the life-span $\tilde{T}(\varepsilon)$ of $u = u(t, x)$ satisfies

$$\lim_{\varepsilon\to 0^+}(\varepsilon^{\alpha+1}\tilde{T}(\varepsilon)) = M_0, \tag{1.22}$$

where M_0 is a positive constant independent of ε, given by

$$M_0 = \left(\max_{i\in J_1}\sup_{x\in\mathbf{R}}\left\{-\frac{1}{\alpha!}\left.\frac{d^{\alpha+1}\lambda_i(u^{(i)}(s))}{ds^{\alpha+1}}\right|_{s=0}\cdot (l_i(0)\psi(x))^\alpha l_i(0)\psi'(x)\right\}\right)^{-1}, \tag{1.23}$$

in which $u = u^{(i)}(s)$ is defined by (1.8).

Theorem 1.2 *Under the assumptions of Theorem 1.1, on the existence domain $0 \le t < \tilde{T}(\varepsilon)$ of the C^1 solution $u = u(t, x)$ to Cauchy problem* (1.1) *and* (1.7), *the solution itself remains bounded and small, but the first order derivative u_x of $u = u(t, x)$ tends to infinity as $t \uparrow \tilde{T}(\varepsilon)$. Moreover, the singularity occurs at the starting point of the envelope of characteristics of the same family, i.e., at the point with minimum t-value on the envelope.*

Theorem 1.3 *Under the assumptions of Theorem* 1.1, *for each $i \notin J_1$, the family of i-th characteristics never forms any envelope on the domain $0 \le t \le \tilde{T}(\varepsilon)$. In* *particular, each family of weakly linearly degenerate characteristics and then each family of linearly degenerate characteristics never forms any envelope on* $0 \le t \le T(\varepsilon)$.

Remark 1.1 Theorems 1.2–1.4 still hold if

$$\varphi(x) = \varepsilon\psi(x) + \psi_1(x, \varepsilon), \tag{1.24}$$

where $\psi_1(x,\varepsilon)$ has the same decay property as in (1.18) and

$$\psi_1(x, \varepsilon), \frac{\partial \psi_1(x,\varepsilon)}{\partial x} = O(\varepsilon^2). \tag{1.25}$$

Remark 1.2 By Theorem 2.2 in [4], suppose that $\mu = 0$ in Theorem A, the C^1 solution $u = u(t,x)$ to Cauchy problem (1.1) and (1.7) may blow up in a finite time.

Remark 1.3 Theorem A can be proved in a simpler way by using Lemma 2.1 given in Section 2.

Remark 1.4 Suppose that in a neighbourhood of $u = 0$, the system (1.1) is strictly hyperbolic, $F(u)$ satisfies the matching condition and $A(u) \in C^1, F(u) \in C^1$. Suppose furthermore that for any given small and decaying initial data $\varphi(x) \in C^1$, Cauchy problem (1.1) and (1.8) always admits a unique global C^1 solution on $t \ge 0$. Then the system (1.1) must be weakly linearly degenerate.

Remark 1.4 can be proved as in the case $F(u) \equiv 0$ (cf. [7]).

2. Preliminaries

Let

$$v_i = l_i(u)u, \quad i = 1, \cdots, n \tag{2.1}$$

and

$$w_i = l_i(u)u_x, \quad i = 1, \cdots, n, \tag{2.2}$$

where $l_i(u) = (l_{i1}(u), \cdots, l_{in}(u))$ denotes the i-th left eigenvector.

By (1.5), it is easy to see that

$$u = \sum_{k=1}^{n} v_k r_k(u) \tag{2.3}$$

and

$$u_x = \sum_{k=1}^{n} w_k r_k(u). \tag{2.4}$$

Let

$$\frac{d}{d_i t} = \frac{\partial}{\partial t} + \lambda_i(u)\frac{\partial}{\partial x} \tag{2.5}$$

be the directional derivative with respect to t along the i-th characteristic. We have (cf.[8], [5] and [6])

$$\frac{dv_i}{d_i t} = \sum_{j=1}^{n}\left(\sum_{k=1}^{n} B_{ijk}(u)\rho_k(u)\right) v_j + \sum_{j,k=1}^{n} \beta_{ijk}(u) v_j w_k + \rho_i(u), \quad i = 1, \cdots, n, \tag{2.6}$$

where

$$B_{ijk}(u) = -l_i(u)\nabla r_j(u) r_k(u), \tag{2.7}$$

$$\rho_k(u) = l_k(u)F(u) \tag{2.8}$$

and

$$\beta_{ijk}(u) = (\lambda_k(u) - \lambda_i(u))l_i(u)\nabla r_j(u) r_k(u). \tag{2.9}$$

Hence, we have

$$\beta_{iji}(u) \equiv 0, \ \forall i, j, \tag{2.10}$$

and by (1.12), in normalized coordinates we have

$$\beta_{ijj}(u_j e_j) = 0, \ \ \forall |u_j| \text{ small}, \ \forall i, j. \tag{2.11}$$

On the other hand, we have (cf.[8], [5] and [6]),

$$\frac{dw_i}{d_i t} = \sum_{j=1}^{n}\left(\sum_{k=1}^{n} B_{ijk}(u)\rho_k(u) + \nu_{ij}(u)\right) w_j + \sum_{j,k=1}^{n} \gamma_{ijk}(u) w_j w_k, \ i = 1, \cdots, n, \tag{2.12}$$

where

$$\nu_{ij}(u) = l_i(u)\nabla F(u) r_j(u) \tag{2.13}$$

and

$$\gamma_{ijk}(u) = \frac{1}{2}\{(\lambda_j(u) - \lambda_k(u))l_i(u)\nabla r_k(u) r_j(u) - \nabla\lambda_k(u) r_j(u)\delta_{ik} + (j|k)\}, \tag{2.14}$$

where $(j|k)$ denotes all the terms obtained by changing j and k in the revious terms.

Hence

$$\gamma_{ijj}(u) \equiv 0, \ \forall j \neq i, \ \ i, j = 1, \cdots, n \tag{2.15}$$

and

$$\gamma_{iii}(u) = -\nabla\lambda_i(u) r_i(u), \ \ i = 1, \cdots, n. \tag{2.16}$$

When the i-th characteristic $\lambda_i(u)$ is linearly degenerate in the sense of Lax, we have

$$\gamma_{iii}(u) \equiv 0; \tag{2.17}$$

while, $\lambda_i(u)$ is weakly linearly degenerate, in normalized coordinates we have

$$\gamma_{iii}(u_i e)i) \equiv 0, \ \ \forall |u_i| \text{ small}. \tag{2.18}$$

Lemma 2.1 *Suppose that* $A(u) \in C^2$*, the system* (1.1) *is strictly hyperbolic and* $F(u) \in C^2$ *satisfies the matching condition in a neighbourhood of* $u = 0$*. Then in normalized coordinates, we have*

$$\left|\sum_{j=1}^{n}\left(\sum_{k=1}^{n} B_{ijk}(u)\rho_k(u)\right) v_j + \rho_i(u)\right| \leq \sum_{j,k=1}^{n} |P_{ijk}(u) v_j v_k|, \ \ \forall |u| \text{ small}, \ \forall i, \tag{2.19}$$

where $P_{ijk}(u)$ *is bounded in a neighbourhood of* $u = 0$ *and*

$$P_{ijj}(u_j e_j) \equiv 0, \quad \forall |u_j| \text{ small}, \ \forall i, j; \tag{2.20}$$

on the other hand, we have

$$\left| \sum_{j=1}^{n} \left(\sum_{k=1}^{n} B_{ijk}(u)\rho_k(u) + \nu_{ij}(u) \right) w_j \right| \le \sum_{j,k=1}^{n} |Q_{ijk}(u) v_k w_j|, \ \forall |u| \text{ small}, \ \forall i, \tag{2.21}$$

where $Q_{ijk}(u)$ *is bounded in a neighbourhood of* $u = 0$ *and*

$$Q_{ijj}(u_j e_j) \equiv 0, \quad \forall |u_j| \text{ small}, \ \forall i, j. \tag{2.22}$$

Proof First we prove that

$$|\rho_i(u)| \le \sum_{j,k=1}^{n} |f_{ijk}(u) v_j v_k|, \quad \forall |u| \text{ small}, \ \forall i, \tag{2.23}$$

where $f_{ijk}(u)$ is bounded in a neighbourhood of $u = 0$ and

$$f_{ijj}(u_j e_j) \equiv 0, \quad \forall |u_j| \text{ small}, \ \forall i, j. \tag{2.24}$$

Note (1.10), (1.12) and (2.8), it is easy to see that

$$\rho_i(u_k e_k) \equiv 0, \quad \forall |u_k| \text{ small}, \ \forall i, k \tag{2.25}$$

and then

$$|\rho_i(u)| \le \sum_{m,l=1}^{n} |\tilde{f}_{iml} u_m u_l|, \quad \forall |u| \text{ small}, \ \forall i, \tag{2.26}$$

where $\tilde{f}_{iml}(i, m, l = 1, \cdots, n, m \ne l)$ are constants and

$$\tilde{f}_{imm} = 0, \quad i, m = 1, \cdots, n. \tag{2.27}$$

By (2.3) and (2.26), we get

$$\begin{aligned} |\rho_i(u)| &\le \sum_{m,l=1}^{n} |\tilde{f}_{iml}| \sum_{j=1}^{n} |v_j r_{jm}(u)| \sum_{k=1}^{n} |v_k r_{kl}(u)| \\ &= \sum_{j,k=1}^{n} \left(\sum_{m,l=1}^{n} |\tilde{f}_{iml} r_{jm}(u) r_{kl}(u)| \right) |v_j v_k|, \ \forall |u| \text{ small}, \ \forall i. \end{aligned} \tag{2.28}$$

Taking

$$f_{ijk}(u) = \sum_{m,l=1}^{n} |\tilde{f}_{iml} r_{jm}(u) r_{kl}(u)|, \quad i, j, k = 1, \cdots, n, \tag{2.29}$$

we get (2.23).

Note (1.12) and (2.27), it follows from (2.29) that

$$|f_{ijj}(u_je_j)| = \sum_{m,l=1}^n |\tilde{f}_{iml} r_{jm}(u_je_j) r_{jl}(u_je_j)| = \sum_{m,l=1}^n |\tilde{f}_{iml}\delta_{jm}\delta_{jl}|$$
$$= |\tilde{f}_{ijj}| = 0, \quad \forall |u| \text{ small}, \ \forall i, j, \tag{2.30}$$

which is nothing but (2.24).

Let

$$\tilde{\nu}_{ij}(u) = \sum_{k=1}^n B_{ijk}(u)\rho_k(u), \quad i, j = 1, \cdots, n. \tag{2.31}$$

Now we prove

$$|\tilde{\nu}_{ij}(u)| \le \sum_{k=1}^n |\tilde{P}_{ijk}(u) v_k|, \quad \forall |u| \text{ small}, \ \forall i, j, \tag{2.32}$$

where $\tilde{P}_{ijk}(u)$ is bounded in a neighbourhood of $u = 0$ and

$$\tilde{P}_{ijj}(u_je_j) \equiv 0, \quad \forall |u_j| \text{ small}, \ \forall i, j. \tag{2.33}$$

By (2.25), we have

$$\tilde{\nu}_{ij}(u_je_j) \equiv 0, \quad \forall |u_j| \text{ small}, \ \forall i, j \tag{2.34}$$

and then

$$|\tilde{\nu}_{ij}(u)| \le \sum_{m=1}^n |\hat{P}_{ijm} u_m|, \quad \forall |u| \text{ small}, \ \forall i, j, \tag{2.35}$$

where $\hat{P}_{ijm} (i, j, m = 1, \cdots, n, j \ne m)$ are constants and

$$\hat{P}_{ijj} = 0, \quad i, j = 1, \cdots, n. \tag{2.36}$$

Note (2.3), it follows from (2.35) that

$$|\tilde{\nu}_{ij}(u)| \le \sum_{m=1}^n |\hat{P}_{ijm}| \sum_{k=1}^n |v_k r_{km}(u)|$$
$$= \sum_{k=1}^n \left(\sum_{m=1}^n |\hat{P}_{ijm} r_{km}(u)| \right) |v_k|, \quad \forall |u| \text{ small}, \ \forall i, j. \tag{2.37}$$

Taking

$$\tilde{P}_{ijk}(u) = \sum_{m=1}^n |\hat{P}_{ijm} r_{km}(u)|, \quad i, j, k = 1, \cdots, n, \tag{2.38}$$

we get (2.32).

Note (1.12) and (2.36), it follows from (2.38) that

$$\tilde{P}_{ijj}(u_je_j) = \sum_{m=1}^n |\hat{P}_{ijm} r_{jm}(u_je_j)| = \sum_{m=1}^n |\hat{P}_{ijm}\delta_{jm}|$$
$$= |\hat{P}_{ijj}| = 0, \quad \forall |u_j| \text{ small}, \ \forall i, j, \tag{2.39}$$

which is just (2.33).

Let

$$P_{ijk}(u)=|f_{ijk}(u)|+|\tilde{P}_{ijk}(u)|,\quad i,j,k=1,\cdots,n. \tag{2.40}$$

Noting (2.23) and (2.32), we have (2.19); noting (2.24) and (2.33), we have (2.20). The proof of the first part of this lemma is finished.

Note (1.10), (1.12) and (2.13), it is easy to see that

$$\nu_{ij}(u_je_j)\equiv 0,\quad \forall |u_j| \text{ small},\ \forall i,j. \tag{2.41}$$

Let

$$\hat{\nu}_{ij}(u)=\sum_{k=1}^{n}B_{ijk}(u)\rho_k(u)+\nu_{ij}(u),\quad i,j=1,\cdots,n. \tag{2.42}$$

Noting (2.31), (2.34) and (2.41), we have

$$\hat{\nu}_{ij}(u_je_j)\equiv 0,\quad \forall |u_j| \text{ small},\ \forall i,j. \tag{2.43}$$

By (2.43), similarly to the proof of (2.32)–(2.33), we get

$$|\hat{\nu}_{ij}(u)|\le\sum_{k=1}^{n}|Q_{ijk}(u)v_k|,\quad \forall |u| \text{ small},\ \forall i,j, \tag{2.44}$$

where $Q_{ijk}(u)$ is bounded in a neighbourhood of $u=0$ and (2.22) holds.

This proves the last part of this lemma. The proof of Lemma 2.1 is finished.

For any given $y\ge 0$, on the existence domain of C^1 solution, let $x=\tilde{x}_i(t,y)$ denote the i-th characteristic passing through a point $(y/a,y)(a>0$, constant), we have

$$\begin{cases}\dfrac{d\tilde{x}_i(t,y)}{dt}=\lambda_i(u(t,\tilde{x}_i(t,y))),\\ \tilde{x}_i\left(\dfrac{y}{a},y\right)=y.\end{cases} \tag{2.45}$$

Let $p_i(t,x)$ be defined by

$$p_i(t,\tilde{x}_i(t,y))=v_i(t,\tilde{x}_i(t,y))\frac{\partial\tilde{x}_i(t,y)}{\partial y}. \tag{2.46}$$

It is easy to see that along the i-th characteristic $x=\tilde{x}_i(t,y)$, we have

$$\begin{aligned}\frac{dp_i}{d_it}=&\sum_{j=1}^{n}\left(\sum_{k=1}^{n}B_{ijk}(u)\rho_k(u)\right)v_j\frac{\partial\tilde{x}_i(t,y)}{\partial y}+\sum_{j,k=1}^{n}\tilde{\beta}_{ijk}(u)\frac{\partial\tilde{x}_i(t,y)}{\partial y}v_jw_k\\ &+\rho_i(u)\frac{\partial\tilde{x}_i(t,y)}{\partial y},\end{aligned} \tag{2.47}$$

where

$$\tilde{\beta}_{ijk}(u)=\beta_{ijk}(u)+\nabla\lambda_i(u)r_k(u)\delta_{ij}, \tag{2.48}$$

By (2.11), in normalized coordinates we have

$$\tilde{\beta}_{ijj}(u_je_j) \equiv 0, \quad \forall |u_j| \text{ small}, \ \forall j \neq i \tag{2.49}$$

and, when the i-th characteristic $\lambda_i(u)$ is weakly linearly degenerate, we have

$$\tilde{\beta}_{iii}(u_ie_i) \equiv 0, \quad \forall |u_i| \text{ small}. \tag{2.50}$$

Moreover, by (2.10), we have

$$\tilde{\beta}_{iji}(u) \equiv 0, \quad \forall j \neq i; \tag{2.51}$$

while

$$\tilde{\beta}_{iii}(u) = \nabla\lambda_i(u)r_i(u), \tag{2.52}$$

which identically vanishes only in the case that $\lambda_i(u)$ is linearly degenerate in the sense of Lax.

Similarly, define $q_i(t,x)$ by

$$q_i(t, \tilde{x}_i(t,y)) = w_i(t, \tilde{x}_i(t,y))\frac{\partial \tilde{x}_i(t,y)}{\partial y}. \tag{2.53}$$

We have

$$\begin{aligned}\frac{dq_i}{d_it} =& \sum_{j=1}^{n}\left(\sum_{k=1}^{n} B_{ijk}(u)\rho_k(u) + \nu_{ij}(u)\right) w_j \frac{\partial \tilde{x}_i(t,y)}{\partial y} \\ &+ \sum_{j,k=1}^{n} \tilde{\gamma}_{ijk}(u)w_jw_k\frac{\partial \tilde{x}_i(t,y)}{\partial y},\end{aligned} \tag{2.54}$$

where

$$\tilde{\gamma}_{ijk}(u) = \gamma_{ijk}(u) + \nabla\lambda_i(u)r_k(u)\delta_{ij}. \tag{2.55}$$

By (2.15)–(2.16), we have

$$\tilde{\gamma}_{ijj}(u) \equiv 0, \quad \forall i. \tag{2.56}$$

3. Proof of Theorem 1.1

Consider the following Cauchy problem

$$\begin{cases} \dfrac{\partial u}{\partial t} + A(u)\dfrac{\partial u}{\partial x} = F(u), \\ t = 0: \quad u = \varepsilon\psi(x), \end{cases} \tag{3.1}$$

where $\varepsilon > 0$ is a small parameter and $\psi(x)$ is a C^1 vector function satisfying

$$\sup_{x\in\mathbf{R}}\{(1+|x|)(|\psi(x)| + |\psi'(x)|)\} < \infty. \tag{3.2}$$

Using Lemma 2.1, we will prove Theorem 1.1 in a way similar to the proof of Theorem 1.3 in [4].

As in [1], we may suppose that

$$0 < \lambda_1(0) < \lambda_2(0) < \cdots < \lambda_n(0). \tag{3.3}$$

By (3.3), there exist positive constants $\delta > 0$ and $\delta_0 > 0$ so small that

$$\lambda_{i+1}(u) - \lambda_i(v) \geq 4\delta_0, \quad \forall |u|, |v| \leq \delta, \quad i = 1, \cdots, n-1 \tag{3.4}$$

and

$$|\lambda_i(u) - \lambda_i(v)| \leq \frac{\delta_0}{2}, \quad \forall |u|, |v| \leq \delta, \ i = 1, \cdots, n. \tag{3.5}$$

For the time being we suppose that on the existence domain $0 \leq t \leq T$(with $T\varepsilon^{\alpha+\frac{4}{3}} \leq 1$) of the C^1 solution $u = u(t,x)$ to Cauchy problem (3.1) we have

$$|u(t,x)| \leq \delta. \tag{3.6}$$

At the end of the proof of Lemma 3.3, we will explain that this hypothesis is reasonable.

By (3.4) and (3.6), on the existence domain $0 \leq t \leq T$ (with $T\varepsilon^{\alpha+\frac{4}{3}} \leq 1$) of the C^1 solution $u = u(t,x)$ to (3.1) we have

$$0 < \lambda_1(u) < \lambda_2(u) < \cdots < \lambda_n(u). \tag{3.7}$$

For any fixed $T > 0$, let

$$D_+^T = \{(t,x) | 0 \leq t \leq T, x \geq (\lambda_n(0) + \delta_0)t\}, \tag{3.8}$$

$$D_-^T = \{(t,x) | 0 \leq t \leq T, x \leq (\lambda_1(0) - \delta_0)t\}, \tag{3.9}$$

$$D^T = \{(t,x) | 0 \leq t \leq T, (\lambda_1(0) - \delta_0)t \leq x \leq (\lambda_n(0) + \delta_0)t\} \tag{3.10}$$

and

$$\begin{aligned} D_i^T =& \{(t,x) | 0 \leq t \leq T, -[\delta_0 + \eta(\lambda_i(0) - \lambda_1(0))]t \leq x - \lambda_i(0)t \\ \leq& [\delta_0 + \eta(\lambda_n(0) - \lambda_i(0))]t\}, \ i = 1, \cdots, n, \end{aligned} \tag{3.11}$$

where $\eta > 0$ is suitably small.

Note that $\eta > 0$ is small, by (3.4) it is easy to see that

$$D_i^T \bigcap D_j^T = \emptyset, \quad \forall i \neq j \tag{3.12}$$

and

$$\bigcup_{i=1}^n D_i^t \subset D^T. \tag{3.13}$$

Let

$$V(D_{\pm}^T) = \max_{i=1,\cdots,n} \|(1+|x|)v_i(t,x)\|_{L^\infty(D_{\pm}^T)}, \tag{3.14}$$

$$W(D_{\pm}^T) = \max_{i=1,\cdots,n} \|(1+|x|)w_i(t,x)\|_{L^\infty(D_{\pm}^T)}, \tag{3.15}$$

$$V_\infty^c(T) = \max_{i=1,\cdots,n} \sup_{(t,x)\in D^T\backslash D_i^T} (1+|x-\lambda_i(0)t|)|v_i(t,x)|, \tag{3.16}$$

$$U_\infty^c(T) = \max_{i=1,\cdots,n} \sup_{(t,x)\in D^T\backslash D_i^T} (1+|x-\lambda_i(0)t|)|u_i(t,x)|, \tag{3.17}$$

$$W_\infty^c(T) = \max_{i=1,\cdots,n} \sup_{(t,x)\in D^T\backslash D_i^T} (1+|x-\lambda_i(0)t|)|w_i(t,x)|, \tag{3.18}$$

$$V_1(T) = \max_{i=1,\cdots,n} \sup_{0\le t\le T} \int_{D_i^T(t)} |v_i(t,x)|dx, \tag{3.19}$$

$$W_1(T) = \max_{i=1,\cdots,n} \sup_{0\le t\le T} \int_{D_i^T(t)} |w_i(t,x)|dx, \tag{3.20}$$

$$V_\infty(T) = \max_{i=1,\cdots,n} \sup_{\substack{0\le t\le T\\ x\in\mathbf{R}}} |v_i(t,x)| \tag{3.21}$$

and

$$W_\infty(T) = \max_{i=1,\cdots,n} \sup_{\substack{0\le t\le T\\ x\in\mathbf{R}}} |w_i(t,x)|, \tag{3.22}$$

where $D_i^T(t)(t\ge 0)$ denotes the t-section of D_i^T:

$$D_i^T(t) = \{(\tau,x)|\tau=t,(\tau,x)\in D_i^T\}. \tag{3.23}$$

Obviously, $V_\infty(T)$ is equivalent to

$$U_\infty(T) = \max_{i=1,\cdots,n} \sup_{\substack{0\le t\le T\\ x\in\mathbf{R}}} |u_i(t,x)|. \tag{3.24}$$

It is easy to see that Lemma 3.1 in [1] is still valid, namely

Lemma 3.1 *For each $i=1,\cdots,n$, on the domain $D^T\backslash D_i^T$, we have*

$$ct \le |x-\lambda_i(0)t| \le Ct, \quad cx \le |x-\lambda_i(0)t| \le Cx, \tag{3.25}$$

where c and C are positive constants independent of (t,x) and T.

In the present situation, Lemma 3.2 in [4] is still valid, namely

Lemma 3.2 *Suppose that in a neighbourhood of $u=0, A(u)\in C^2$, the system (1.1) is strictly hyperbolic and $F(u)\in C^2$ satisfies the matching condition. Then in normalized coordinates there exists $\varepsilon_0>0$ so small that for any fixed $\varepsilon\in(0,\varepsilon_0]$, on any given existence domain $0\le t\le T$ of the C^1 solution $u=u(t,x)$ to the Cauchy problem (3.1), there exists a positive constant k_1 independent of ε and T, such that the following uniform a priori estimates hold:*

$$V(D_{\pm}^T), W(D_{\pm}^T) \le k_1\varepsilon. \tag{3.26}$$

Lemma 3.3 *Under the assumptions of Theorem* 1.1*, in normalized coordinates there exists* $\varepsilon_0 > 0$ *so small that for any fixed* $\varepsilon \in (0, \varepsilon_0]$*, on any given existence domain* $0 \le t \le T$ *of the* C^1 *solution* $u = u(t,x)$ *to Cauchy problem* (3.1)*, there exist positive constants* $k_i (i = 2, \cdots, 8)$ *independent of* ε *and* T*, such that the following uniform a priori estimates hold:*

$$W_\infty^c(T) \le k_2 \varepsilon, \tag{3.27}$$

$$W_1(T) \le k_3 \varepsilon |\log \varepsilon|, \tag{3.28}$$

$$V_\infty^c(T) \le k_4 \varepsilon, \tag{3.29}$$

$$V_1(T) \le K_5 \varepsilon |\log \varepsilon| + k_6 (\varepsilon |\log \varepsilon|)^{2+\alpha} T, \tag{3.30}$$

$$V_\infty(T), U_\infty(T) \le k_7 \varepsilon |\log \varepsilon|, \tag{3.31}$$

where

$$T \varepsilon^{\alpha + \frac{4}{3}} \le 1. \tag{3.32}$$

Moreover,

$$W_\infty(T) \le k_8 \varepsilon, \tag{3.33}$$

where

$$T \varepsilon^{\alpha + \frac{3}{4}} \le 1. \tag{3.34}$$

Proof This lemma will be proved in a way similar to the proof of Lemma 5.1 and Lemma 5.2 in [4]. In what follows we only point out the essentially different part in the proof and $\varepsilon_0 > 0$ is always supposed to be suitably small.

Similarly to (3.60) in [4], on any given existence domain of the C^1 solution, when $\delta > 0$ is suitably small, we have

$$U_\infty^c(T) \le c_1 V_\infty^c(T), \tag{3.35}$$

henceforth, $c_j (j = 1, 2, \cdots)$ will denote positive constants independent of ε and T.

Let

$$\tilde{V}_1(T) = \max_{i=1,\cdots,n} \max_{j \ne i} \sup_{\tilde{c}_j} \int_{\tilde{c}_j} |v_i(t,x)| dt, \tag{3.36}$$

$$\tilde{W}_1(T) = \max_{i=1,\cdots,n} \max_{j \ne i} \sup_{\tilde{c}_j} \int_{\tilde{c}_j} |w_i(t,x)| dt|, \tag{3.37}$$

where $\tilde{c}_j (j \ne i)$ stands for any given j-th characteristic in D_i^T.

As in the proof of Lemma 3.3 in [4], we first estimate $\tilde{W}_1(T)$. Using (2.54) and

Lemma 2.1, instead of (3.32) in [4] we have

$$
\begin{aligned}
|q_i(t,\tilde{x}_i(t,y))|_{t=t(y)} \le & \left| w_i\left(\frac{y}{\lambda_n(0)+\delta_0}, y\right)\right| \\
& + c_2 \Bigg\{ W_\infty^c(T)(W_\infty^c(T)+V_\infty^c(T)) \int_{y/(\lambda_n(0)+\delta_0)}^{t(y)} (1+s)^{-1}(1+|x_i(s,y)|)^{-1}\frac{\partial \tilde{x}_i(s,y)}{\partial y} ds \\
& + (W_\infty^c(T)+V_\infty^c(T)) \sum_{k=1}^n \int_{(s,\tilde{x}_i(s,y))\in D_k^T} (1+s)^{-1}|w_k(s,\tilde{x}_i(s,y))| \frac{\partial \tilde{x}_i(s,y)}{\partial y} ds \\
& + W_\infty^c(T) \sum_{j=1}^n \int_{(s,\tilde{x}_i(s,y))\in D_j^T} (1+s)^{-1}|v_j(s,\tilde{x}_i(s,y))| \frac{\partial \tilde{x}_i(s,y)}{\partial y} ds \Bigg\}. \qquad (3.38)
\end{aligned}
$$

Thus, using Lemma 3.2, similarly to (5.5) in [4], we get

$$
\begin{aligned}
\tilde{W}_1(T) \le & c_3\{k_1\varepsilon \log(1+T) + (W_\infty^c(T)\log(1+T))^2 \\
& + W_\infty^c(T)W_1(T)\log(1+T) + W_\infty^c(T)V_\infty^c(T)(\log(1+T))^2 \\
& + V_\infty^c(T)W_1(T)\log(1+T) + W_\infty^c(T)V_1(T)\log(1+T)\}, \qquad (3.39)
\end{aligned}
$$

where k_1 is given by Lemma 3.2.

Similarly, usinig Lemma 2.1 and Lemma 3.2, instead of (5.6)–(5.7) and (5.15)–(5.17) in [4] we have

$$
\begin{aligned}
W_1(T) \le & c_4\{k_1\varepsilon \log(1+T) + (W_\infty^c(T)\log(1+T))^2 \\
& + W_\infty^c(T)W_1(T)\log(1+T) + W_\infty^c(T)V_\infty^c(T)(\log(1+T))^2 \\
& + V_\infty^c(T)W_1(T)\log(1+T) + W_\infty^c(T)V_1(T)\log(1+T)\}, \qquad (3.40) \\
W_\infty^c(T) \le & c_5\{k_1\varepsilon + (W_\infty^c(T))^2\log(1+T) + W_\infty^c(T)\tilde{W}_1(T) \\
& + W_\infty^c(T)V_\infty^c(T)\log(1+T) + W_\infty^c(T)\tilde{V}_1(T) + V_\infty^c(T)\tilde{W}_1(T)\}, \qquad (3.41) \\
\tilde{V}_1(T) \le & c_6\{k_1\varepsilon \log(1+T) + W_\infty^c(T)V_\infty^c(T)(\log(1+T))^2 \\
& + W_\infty^c(T)V_1(T)\log(1+T) + V_\infty^c(T)W_1(T)\log(1+T) \\
& + (V_\infty^c(T)\log(1+T))^2 + V_\infty^c(T)V_1(T)\log(1+T) \\
& + (V_\infty(T))^{1+\alpha}(W_\infty^c(T)\log(1+T) + W_1(T))T\}, \qquad (3.42) \\
V_1(T) \le & c_7\{k_1\varepsilon \log(1+T) + W_\infty^c(T)V_\infty^c(T)(\log(1+T))^2 \\
& + W_\infty^c(T)V_1(T)\log(1+T) + V_\infty^c(T)W_1(T)\log(1+T) \\
& + (V_\infty^c(T)\log(1+T))^2 + V_\infty^c(T)V_1(T)\log(1+T) \\
& + (V_\infty(T))^{1+\alpha}(W_\infty^c(T)\log(1+T) + W_1(T))T\} \qquad (3.43)
\end{aligned}
$$

and

$$
\begin{aligned}
V_\infty^c(T) \le & c_8\{k_1\varepsilon + W_\infty^c(T)V_\infty^c(T)\log(1+T) + W_\infty^c(T)\tilde{V}_1(T) \\
& + V_\infty^c(T)\tilde{W}_1(T) + (V_\infty^c(T))^2\log(1+T) + V_\infty^c(T)\tilde{V}_1(T)\}. \qquad (3.44)
\end{aligned}
$$

When T satisfies (3.32), it follows from (3.39)–(3.44) that

$$\begin{aligned}\tilde{W}_1(T), W_1(T) \leq & c_9\{k_1\varepsilon|\log\varepsilon| + (W^c_\infty(T)|\log\varepsilon|)^2 + W^c_\infty(T)W_1(T)|\log\varepsilon| \\ & + W^c_\infty(T)V^c_\infty(T)|\log\varepsilon|^2 + V^c_\infty(T)W_1(T)|\log\varepsilon| \\ & + W^c_\infty(T)V_1(T)|\log\varepsilon|\}, \qquad (3.45)\\ W^c_\infty(T) \leq & c_{10}\{k_1\varepsilon + (W^c_\infty(T))^2|\log\varepsilon| + W^c_\infty(T)\tilde{W}_1(T) \\ & + W^c_\infty(T)V^c_\infty(T)|\log\varepsilon| + W^c_\infty(T)\tilde{V}_1(T) + V^c_\infty(T)\tilde{W}_1(T)\}, \qquad (3.46)\\ \tilde{V}_1(T), V_1(T) \leq & c_{11}\{k_1\varepsilon|\log\varepsilon| + W^c_\infty(T)V^c_\infty(T)|\log\varepsilon|^2 + W^c_\infty(T)V_1(T)|\log\varepsilon| \\ & + V^c_\infty(T)W_1(T)|\log\varepsilon| + (V^c_\infty(T)|\log\varepsilon|)^2 + V^c_\infty(T)V_1(T)|\log\varepsilon| \\ & + (V_\infty(T))^{1+\alpha}(W^c_\infty(T)|\log\varepsilon| + W_1(T))T\} \qquad (3.47)\end{aligned}$$

and

$$\begin{aligned}V^c_\infty(T) \leq & c_{12}\{k_1\varepsilon + W^c_\infty(T)V^c_\infty(T)|\log\varepsilon| + W^c_\infty(T)\tilde{V}_1(T) + V^c_\infty(T)\tilde{W}_1(T) \\ & + (V^c_\infty(T))^2|\log\varepsilon| + V^c_\infty(T)\tilde{V}_1(T)\}. \qquad (3.48)\end{aligned}$$

Noting Lemma 3.2, similarly to (3.48) in [4], we have

$$V_\infty(T), U_\infty(T) \leq c_{13}(k_1\varepsilon + W^c_\infty(T) + W_1(T)). \qquad (3.49)$$

By (3.2), it is easy to see that

$$\begin{aligned}& V^c_\infty(0), W^c_\infty(0) \leq c_{14}\varepsilon, \quad V_1(0) = \tilde{V}_1(0) = W_1(0) = \tilde{W}_1(0) = 0, \\ & V_\infty(0), U_\infty(0) \leq c_{15}\varepsilon.\end{aligned} \qquad (3.50)$$

Then, by continuity, there exist positive constants $k_i (i = 2, 3, \cdots, 7)$ independent of ε, such that (3.27)–(3.31) and

$$\tilde{V}_1(T) \leq k_5\varepsilon|\log\varepsilon| + k_6(\varepsilon|\log\varepsilon|)^{2+\alpha}T, \qquad (3.51)$$

$$\tilde{W}_1(T) \leq k_3\varepsilon|\log\varepsilon|, \qquad (3.52)$$

hold at least for $T > 0$ suitably small.

Thus, in order to prove (3.27)–(3.31), it suffices to show that we can choose $k_i(i = 2, 3, \cdots, 7)$ in such a way that for any fixed $T_0 (0 < T_0 \leq T$ with $T_0\varepsilon^{\alpha+\frac{4}{3}} \leq 1)$,such that

$$W^c_\infty(T_0) \leq 2k_2\varepsilon, \qquad (3.53)$$

$$W_1(T_0), \tilde{W}_1(T_0) \leq 2k_3\varepsilon|\log\varepsilon|, \qquad (3.54)$$

$$V^c_\infty(T_0) \leq 2k_4\varepsilon, \qquad (3.55)$$

$$V_1(T_0), \tilde{V}_1(T_0) \leq 2(k_5\varepsilon|\log\varepsilon| + k_6(\varepsilon|\log\varepsilon|)^{2+\alpha}T_0) \qquad (3.56)$$

and

$$V_\infty(T_0), U_\infty(T_0) \leq 2k_7\varepsilon|\log\varepsilon|, \qquad (3.57)$$

we have

$$W_\infty^c(T_0) \le k_2\varepsilon, \tag{3.58}$$
$$W_1(T_0), \tilde W_1(T_0) \le k_3\varepsilon|\log\varepsilon|, \tag{3.59}$$
$$V_\infty^c(T_0) \le k_4\varepsilon, \tag{3.60}$$
$$V_1(T_0), \tilde V_1(T_0) \le k_5\varepsilon|\log\varepsilon| + k_6(\varepsilon|\log\varepsilon|)^{2+\alpha}T_0 \tag{3.61}$$

and

$$V_\infty(T_0), U_\infty(T_0) \le k_7\varepsilon|\log\varepsilon|. \tag{3.62}$$

Substituting (353)–(3.57) into the right hand side of (3.45)–(3.49) (in which we take $T = T_0$), we get

$$W_1(T_0), \tilde W_1(T_0) \le c_9\left\{2k_1 + 4k_2k_6\varepsilon^{\frac{2}{3}}|\log\varepsilon|^{2+\alpha}\right\}\varepsilon|\log\varepsilon| \le 3c_9k_1\varepsilon|\log\varepsilon|, \tag{3.63}$$
$$W_\infty^c(T_0) \le c_{10}\left\{2k_1 + 4k_2k_6\varepsilon^{\frac{2}{3}}|\log\varepsilon|^{2+\alpha}\right\}\varepsilon \le 3c_{10}k_1\varepsilon, \tag{3.64}$$
$$\begin{aligned} V_1(T_0), \tilde V_1(T_0) \le & c_{11}\{[2k_1 + 4(k_2+k_4)k_6\varepsilon^{\frac{2}{3}}|\log\varepsilon|^{2+\alpha}]\varepsilon|\log\varepsilon| \\ & + 2(k_2+k_3)(2k_7)^{1+\alpha}(\varepsilon|\log\varepsilon|)^{2+\alpha}T_0\} \\ \le & c_{11}\{3k_1\varepsilon|\log\varepsilon| + 2(k_2+k_3)(2k_7)^{1+\alpha}(\varepsilon|\log\varepsilon|)^{2+\alpha}T_0\}, \end{aligned} \tag{3.65}$$
$$V_\infty^c(T_0) \le c_{12}\left\{2k_1 + 4(k_2+k_4)k_6\varepsilon^{\frac{2}{3}}|\log\varepsilon|^{2+\alpha}\right\}\varepsilon \le 3c_{12}k_1\varepsilon \tag{3.66}$$

and

$$V_\infty(T_0), U_\infty(T_0) \le c_{13}\{k_1\varepsilon + 2k_2\varepsilon + 2k_3\varepsilon|\log\varepsilon|\} \le 3c_{13}k_3\varepsilon|\log\varepsilon|. \tag{3.67}$$

Hence, taking

$$\begin{aligned} & k_2 \ge 3c_{10}k_1, k_3 \ge 3c_9k_1, k_4 \ge 3c_{12}k_1, k_5 \ge 3c_{11}k_1, \\ & k_7 \ge 3c_{13}k_3, k_6 \ge 2c_{11}(k_2+k_3)(2k_7)^{1+\alpha}, \end{aligned}$$

we get (3.58)–(3.62). This proves (3.27)–(3.31).

Here we point out that when $\varepsilon_0 > 0$ is suitably small, we have

$$U_\infty(T) \le k_7\varepsilon|\log\varepsilon| \le k_7\varepsilon_0|\log\varepsilon_0| \le \frac{1}{2}\delta. \tag{3.68}$$

This implies the validity of hypothesis (3.6).

Finally, we prove (3.33). By (2.12) and Lemma 2.1, instead of (5.20) in [4] we have

$$\begin{aligned} |w_i(t,x)| \le & c_{16}\{W(D_+^T) + [1+V_\infty(T)](W_\infty^c(T))^2 \\ & + [1+V_\infty(T)]W_\infty^c(T)V_\infty^c(T) + W_\infty^c(T)V_\infty(T)\log(1+T) \\ & + [W_\infty^c(T)\log(1+T) + V_\infty^c(T)\log(1+T) + V_\infty^c(T)V_\infty(T)\log(1+T)]W_\infty(T) \\ & + V_\infty^c(T)(W_\infty(T))^2\log(1+T) + (V_\infty(T))^\alpha(W_\infty(T))^2T\}. \end{aligned} \tag{3.69}$$

Noting Lemma 3.2, by (3.27), (3.29) and (3.31), we get (5.21) in [4] from (3.69), namely

$$W_\infty(T) \leq c_{17}\{\varepsilon(1+|\log\varepsilon|W_\infty(T)+|\log\varepsilon|(W_\infty(T))^2)+(\varepsilon|\log\varepsilon|)^\alpha T(W_\infty(T))^2\} \quad (3.70)$$

where $T \leq \varepsilon^{-\left(\alpha+\frac{4}{3}\right)}$.

Then, completely repeating the procedure of proving (5.13) in [4], we can prove (3.33). This completes the proof of Lemma 3.3.

By the existence and uniqueness of local C^1 solution to Cauchy problem(cf.[9]), by Lemma 3.3, similarly to Remark 5.1 and Remark 5.2 in [4], we have

Remark 3.1 When $\varepsilon_0 > 0$ is suitably small, for any fixed $\varepsilon \in (0, \varepsilon_0]$, Cauchy problem (3.1) admits a unique C^1 solution $u = u(t, x)$ on $0 \leq t \leq \varepsilon^{-\left(\alpha+\frac{3}{4}\right)}$. Hence, we get the following lower bound on the life-span $\tilde{T}(\varepsilon)$ of C^1 solution

$$\tilde{T}(\varepsilon) \geq \varepsilon^{-\left(\alpha+\frac{3}{4}\right)}. \quad (3.71)$$

Remark 3.2 Similarly to the proof of (3.33) under (3.34), we can easily prove that for any fixed $\mu \in \tilde{(}0, 1)$, there exists $\varepsilon_0 = \varepsilon_0(\tilde{\mu}) > 0$ so small that for any fixed $\varepsilon \in (0, \varepsilon_0]$, Cauchy problem (3.1) admits a unique C^1 solution $u = u(t, x)$ on $0 \leq t \leq \varepsilon^{-(\alpha+\tilde{\mu})}$. Hence, we have

$$\tilde{T}(\varepsilon) \geq \varepsilon^{-(\alpha+\tilde{\mu})} \quad (3.72)$$

where $0 < \tilde{\mu} < 1$.

Using Lemma 3.2, Lemma 3.3 and Remark 3.1, almost completely repeating the proof of Theorem 1.1 in [2] or Theorem 1.3 in [4], we can easily show Theorem 1.1. In what follows we only point out the essentially different part in the proof and will directly use the notation and results given in [2] and [4].

Proof of Theorem 1.1 As in [4], it suffices to prove Lemma 3.1 in [2], i.e., in normalized coordinates $\tilde{u}$ satisfying (1.14), to prove

$$\overline{\lim_{\varepsilon\to 0}}\{\varepsilon^{\alpha+1}\tilde{T}(\varepsilon)\} \leq M_0 \quad (1.22a)$$

and

$$\underline{\lim_{\varepsilon\to 0}}\ \{\varepsilon^{\alpha+1}\tilde{T}(\varepsilon)\} \geq M_0, \quad (1.22b)$$

where M_0 is given by (1.23). As in [2] and [4], we still denote $\tilde{u}$ by u. Moreover, in the proof, $\varepsilon_0 > 0$ is always supposed to be suitably small.

(1) **Proof of (1.22a)**

By Remark 3.1, there exists $\varepsilon_0 > 0$ so small that for any fixed $\varepsilon \in (0, \varepsilon_0]$, Cauchy problem (3.1) admits a unique C^1 solution $u = u(t, x)$ on the domain $0 \leq t \leq T_1$, where

$$T_1 \triangleq \varepsilon^{-\left(\alpha+\frac{3}{4}\right)} \leq \tilde{T}(\varepsilon) - 1 \triangleq \bar{T}. \quad (3.73)$$

Similarly to (5.26) in [4], we may suppose that

$$\tilde{T}(\varepsilon)\varepsilon^{\alpha+\frac{4}{3}} \leq 1. \quad (3.74)$$

Thus, in what follows we only discuss the problem in the domain $0 \le t \le \varepsilon^{-(\alpha+\frac{4}{3})}$. By Lemma 3.3, on the domain $0 \le t \le \varepsilon^{-(\alpha+\frac{4}{3})}$, we have

$$u(t,x), v(t,x) = O(\varepsilon|\log\varepsilon|). \tag{3.75}$$

By (3.2), there exist $i_0 \in J_1$ and $x_0 \in R$ such that (3.19) in [2] is still valid. Without loss of generality, we may assume $i_0 = 1 \in J_1$.

By (2.12), on the existence domain of C^1 solution, along the 1st characteristic $x = x_1(t, x_0)$ passing through point $(0, x_0)$, (3.22) in [2] should be rewritten as

$$\frac{dw_1}{d_1 t} = \bar{a}_0(t)w_1^2 + \bar{a}_1(t)w_1 + \bar{a}_2(t), \tag{3.76}$$

where

$$\begin{cases} \bar{a}_0(t) = a_0(t), \\ \bar{a}_1(t) = a_1(t) + \displaystyle\sum_{k=1}^{n} B_{11k}(u)\rho_k(u) + \nu_{11}(u), \\ \bar{a}_2(t) = a_2(t) + \displaystyle\sum_{j=2}^{n}\left(\sum_{k=1}^{n} B_{1jk}(u)\rho_k(u) + \nu_{1j}(u)\right) w_j, \end{cases} \tag{3.77}$$

in which $a_0(t), a_1(t)$ and $a_2(t)$ are defined by (3.23) in [2].

When $\varepsilon_0 > 0$ is suitably small, by (3.73), we have

$$T_1 > \varepsilon^{-\alpha} \overset{\triangle}{=} T_0 > t_0 \tag{3.78}$$

(see [2] for the definition of t_0).

By Lemma 3.3, on $0 \le t \le T_0$, we have

$$w(t,x), \frac{\partial u}{\partial x}(t,x) = O(\varepsilon). \tag{3.79}$$

Integrating

$$\frac{dw_1}{d_1 t} = \sum_{j=1}^{n}\left(\sum_{k=1}^{n} B_{1jk}(u)\rho_k(u) + \nu_{1j}(u)\right) w_j + \sum_{j,k=1}^{n} \gamma_{1jk}(u)w_j w_k, \tag{3.80}$$

along the characteristic $x = x_1(t, x_0)$, noting Lemma 2.1, we get

$$\begin{aligned} |w_1(t, x_1(t,x_0)) - w_1(0,x_0)| \le & \left|\int_0^t \sum_{j,k=1}^{n} \gamma_{1jk}(u)w_j w_k ds\right| + \int_0^{t_0} \sum_{j,k=1}^{n} |Q_{1jk}(u)v_k w_j| ds \\ & + \int_{t_0}^{t} \sum_{j\ne k} |Q_{1jk}(u)v_k w_j| ds + \int_{t_0}^{t} \sum_{j=1}^{n} |(Q_{1jj}(u) \\ & - Q_{1jj}(u_j e_j))v_j w_j| ds, \quad \forall t \in [0, T_0]. \end{aligned} \tag{3.81}$$

By Lemma 3.2 and Lemma 3.3, noting (3.35), similarly to (4.62) in [1] and (3.25) in [2], instead of (5.28) iin [4] we have

$$\begin{aligned}
&|w_1(t,x_1(t,x_0))-w_1(0,x_0)|\\
&\quad\le c_{18}\{\varepsilon^2+W_\infty(t)W_\infty^c(t)\log(1+t)+(W_\infty(t))^2V_\infty^c(t)\log(1+t)\\
&\qquad+(V_\infty(t))^\alpha(W_\infty(t))^2t+V_\infty(t)W_\infty(t)+V_\infty^c(t)W_\infty(t)\log(1+t)\\
&\qquad+W_\infty^c(t)V_\infty(t)\log(1+t)+V_\infty^c(t)V_\infty(t)W_\infty(t)\log(1+t)\}\\
&\quad\le c_{19}\varepsilon^2|\log\varepsilon|^{\max\{2,\alpha\}},\quad \forall t\in[0,T_0]
\end{aligned}\tag{3.82}$$

and then, noting (3.15) in [2], instead of (5.29) in [4] we have

$$w_1(T_0,x_1(T_0,x_0))=\varepsilon l_1(0)\psi'(x_0)+O(\varepsilon^2|\log\varepsilon|^{\max\{2,\alpha\}}).\tag{3.83}$$

Noting (2.6), (2.10), using Lemma 2.1 and Lemma 3.3, similarly to (4.49) in [1], we have

$$\begin{aligned}
&|v_1(t,x_1(t,x_0))-v_1(0,x_0)|\\
&\le\left|\int_0^t\sum_{k\neq1}\beta_{1jk}(u)v_jw_ks,x_1(s,x_0))ds\right|+\int_0^t\sum_{j,k=1}^n|P_{1jk}(u)v_jv_k|(s,x_1(s,x_0))|ds\\
&\le c_{20}\varepsilon^2|\log\varepsilon|^2+\left|\int_{t_0}^t\sum_{k\neq1}\beta_{1jk}(u)v_jw_kds\right|+\int_{t_0}^t\sum_{j\neq k}|P_{1jk}(u)v_jv_k|ds\\
&\quad+\int_{t_0}^t\sum_{j=1}^n|(P_{1jj}(u)-P_{1jj}(u_je_j))v_j^2|ds.
\end{aligned}\tag{3.84}$$

Then, by Lemma 3.1 and Lemma 3.3, noting (3.35), instead of (4.50) in [1] we have

$$\begin{aligned}
|v_1(t,x_1(t,x_0))-v_1(0,x_0)|\le&c_{20}\varepsilon^2|\log\varepsilon|^2+c_{21}\{V_\infty(t)W_\infty^c(t)\log(1+t)\\
&+V_\infty(t)V_\infty^c(t)\log(1+t)+(V_\infty(t))^2V_\infty^c(t)\log(1+t)\}\\
\le&c_{22}\varepsilon^2|\log\varepsilon|^2.
\end{aligned}\tag{3.85}$$

Noting (4.51) in [1] and (3.75), instead of (4.52) in [1] we have

$$|u_1(t,x)-v_1(t,x)|\le c_{23}\varepsilon^2|\log\varepsilon|^2.\tag{3.86}$$

Hence, noting (3.14) in [2], instead of (5.33) in [4] we have

$$|u_1(t,x_1(t,x_0))-\varepsilon l_1(0)\psi(x_0)|\le c_{24}\varepsilon^2|\log\varepsilon|^2.\tag{3.87}$$

By (3.87), repeating the corresponding procedure in [4], we can easily get (5.35) in [4], namely

$$\bar a_0(t)=a_0(t)\ge\frac12 b\varepsilon^\alpha>0,\quad\forall t\in[T_0,\bar T],\tag{3.88}$$

where b is defined by (3.21) in [2].

Noting (5.36)–(5.37) in [4], (3.77) and (3.35), using Lemma 2.1 and Lemma 3.3, similarly to (5.36)–(5.39) in [4], we have

$$\begin{aligned}\int_{T_0}^{\bar{T}} |\bar{a}_1(t)|dt \leq& \int_{T_0}^{\bar{T}} |a_1(t)|dt + \int_{T_0}^{\bar{T}} \sum_{k=2}^{n} |Q_{11k}(u)v_k(t, x_1(t, x_0))|dt \\ &+ \int_{T_0}^{\bar{T}} |(Q_{111}(u) - Q_{111}(u_1 e_1))v_1(t, x_1(t, x_0))|dt \\ \leq& C_{25}\varepsilon|\log\varepsilon| + c_{26}\left\{V_\infty^c(\bar{T})\int_{T_0}^{\bar{T}}(1+t)^{-1}dt + V_\infty^c(\bar{T})V_\infty(\bar{T})\int_{T_0}^{\bar{T}}(1+t)^{-1}dt\right\} \\ \leq& c_{27}\varepsilon|\log\varepsilon| \qquad (3.89)\end{aligned}$$

$$\begin{aligned}\int_{T_0}^{\bar{T}} |\bar{a}_2(t)|dt \leq& \int_{T_0}^{\bar{T}} |a_2(t)|dt + \int_{T_0}^{\bar{T}} \sum_{j=2}^{n}\sum_{k=1}^{n} |Q_{1jk}(u)v_k w_j(t, x_1(t, x_0))|dt \\ \leq& C_{28}\varepsilon^2 + c_{29}W_\infty^c(\bar{T})V_\infty(\bar{T})\int_{T_0}^{\bar{T}}(1+t)^{-1}dt \\ \leq& c_{30}\varepsilon^2|\log\varepsilon|^2, \qquad (3.90)\end{aligned}$$

$$\begin{aligned}K \triangleq& \int_{T_0}^{\bar{T}} |\bar{a}_2(t)|\exp\left(-\int_{T_0}^{t}\bar{a}_1(s)ds\right)dt \leq \int_{T_0}^{\bar{T}}|\bar{a}_2(t)|dt\exp\left(\int_{T_0}^{\bar{T}}|\bar{a}_1(t)|dt\right) \\ \leq& c_{31}\varepsilon^2|\log\varepsilon|^2 \qquad (3.91)\end{aligned}$$

and

$$w_1(T_0, x_1(T_0, x_0)) > c_{31}\varepsilon^2|\log\varepsilon|^2 \geq K. \qquad (3.92)$$

Applying Lemma 2.1 in [2] and completely repeating the rest of the proof (1.25a) in [4], we get (1.22a) immediately.

(2) **Proof of (1.22b)**

We can prove (1.22b) by almost completely repeating the procedure of proving (1.25b) in [4] and only changing (5.49)–(5.51) and (5.53) in [4].

Using Lemma 2.1 and Lemma 3.3, noting (5.49)–(5.50) in [4], (3.77) and (3.35), instead of (5.49)–(5.51) in [4] we have

$$\begin{aligned}\int_0^T |\bar{a}_1(t)|dt \leq& \int_0^T |a_1(t)|dt + \sum_{k=2}^{n}\int_0^T |Q_{11k}(u)v_k|dt + \int_0^T |(Q_{111}(u) - Q_{111}(u_1e_1))v_1|dt \\ \leq& c_{32}\left\{\int_0^T |a_1(t)|dt + \sum_{k=1}^{n}\left(\int_0^{T_y}|v_k|dt + \int_{T_y}^{\tilde{T}_y}|v_k|dt\right)\right. \\ &\left. + \sum_{k=2}^{n}\int_{\tilde{T}_y}^{T}|v_k|dt + \int_{\tilde{T}_y}^{T}|(Q_{111}(u) - Q_{111}(u_1e_1))v_1|dt\right\}\end{aligned}$$

$$\begin{aligned}&\le c_{33}(\varepsilon|\log\varepsilon| + V(D_+^T)\log(1+T) + V(D_-^T)\log(1+T)\\&\quad + \tilde V_1(T) + V_\infty^c(T)\log(1+T) + V_\infty^c(T)V_\infty(T)\log(1+T))\\&\le c_{34}(\varepsilon|\log\varepsilon| + k_6(\varepsilon|\log\varepsilon|)^{2+\alpha}\varepsilon^{-(1+\alpha)} + \varepsilon^2|\log\varepsilon|^2)\\&\le c_{35}\varepsilon|\log\varepsilon|^{2+\alpha},\end{aligned} \tag{3.93}$$

$$\begin{aligned}\int_0^T|\bar a_2(t)|dt \le& \int_0^T|a_2(t)|dt + \sum_{j=2}^n\int_0^{T_y}\sum_{k=1}^n|Q_{1jk}(u)v_kw_j|dt\\&+\sum_{j=2}^n\int_{T_y}^{\tilde T_y}\sum_{k=1}^n|Q_{1jk}(u)v_kw_j|dt + \sum_{j=2}^n\int_{\tilde T_y}^{T}\sum_{k=1}^n|Q_{1jk}(u)v_kw_j|dt\\&\le c_{36}(\varepsilon^2|\log\varepsilon| + (W(D_+^T)+W(D_-^T)+V(D_+^T)+V(D_-^T))^2\\&\quad + W_\infty^c(T)V_\infty(T)\log(1+T) + \tilde W_1(T)V_\infty(T))\\&\le c_{37}\varepsilon^2|\log\varepsilon|^2\end{aligned} \tag{3.94}$$

and

$$\begin{aligned}K \triangleq& \int_0^T|\bar a_2(t)|\exp\left(-\int_0^t\bar a_1(s)ds\right)dt \le \int_0^T|\bar a_2(t)|dt\exp\left(\int_0^T|\bar a_1(t)|dt\right)\\&\le c_{38}\varepsilon^2|\log\varepsilon|^2.\end{aligned} \tag{3.95}$$

Instead of (5.53) in [4] we have

$$\begin{aligned}(w_1(0,y)+K)\int_0^T\bar a_0^+(t)dt \le& \frac{M}{M_0} + c_{39}\varepsilon|\log\varepsilon|^{1+\alpha} + K\int_0^T|\bar a_0|(t)dt\\ \le& \frac{M}{M_0} + c_{40}\varepsilon|\log\varepsilon|^{\max\{1+\alpha,2\}} < 1.\end{aligned} \tag{3.96}$$

Then completely repeating the rest of the proof of (1.25b) in [4], we obtain (1.22b). This finishes the proof of Theorem 1.1.

4. Proof of Theorem 1.2 and Theorem 1.3

We can prove Theorem 1.2 and Theorem 1.3 by almost completely repeating the procedure of proving Theorem 1.4 and Theorem 1.5 in [4] and only changing (6.17)–(6.19) and (6.24) in [4].

In the present situation, by (3.82), instead of (6.17) in [4] we have

$$\begin{aligned}&|w_i(t,x_i(t,y_i)) - \varepsilon l_i(0)\tilde\psi_i'(y_i)|\\&\quad\le c_{41}\{\varepsilon^2 + W_\infty(t)W_\infty^c(t)\log(1+t) + V_\infty(t)W_\infty(t)\\&\qquad + (W_\infty(t))^2V_\infty^c(t)\log(1+t) + (V_\infty(t))^\alpha(W_\infty(t))^2t + V_\infty^c(t)W_\infty(t)\log(1+t)\\&\qquad + W_\infty^c(t)V_\infty(t)\log(1+t) + V_\infty^c(t)V_\infty(t)W_\infty(t)\log(1+t)\}\\&\quad\le c_{42}(\varepsilon^2 + \varepsilon^2|\log\varepsilon| + \varepsilon^2|\log\varepsilon|^2 + \varepsilon^{2+\alpha}|\log\varepsilon|^\alpha t)\}\\&\quad\le c_{43}(\varepsilon^2 + \varepsilon^2|\log\varepsilon| + \varepsilon^2|\log\varepsilon|^2 + \varepsilon^{\frac54}|\log\varepsilon|^\alpha)\\&\quad\le c_{44}\varepsilon^{\frac98},\quad \forall t\in[0,T_1].\end{aligned} \tag{4.1}$$

Then we get (6.18) in [4].

On the other hand, noting Lemma 2.1 and Lemma 3.3, along $x = x_i(t, y_i)$ we have

$$\begin{aligned}
&\left|\sum_{j=1}^{n}\left(\sum_{k=1}^{n} B_{ijk}(u)\rho_k(u) + \nu_{ij}(u)\right) w_j\right| \\
&\quad \le \sum_{j\neq k} |Q_{ijk}(u)v_k w_j| + \sum_{j\neq i} |Q_{ijj}(u)v_j w_j| + |(Q_{iii}(u) - Q_{iii}(u_i e_i))v_i w_i| \\
&\quad \le c_{45}(1+t)^{-1}\{V^c_\infty(t)|w_i| + V^c_\infty(t)W^c_\infty(t)(1+t)^{-1} + W^c_\infty(t)V_\infty(t) + V^c_\infty(t)V_\infty(t)|w_i|\} \\
&\quad \le c_{46}\left\{|w_i|\varepsilon^{1+\left(\frac{3}{4}+\alpha\right)} + \varepsilon^{2\left(1+\left(\frac{3}{4}+\alpha\right)\right)} + \varepsilon^{2+\left(\frac{3}{4}+\alpha\right)}|\log\varepsilon| + |w_i|\varepsilon^{2+\left(\frac{3}{4}+\alpha\right)}|\log\varepsilon|\right\} \\
&\quad \le c_{47}\left\{|w_i|\varepsilon^{\frac{7}{4}+\alpha} + \varepsilon^{\frac{11}{4}+\alpha}|\log\varepsilon|\right\}, \quad \forall t \in [T_1, t^*).
\end{aligned} \tag{4.2}$$

Moreover, similarly to (4.68)–(4.70) in [1], along $x = x_i(t, y_i)$ we have

$$\gamma_{iii}(u_i e_i) \ge \frac{1}{2} a_i \left(\tilde{\psi}_i(y_i)\right)^\alpha > 0, \tag{4.3}$$

$$|\gamma_{iii}(u) - \gamma_{iii}(u_i e_i)| \le c_{48}\varepsilon^{\alpha+\frac{7}{4}}, \quad \forall t \in [T_1, t^*) \tag{4.4}$$

and

$$\left|\sum_{j\neq k} \gamma_{ijk}(u) w_j w_k\right| \le c_{49}\left\{|w_i|\varepsilon^{\alpha+\frac{7}{4}} + \varepsilon^{2\left(\alpha+\frac{7}{4}\right)}\right\}, \quad \forall t \in [T_1, t^*). \tag{4.5}$$

Then, instead of (4.71) in [1] we have

$$\frac{dw_i}{d_i t} \ge \frac{1}{4} a_i \left(\tilde{\psi}_i(y_i)\right)^\alpha \varepsilon^\alpha w_i^2 - c_{50}\varepsilon^{\frac{11}{4}+\alpha}|\log\varepsilon|, \quad \forall t \in [T_1, t^*). \tag{4.6}$$

Hence, as in [1], noting (2.12), we have $w_i(t, x_i(t, y_i))$ is strictly increasing function of t for $t \ge T_1$; then we get (6.19) in [4].

Instead of (6.24) in [4] we have

$$Q_i(t) \le Q_i(T_1) + C_{51}(V_\infty(t) + W^c_\infty(t)) \int_{T_1}^t (1+\tau)^{-1} Q_i(\tau) d\tau, \quad \forall t \in [T_1, t^*). \tag{4.7}$$

The rest of the proof of Theorem 1.2 and Theorem 1.3 is the same as that of Theorem 1.4 and Theorem 1.5 in [4]. This finishes the proof of Theorem 1.2 and Theorem 1.3.

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