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# SEMILINEAR ELLIPTIC EQUATIONS WITH SINGULARITY ON THE BOUNDARY*

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**Abstract** In this paper, we consider the existence and nonexistence of positive solutions to semilinear elliptic equation $-\Delta u = K(x)(1-|x|)^{-\lambda}u^q$ in the unit ball $B$ with 0-Dirichlet boundary condition. Our main tools are based on the interior estimates of the Schauder type, the Schauder fixed point theorem and the pointwise estimates for Green functions.
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## 1. Introduction

Let $B$ be the unit ball $\{x \in R^n : |x| < 1\}$ in $R^n$ and consider the semilinear elliptic problem

$$\begin{cases} -\Delta u = K(x)(1-|x|)^{-\lambda}u^q, & x \in B, \\ u(x) > 0, & x \in B, \\ u(x) = 0, & x \in \partial B, \end{cases} \tag{1.1}$$

where $\lambda > 0,\ q > 1$ and $K(x)$ is a given nonnegative $\alpha$-Hölder continuous function on $\overline{B}$. As a matter,of course, this kind of problems which allow $\lambda \le 0$ has been investigated extensively.

When $K(\cdot)$ is a given nonnegative continuous radial function on $\overline{B}$, this problem was already studied by Seuba-Ebihare-Furusho [1] within the framework of the theory of ODE. They obtained the existence of positive radial solutions in $C^2(B) \cap C^1(\overline{B})$ for the case $0 < \lambda < 2$ and $1 < q < (n+2)/(n-2)$. Hayashida-Nakatani [2] also studied some similar problems and discussed some mathematical backgrounds for (1.1). In a recent paper [3], Hashimoto-$\hat{O}$tani studied this problem by the variational method. They showed the existence of positive radial solutions in $C^2(B) \cap C^1(\overline{B})$ for the case

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$0 < \lambda < 1 + (q+1)/2$ and $1 < q < (n+2)/(n-2)$. Meanwhile they obtained the nonexistence theorem of positive solutions in $C^2(B) \cap C^1(\overline{B})$ for the case $\lambda \geq 1 + q$.

As was pointed in [3], it would be interesting to investigate the existence of (not necessarily classical) positive solutions of (1.1) for the case $1 + (1+q)/2 \leq \lambda < 1 + q$.

The main purpose of this paper is devoted to the existence and nonexistence of positive solutions in $C^0(\overline{B}) \cap C^{2,\alpha}(B)$ of (1.1) from the viewpoint of the theory of nonlinear PDE. However, our method can deal with the sharper singular case (i.e., $0 < \lambda < 1 + \alpha + (q-1)\beta$, for some $0 < \alpha \leq \beta,\ 0 < \beta < 1$) than those in Hashimoto-$\hat{O}$tanni [3]. Our argument is based on the interior estimates of the Schauder type, the Schauder fixed point theorem and the pointwise estimates for Green function. However, our argument does not require the symmetry of both $K(\cdot)$ and the solution. Moreover, we have droped out the subcritical condition $q < (n+2)/(n-2)$.

The main results are stated in the next section, and their proofs will be given in Section 3 and Section 4.

## 2. Main Result

Throughout this paper, the following condition will be imposed on $K(\cdot)$

$$(K_\alpha) \begin{cases} K(x) \in C^\alpha(\overline{B}), \\ K(x) \geq 0,\ x \in B, \\ K(x) > 0,\ x \in \partial B, \end{cases} \tag{2.1}$$

where $0 < \alpha \leq \beta$ for some $0 < \beta < 1$.

The main results of this paper are the following two theorems.

**Theorem 2.1** (Existence theorem)*Let $K(\cdot)$satisfy condition ($K_\alpha$) and $q > 1, 0 < \lambda < 1 + \alpha + (q-1)\beta$, then (1.1) has at least one positive solution $u(x)$ belonging to $C^0(\overline{B}) \cap C^{1,\alpha}(B)$. Furthermore, there exist positive constants $c_1, c_2, c_3$ and $\epsilon (0 < \epsilon < 1)$ such that*

$$|u|^{(-\beta)}_{0,\alpha;B} \leq c_1,\ |u|^{(-\beta)}_{2,\alpha;B} \leq c_2 c_1^q |K|_{0,\alpha;B},$$

$$u(x) \geq c_3(1 - |x|),\ \forall\ x \in B_{1-\epsilon},$$

*where $B_r$ is an open ball with radius $r$ centered at origin.*

**Remark 2.1** As pointed out in the introduction, we do not need the subcritical condition $q < (n+2)/(n-2)$.

**Remark 2.2** If we have $\alpha = \beta \to 1$, then $\overline{q} := 1 + \alpha + (q-1)\beta \to 1 + q$. Combining this with a nonexistence result of Hayashimoto-$\hat{O}$tani [3], we know that our result, in some sense, is essentially optimal.

**Remark 2.3** For the case of $\alpha = 0$, i.e., $K(\cdot)$ only continuous, the existence of (not necessary classical) positive solutions of Dirichlet problem (1.1) is still open.

As usual, a function $u \in C^0(\overline{B})$ is called a mild solution of Dirichlet problem (1.1) if $u$ satisfy the following integral equation

$$u(x) = \int_B G(x,y)\frac{K(y)u^q(y)}{(1-|y|)^\lambda}\,dy, \tag{2.2}$$

where $G(x,y)$ is the Green's function of $-\Delta$ in $B$ with zero Dirichelt boundary condition.

**Theorem 2.2** (Nonexistence theorem) *Let $K(\cdot)$ satisfy condition $(K_\alpha)$ and $q > 1, \lambda \geq 1+q$, then Dirichlet problem (1.1) has no positive mild solution $u$ such that*

$$u(x) \geq c(1-|x|), \forall\ x \in \Gamma_{R_0}, \tag{2.3}$$

*where $\Gamma_{R_0} = \{y \in R^n | R_0 \leq |y| \leq 1\}$.*

**Corollary 2.3** *Let $K(\cdot)$ satisfy condition $(K_0)$ and $q > 1, \lambda \geq 1+q$, then the Dirichlet problem (1.1) has no positive solution in $C^1(\overline{B})$.*

**Remark 2.4** Corollary 2.3 is an extension of both Theorem 2 and Remark 2 (ii) in [3].

## 3. Proof of Theorem

For $0 < \alpha \leq \beta, 0 < \beta < 1$. Let

$$\begin{aligned} W_1 =& \{u \in C^{2,\alpha}(B):\ |u|^{(-\beta)}_{2,\alpha;B} < \infty\}, \\ W_2 =& \{u \in C^{\alpha}(B):\ |u|^{(-\beta)}_{0,\alpha;B} < \infty\}, \\ W_3 =& \{u \in C^{\alpha}(B):\ |u|^{(2-\beta)}_{0,\alpha;B} < \infty\}, \end{aligned}$$

where the norms are interior norms of the functions in $C^{k,\alpha}$(here $k = 0, 1$ respectively), we refer readers to Charpter 6.1 in [4]. It is well known that $W_i$ (i=1,2,3) are the Banach spaces. Since $\beta > 0$, the condition that $|u|^{(-\beta)}_{0,\alpha;B}$ is finite obviously requires that $u = 0$ on $\partial B$.

**Lemma 3.1** (Pointwise estimates for Green functions [5]) *Let*

$$L = -\sum_{i,j=1}^{n} a_{ij}(\cdot)\frac{\partial^2}{\partial x_i \partial_j} + \sum_{i=1}^{n} b_i(\cdot)\frac{\partial}{\partial x_i} + c(\cdot) \tag{3.1}$$

*be a strictly elliptic operator on $R^n(n > 2)$ with Hölder-continuous coefficients and $c \geq 0$. Let $\Omega$ be a bounded $C^{1,1}$-domain and $G$ denote the Green function of $L$ in $\Omega$ with zero Dirichlet boundary condition. Then there exists a constant $\alpha > 0$ such that*

$$\alpha^{-1}F_n(x,y) \leq G(x,y) \leq \alpha F_n(x,y),\ \ x, y \in \Omega \tag{3.2}$$

*with*

$$F_n(x,y) = |x-y|^{-n}\min(d_x, d_y, |x-y|^2), d_x = \mathrm{dist}(x, \partial\Omega).$$

**Lemma 3.2** *Suppose that* $K(x) \in C^{\alpha}(\overline{B})$, $q > 1$ *and* $\lambda < 1 + \alpha + (q-1)\beta$. *Then for any* $v \in W_2$,*we have*

$$f(x) \equiv K(x)|v(x)|^q/(1-|x|)^{\lambda} \in W_3$$

*and*

$$|f|^{(2-\beta)}_{0,\alpha;B} \le C_1 |K|_{0,\alpha;B} (|v|^{(-\beta)}_{0,\alpha;B})^q, \tag{3.3}$$

*where* $C_1 = C_1(\lambda, q)$ *is independent of* $v$.

**Proof** For $v \in W_2$, denoting $|v|^{(-\beta)}_{0,\alpha;B}$ by $C_0$, we have $|v(x)| \le C_0 d_x^{\beta}$, $d_x = 1 - |x|$. Hence,

$$\begin{aligned} \sup_{x\in B} d_x^{(2-\beta)} |f(x)| \equiv & \sup_{x \in B} (1-|x|)^{(2-\beta)} \frac{K(x)|v(x)|^q}{(1-|x|)^{\lambda}} \\ \le & |K|_{0,0;B} C_0^q \sup_{x\in B} (1-|x|)^{2+(q-1)\beta-\lambda} \\ \le & |K|_{0,0;B} C_0^q. \end{aligned} \tag{3.4}$$

Next, let $x, y$ be two distinct points in $B$, with $d_x < d_y$, then we have

$$\begin{aligned} d_{x,y}^{2+\alpha-\beta} \frac{|f(x)-f(y)|}{|x-y|^{\alpha}} = & \frac{(1-|x|)^{2+\alpha-\beta}}{|x-y|^{\alpha}} \left| \frac{K(x)|v(x)|^q}{(1-|x|)^{\lambda}} - \frac{K(y)|v(y)|^q}{(1-|y|)^{\lambda}} \right| \\ \le & \frac{(1-|x|)^{2+\alpha-\beta}}{|x-y|^{\alpha}} \{ \frac{K(y)}{(1-|y|)^{\lambda}} ||v(x)|^q - |v(y)|^q| \\ & + K(y)|v(x)|^q \left| (1-|x|)^{-\lambda} - (1-|y|)^{-\lambda} \right| \\ & + \frac{|v(x)|^q}{(1-|x|)^{\lambda}} |K(x) - K(y)| \} \\ \le & C|K|_{0,0} C_0^{q-1} [v]^{(-\beta)}_{0,\alpha} (1-|y|)^{2+(q-1)\beta-\lambda} \\ & + \lambda |K|_{0,0} C_0^q (1-|x|)^{1+\alpha+(q-1)\beta-\lambda} \\ & + [K]_{0,\alpha;B} C_0^q (1-|x|)^{2+\alpha+(q-1)\beta-\lambda} \\ \le & C|K|_{0,\alpha;B} C_0^q. \end{aligned} \tag{3.5}$$

Taking the supremum with respect to $x, y$, we obtain

$$\sup_{\substack{x,y\in B \\ x\ne y}} d_{x,y}^{2+\alpha-\beta} \frac{|f(x)-f(y)|}{|x-y|^{\alpha}} \le C|K|_{0,\alpha;B} C_0^q.$$

Combining (3.4) and (3.5), we get the desired result.

Now, we give the proof of Theorem 2.1.

**Proof of theorem 2.1** First, for any $v \in W_2$, we consider the following linear Dirichlet problem

$$\begin{cases} -\Delta u = \frac{K(x)|v(x)|^{q-1}v(x)}{(1-|x|)^{\lambda}}, & \text{in } \ B, \\ u(x) = 0, & \text{on } \ \partial B. \end{cases} \tag{3.6}$$

By Lemma 3.2 and Theorem 6.22 of [4], there exists a unique solution $u$ of Dirichlet problem (3.6) which satisfies the interior estimate

$$|u|^{(-\beta)}_{2,\alpha;B} \le C|K|_{0,\alpha;B}(|v|^{(-\beta)}_{0,\alpha;B})^q, \tag{3.7}$$

where $C$ is independent of $v$ and $K(\cdot)$.

Hence, for any $v \in W_2$, the operator $T : W_2 \longrightarrow W_2$ defined by letting $u = Tv$ be the unique solution $u \in C^0(\overline{B}) \cap C^{(2,\alpha)}(B)$ of the linear Dirichlet problem (3.6) is well defined.

Next, for some positive constant $a$ and $\epsilon < 1$, let

$$V = \{u \in W_2 :\ |u|^{(-\beta)}_{0,\alpha;B} \le a, u(x) \ge c_3(1-|x|), \forall x \in B_{1-\epsilon}\}, \tag{3.8}$$

where $c_3$ is a suitable constant. Obviously, $V$ is a closed convex subset in $W_2$.

On the one hand, from (3.7), if $v \in V$, we can take $a$ sufficiently small such that $C|K|_{0,\alpha;B}a^{q-1} \le 1$ (since $q > 1$), then we get

$$|u|^{(-\beta)}_{0,\alpha;B} \le |u|^{(-\beta)}_{2,\alpha;B} \le a. \tag{3.9}$$

On the other hand, under the conditions of Theorem 2.1, the solution $u \in C^0(\overline{B}) \cap C^{2,\alpha}(B)$ of Dirichlet problem (3.6) must be a mild solution in $B$. Then we have the representation

$$u(x) = Tv(x) = \int_B G(x,y)\frac{K(y)v^q(y)}{(1-|y|)^\lambda}\,dy. \tag{3.10}$$

By virtue of condition $(K_\alpha)$, there exists a real number $R_0,\ 0 < R_0 < 1$, such that

$$K(y) \ge \delta > 0,\ y \in \Gamma_{R_0} = \{y \in R^n :\ R_0 \le |y| \le 1\},\ R_0 + \epsilon < 1.$$

Let $0 < R < R_0/2$ and

$$\Gamma_{R_0-R} = \{y \in R^n :\ R_0 - R \le |y| < 1\}.$$

We distinguish two cases below.

**Case (i)** $x \notin \Gamma_{R_0-R}$. Now, we have $R \le |x-y| \le 2$ for $y \in \Gamma_{R_0}$. Hence,with the aid of Lemma 3.1, we obtain

$$\begin{aligned} u(x) &\ge \delta\alpha^{-1}c_3^q\int_{\Gamma_{R_0}\setminus\Gamma_{1-\epsilon}}\frac{(1-|x|)(1-|y|)^{1+q-\lambda}}{2^n}\,dy \\ &= c_4c_3^q(1-|x|), \end{aligned} \tag{3.11}$$

where $c_4 = \dfrac{\delta\alpha^{-1}}{2^n}\displaystyle\int_{\Gamma_{R_0}\setminus\Gamma_{1-\epsilon}}(1-|y|)^{1+q-\lambda}\,dy.$

**Case (ii)** $x \in \Gamma_{R_0-R}\setminus\Gamma_{1-\epsilon}$. We take $x_0$ such that $|x_0| = R_0 - R$ and $x_0$ paralleles $x$. Let $B_R(x_0)$ be the open ball of radius $R$ centered at $x_0$. Let $\Gamma_{x_0,R}$ denote the cone generated by the origin and $B_R(x_0)$. By virtue of Lemma 3.1, we have

$$\begin{array}{rcl} u(x) & \geq & \delta\alpha^{-1}c_3^q \int_{(\Gamma_{R_0}\setminus\Gamma_{1-\epsilon})\setminus\Gamma_{x_0,R}} \frac{(1-|x|)(1-|y|)^{1+q-\lambda}}{2^n}\,dy \\ & = & c_4' c_3^q (1-|x|), \end{array} \tag{3.12}$$

where $c_4' = \dfrac{\delta\alpha^{-1}}{2^n}\displaystyle\int_{(\Gamma_{R_0}\setminus\Gamma_{1-\epsilon})\setminus\Gamma_{x_0,R}} (1-|y|)^{1+q-\lambda}\,dy.$

Obviously, $c_4 \geq c_4'$. combining (3.11) with (3.12) yields

$$u(x) \geq c_4' c_3^q (1-|x|),\ x \in B_{1-\epsilon}. \tag{3.13}$$

Taking $c_3$ such that $c_3^{q-1}c_4' = 1$, we then obtain

$$u(x) \geq c_3(1-|x|),\ x \in B_{1-\epsilon}. \tag{3.14}$$

Now, combining (3.14) with (3.9), we know that $TV \subset V$.

By estimate (3.9), we know that $TV$ is a precompact set in $W_2$.

In order to show the continuity of $T$, we let $v_k \in W_2, v_k \to v \in W_2$. Then $v_k \to v$ in $\overline{B}$ uniformly, and there exists a constant $C$ independent of $k$ such that $|v_k|_{0,\alpha;B}^{(-\beta)} \leq C$. From the estimate (3.7) we have $u_k = Tv_k$ and

$$|u_k|_{2,\alpha;B}^{(-\beta)} \leq C_1, \tag{3.15}$$

where $C_1$ is independent of $k$. For any $\Omega \subset\subset B$, we know that $u_k \in C^{2,\alpha}(\overline{\Omega})$ and there exists a constant $C$ independent of $k$ such that

$$|u_k|_{2,\alpha;\Omega} \leq C,\ \text{and}\ |u_k(x)| \leq C(1-|x|)^\beta,\ \forall x \in B. \tag{3.16}$$

Hence, $\{u_k\}$ is a precompact subset in $C^2(\overline{\Omega})$ and therefore every subsequence has a convergent subsequence. Let $\{u_{k_j}\}$ be such a convergent subsequence with limit $\widetilde{u} \in C^2(\overline{\Omega})$ and $\widetilde{u}|_{\partial B} = 0$ since (3.16).

Passing to the limit in the equation

$$-\Delta u_{k_j} = \frac{K(x)|v_{k_j}(x)|^{q-1}v_{k_j}(x)}{(1-|x|)^\lambda},\ \text{in}\ \overline{\Omega} \tag{3.17}$$

yields

$$-\Delta\widetilde{u} = \frac{K(x)|v(x)|^{q-1}v(x)}{(1-|x|)^\lambda},\ \text{in}\ \overline{\Omega}. \tag{3.18}$$

Obviously, $\widetilde{u}|_{\partial B} = 0$. By the arbitrariness of $\Omega \subset\subset B$, we get

$$\left\{\begin{array}{rcl} -\Delta\widetilde{u} & = & \frac{K(x)|v(x)|^{q-1}v(x)}{(1-|x|)^\lambda},\ \text{in}\ \ B, \\ \widetilde{u}|_{\partial B} & = & 0. \end{array}\right. \tag{3.19}$$

By the uniqueness of the solution,we must have $\widetilde{u} = u = Tv$,and hence the sequence $\{u_k\} = \{Tv_k\}$ itself converges to $u = Tv$.

The Schauder fixed point theorem is now applicable and the theorem is proved.

## 4. Proof of Theorem 2.2

In this section, we give the proof of Theorem 2.2 and Corollary 2.3.

**Proof of Theorem 2.2** For $x \in B$, let $\Omega = \{y \in B : (1-|x|)(1-|y|) > |x-y|^2\}$. If $u(x) \in C^0(\overline{B})$ is a mild solution of Dirichlet problem (1.1) such that

$$u(x) \geq c(1-|x|), \forall x \in \Gamma_{R_0}, \tag{4.1}$$

then, using $\Gamma_{R_0}$ as above and Lemma 3.1, we have

$$\begin{aligned} u(x) =& \int_B G(x,y)\frac{K(y)u^q(y)}{(1-|y|)^\lambda}\, dy \\ \geq& \alpha^{-1}c^q \int_{B\cap\Gamma_{R_0}} \frac{\min(d_x\, d_y, |x-y|^2)K(y)(1-|y|)^{q-\lambda}}{|x-y|^n}\, dy \qquad (4.2) \\ \geq& \alpha^{-1}c^q \int_{\Omega\cap\Gamma_{R_0}} \frac{K(y)(1-|y|)^{q-\lambda}}{|x-y|^{n-2}}\, dy. \qquad (4.3) \end{aligned}$$

The convergence of the last integral above requires $\lambda < 1+q$.

**Proof of Corollary 2.1** By Corollary 1 of [3], we know that there exist numbers $\rho \in (0,1)$ and $c_\rho$ such that

$$u(x) \geq c_\rho(1-|x|),\ \forall x \in \Gamma_\rho.$$

We can now assert the corollary result by Theorem 2.2.

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