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STABILITY AND REGULARITY OF SUITABLY WEAK SOLUTIONS OF n-DIMENSIONAL MAGNETOHYDRODYNAMICS EQUATIONS

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Dedicated to the 80th birthday of Professor Zhou Yulin

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Abstract In this paper, it is shown that the weak solutions of magnetohydrodynamics equations in spaces $L^q(R^+;\ L^p(R^n))$ are stable and regular.

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1. Introduction

The Magnetohydrodynamics equations and the incompressible Navier-Stokes equations play very important roles in nonlinear partial differential equations with dissipation. In this work, the author is concerned with stability and regularity of the following $n(\geq 3)$-dimensional Magnetohydrodynamics equations

$$u_t + (u \cdot \nabla)u - (A \cdot \nabla)A - \triangle u + \nabla p = 0, \qquad \nabla \cdot u = 0, \qquad \text{in } \mathbb{R}^n \times \mathbb{R}^+, \tag{1}$$

$$u(x,0) = u_0(x), \qquad \nabla \cdot u_0 = 0, \qquad \text{in } \mathbb{R}^n, \tag{2}$$

$$A_t + (u \cdot \nabla)A - (A \cdot \nabla)u - \triangle A = 0, \qquad \nabla \cdot A = 0, \qquad \text{in } \mathbb{R}^n \times \mathbb{R}^+, \tag{3}$$

$$A(x,0) = A_0(x), \qquad \nabla \cdot A_0 = 0, \qquad \text{in } \mathbb{R}^n, \tag{4}$$

where $u(x,t) = (u_1(x,t), u_2(x,t), \cdots, u_n(x,t))$ and $A(x,t) = (A_1(x,t), A_2(x,t), \cdots, A_n(x,t))$ are unknown vector-valued functions; and $p = p(x,t)$ is a real-valued function, representing pressure. In addition

$$\nabla \cdot u = \sum_{j=1}^{n} \frac{\partial u_j}{\partial x_j}, \qquad (u \cdot \nabla)u = \sum_{j=1}^{n} u_j \frac{\partial u}{\partial x_j}, \qquad \triangle u = \sum_{j=1}^{n} \frac{\partial^2 u}{\partial x_j^2}.$$

Suppose that the weak solutions $(u,\ A,\ p)$ of the Cauchy problems (1)-(4) satisfy

$$\lim_{|x|\to\infty} \frac{\partial^{\alpha_1+\alpha_2+\cdots+\alpha_n}}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2} \cdots \partial x_n^{\alpha_n}} (u(x,t),\ A(x,t),\ p(x,t)) = 0, \tag{5}$$

where $\alpha_1 \geq 0, \alpha_2 \geq 0, \cdots, \alpha_n \geq 0$ are integers. Notice that if $A_0 \equiv 0$, then a simple argument shows that $A \equiv 0$, and (1) reduces to the following incompressible Navier-Stokes equations

$$u_t + (u \cdot \nabla)u - \triangle u + \nabla p = 0, \qquad \nabla \cdot u = 0 \qquad \text{in } \mathbb{R}^n \times \mathbb{R}^+. \tag{6}$$

Let the initial data $(u_0,\ A_0) \in L^2(\mathbb{R}^n)$. Then the problem (1)-(4) admit a global weak solution $(u,\ A) \in L^\infty(0,T;L^2(\mathbb{R}^n)) \bigcap L^2(0,T;H^1(\mathbb{R}^n))$ and $p \in L^\infty(0,T;L^1(\mathbb{R}^n)) \bigcap L^2(0,T;W^{1,1}(\mathbb{R}^n))$, where $T > 0$ is any constant. This is a well known result, see [1]. However, the weak solutions are not unique in general. On the other hand, under the additional restrictions on the weak solutions: $(u,\ A) \in L^q(\mathbb{R}^+;L^p(\mathbb{R}^n))$, where $n/p + 2/q = 1$ and $p > n \geq 3$, then the global weak solution is unique. Our calculations in this paper show that if such solutions exist, then they are actually very strong. They are almost equivalent to smooth solutions for all $n \geq 3$. Let $(u,\ A,\ p)$ and $(v,\ B,\ q)$ be the solutions of problem (1)-(4) corresponding to the initial data $(u_0,\ A_0)$ and $(v_0,\ B_0)$, respectively, such that the above assumptions are satisfied. Let $(w,\ E,\ \pi) = (u - v,\ A - B,\ p - q)$. Then they satisfy the equations

$$w_t + [(w \cdot \nabla)u + (v \cdot \nabla)w] - [(E \cdot \nabla)A + (B \cdot \nabla)E] - \triangle w + \nabla \pi = 0, \quad \text{in } \mathbb{R}^n \times \mathbb{R}^+, \tag{7}$$
$$E_t + [(w \cdot \nabla)A + (v \cdot \nabla)E] - [(E \cdot \nabla)u + (B \cdot \nabla)w] - \triangle E = 0, \qquad \text{in } \mathbb{R}^n \times \mathbb{R}^+, \tag{8}$$

where $\nabla \cdot w = \nabla \cdot E = 0$ in $\mathbb{R}^n \times \mathbb{R}^+$, together with the initial conditions

$$w(x,0) = w_0(x) \quad = \quad u_0(x) - v_0(x), \qquad \nabla \cdot w_0 = 0, \qquad \text{in } \mathbb{R}^n, \tag{9}$$
$$E(x,0) = E_0(x) \quad = \quad A_0(x) - B_0(x), \qquad \nabla \cdot E_0 = 0, \qquad \text{in } \mathbb{R}^n. \tag{10}$$

Notations Denote by C any positive, time-independent constant, which may be different from one place to another place, and may depend on the initial data (u_0, A_0). Denote by $L^p(\mathbb{R}^n)$ and $H^m(\mathbb{R}^n)$ the usual functional spaces, where $p \in [1, +\infty]$ and $m \geq 1$. Let $f = (f_1, f_2, \cdots, f_n) \in L^\infty(\mathbb{R}^+, L^2(\mathbb{R}^n))$ and $g = (g_1, g_2, \cdots, g_n) \in L^\infty(\mathbb{R}^+, L^2(\mathbb{R}^n))$. Define

$$\begin{aligned}
&x = (x_1, x_2, \cdots, x_n) \in \mathbb{R}^n, \qquad |x|^2 = |x_1|^2 + |x_2|^2 + \cdots + |x_n|^2,\\
&|f|^2 = |f_1|^2 + |f_2|^2 + \cdots + |f_n|^2, \qquad |g|^2 = |g_1|^2 + |g_2|^2 + \cdots + |g_n|^2,\\
&|(f,\ g)|^2 = |f|^2 + |g|^2,\\
&|(f,\ g)(\cdot,t)\|^2 = \|(f,\ g)(\cdot,t)\|^2_{L^2(\mathbb{R}^n)} = \int_{\mathbb{R}^n} \left[|f(x,t)|^2 + |g(x,t)|^2\right] dx,\\
&f \cdot g = f_1 g_1 + f_2 g_2 + \cdots + f_n g_n,\\
&\nabla f \cdot \nabla g = \sum_{i=1}^n \nabla f_i \cdot \nabla g_i = \sum_{i=1}^n \sum_{j=1}^n \frac{\partial f_i}{\partial x_j} \frac{\partial g_i}{\partial x_j}, \ \text{ if } f, g \in L^2(0,T;H^1(\mathbb{R}^n)).
\end{aligned}$$

Let $\varphi(x) \in L^1(\mathbb{R}^n) \bigcap L^2(\mathbb{R}^n)$, define its Fourier transform and inverse Fourier transform by

$$F[\varphi](\xi) \equiv \widehat{\varphi}(\xi) \equiv \int_{\mathbb{R}^n} \varphi(x) \exp[-\ \mathrm{i}\ x \cdot \xi] dx, \ \text{here i } = \sqrt{-1},$$
$$F^{-1}[\varphi](x) \equiv \check{\varphi}(x) \equiv \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \varphi(\xi) \exp[\ \mathrm{i}\ x \cdot \xi] d\xi.$$

Assumption (H) Let $(u_0,\ A_0) \in L^2(\mathbb{R}^n) \bigcap L^p(\mathbb{R}^n)$ and $\nabla \cdot u_0 = \nabla \cdot A_0 = 0$ in $\mathbb{R}^n$, such that the global weak solutions of the Cauchy problems (1)-(4) satisfy $(u,\ A) \in L^q(\mathbb{R}^+; L^p(\mathbb{R}^n))$, for some $p > n \geq 3$ and $q > 2$, which satisfy $n/p + 2/q = 1$. Denote by $\mathcal{A} = \int_0^\infty \|(u,\ A)(\cdot, t)\|^q_{L^p(\mathbb{R}^n)} dt$. All analysis and calculations of this paper are rigorously processed under the Assumption (H). Below are the main results and their proofs.

2. Local Stability of Global Weak Solutions of Problems (1)-(4)

First, we prove that the solution operator induced by the Cauchy problems for the Magnetohydrodynamics equations (1)-(4) is locally Lipschitz continuous under the Assumption (H). Then we show that the subset of the initial data (u_0, A_0) such that Assumption (H) holds is open in the sense of $L^2(\mathbb{R}^n)$-norm. Often we implicitly make use of the differential inequalities and the preliminary results in the appendix.

Theorem 1 *Let $(u_0,\ A_0) \in L^2(\mathbb{R}^n)$ and $(v_0,\ B_0) \in L^2(\mathbb{R}^n)$. Then the following estimates hold for the problems (7)-(10)*

$$\sup_{t \in \mathbb{R}^+} \|(w, E)(\cdot, t)\|^2 \leq C \|(w_0,\ E_0)\|^2, \tag{11}$$
$$\int_0^\infty \|\nabla(w, E)(\cdot, t)\|^2 dt \leq C \|(w_0,\ E_0)\|^2, \tag{12}$$
$$\sup_{t \in \mathbb{R}^+} \|\widehat{\pi}(\cdot, t)\|_{L^\infty(\mathbb{R}^n)} \leq C \|(w_0,\ E_0)\|, \tag{13}$$

where C depends only on $\mathcal{A}$.

Remark 1 It is very interesting and important that the upper bound of these norms depend explicitly on $(w_0,\ E_0)$, but independent of time t. Without the Assumption (H), it is very difficult to study the uniform stability of the solutions to the problem (1)-(4), but we can at least establish the estimates in some functional spaces for $n(\geq\ 3)$-dimensional problem under the Assumption (H). They illustrate that if $(v_0,\ B_0) \to (u_0,\ A_0)$, in some Sobolev space, then the corresponding solutions $(v,\ B,\ q) \to (u,\ A,\ p)$ in another Sobolev space, for all $t > 0$.

Proof It is easy to get the following estimate from (7-8)

$$\begin{aligned}
&\frac{d}{dt}\|(w, E)(\cdot, t)\|^2 + 2\|\nabla(w, E)(\cdot, t)\|^2 \\
&\quad= 2\int_{\mathbb{R}^n} \{w \cdot [(E \cdot \nabla)A - (w \cdot \nabla)u] + E \cdot [(E \cdot \nabla)u - (w \cdot \nabla)A]\} dx \\
&\quad\leq 2\int_{\mathbb{R}^n} [|A||E||\nabla w| + |u||w||\nabla w| + |u||E||\nabla E| + |A||w||\nabla E|] dx,
\end{aligned}$$

and

$$
\begin{aligned}
2\int_{\mathbb{R}^n}|A||E||\nabla w| \leq & 2\|A(\cdot,t)\|_{L^p(\mathbb{R}^n)}\|E(\cdot,t)\|_{L^\nu(\mathbb{R}^n)}\|\nabla w(\cdot,t)\| \\
\leq & C\|A(\cdot,t)\|_{L^p(\mathbb{R}^n)}\|E(\cdot,t)\|^{1-\alpha}\|\nabla E(\cdot,t)\|^{\alpha}\|\nabla w(\cdot,t)\| \\
\leq & \frac{1}{4}\|\nabla(w,E)(\cdot,t)\|^2 + C\|A(\cdot,t)\|_{L^p(\mathbb{R}^n)}^{2/(1-\alpha)}\|(w,E)(\cdot,t)\|^2,
\end{aligned}
$$

where $1/p+1/\nu=1/2$ and $\alpha=n/p=1-2/q$, thus $(1-\alpha)q=2$. Therefore

$$
\|A(\cdot,t)\|_{L^p(\mathbb{R}^n)}^{2/(1-\alpha)} = \|A(\cdot,t)\|_{L^p(\mathbb{R}^n)}^{q} \in L^1(\mathbb{R}^+).
$$

Similarly, we have other estimates. Integrating the inequality in time to give

$$
\begin{aligned}
&\|(w,E)(\cdot,t)\|^2 + \int_0^t \|\nabla(w,E)(s)\|^2 ds \\
&\quad \leq \|(w_0,\ E_0)\|^2 + C\int_0^t \|(u,\ A)(s)\|_{L^p(\mathbb{R}^n)}^q \|(w,E)(s)\|^2 ds.
\end{aligned}
$$

By using Gronwall's inequality, we obtain

$$
\|(w,E)(\cdot,t)\|^2 + \int_0^t \|\nabla(w,E)(s)\|^2 ds \leq \|(w_0,\ E_0)\|^2 \exp\left[C\int_0^\infty \|(u,\ A)(\cdot,t)\|_{L^p(\mathbb{R}^n)}^q dt\right].
$$

Thus

$$
\sup_{t\in\mathbb{R}^+}\|(w,E)(\cdot,t)\| \leq C\|(w_0,\ E_0)\|, \qquad \int_0^\infty \|\nabla(w,E)(\cdot,t)\|^2 dt \leq C\|(w_0,\ E_0)\|^2.
$$

By taking divergence of the equation (7), one obtains

$$
\sum_{i=1}^n\sum_{j=1}^n \frac{\partial^2}{\partial x_i \partial x_j}\left(u_i w_j + v_i w_j - A_i E_j - B_i E_j\right) + \triangle \pi = 0.
$$

Applying the Fourier transform gives

$$
\sum_{i=1}^n\sum_{j=1}^n \xi_i\xi_j F\left[u_i w_j + v_i w_j - A_i E_j - B_i E_j\right](\xi,t) + |\xi|^2\widehat{\pi}(\xi,t) = 0.
$$

Applying triangle inequality and Cauchy-Schwartz inequality gives the estimates

$$
\begin{aligned}
|\xi|^2|\widehat{\pi}| \leq & \sum_{i=1}^n\sum_{j=1}^n |\xi_i||\xi_j|\{\|u_i(\cdot,t)\|\|w_j(\cdot,t)\| + \|v_i(\cdot,t)\|\|w_j(\cdot,t)\|\} \\
& + \sum_{i=1}^n\sum_{j=1}^n |\xi_i||\xi_j|\{\|A_i(\cdot,t)\|\|E_j(\cdot,t)\| + \|B_i(\cdot,t)\|\|E_j(\cdot,t)\|\} \\
\leq & |\xi|^2\{\|u(\cdot,t)\|\|w(\cdot,t)\| + \|v(\cdot,t)\|\|w(\cdot,t)\| + \|A(\cdot,t)\|\|E(\cdot,t)\| + \|B(\cdot,t)\|\|E(\cdot,t)\|\} \\
\leq & |\xi|^2\{\|(u,\ A)(\cdot,t)\| + \|(v,\ B)(\cdot,t)\|\}\|(w,E)(\cdot,t)\|.
\end{aligned}
$$

Therefore, we get

$$\|\widehat{\pi}(\cdot,t)\|_{L^\infty(\mathbb{R}^n)} \le \{\|(u,\ A)(\cdot,t)\| + \|(v,\ B)(\cdot,t)\|\}\,\|(w,E)(\cdot,t)\|.$$

The last estimate follows immediately from

$$\sup_{t\in\mathbb{R}^+}\|\widehat{\pi}(\cdot,t)\|_{L^\infty(\mathbb{R}^n)} \le \{\|(u_0,\ A_0)\| + \|(v_0,\ B_0)\|\}\sup_{t\in\mathbb{R}^+}\|(w,E)(\cdot,t)\|.$$

Theorem 2 *If* $\|(u_0-v_0,\ A_0-B_0)\|_{L^p(\mathbb{R}^n)} < \delta$ *and* $(v_0,\ B_0)\in L^2(\mathbb{R}^n)\bigcap L^p(\mathbb{R}^n)$, *for a sufficiently small constant* $\delta=\delta(n,\ p,\ u_0,\ A_0)>0$, *then a unique strong solution* $(v,\ B)\in L^q(\mathbb{R}^+;L^p(\mathbb{R}^n))$ *of the problem (1)-(4) exists and*

$$\sup_{t\in\mathbb{R}^+}\|(w,E)(\cdot,t)\|_{L^p(\mathbb{R}^n)} \le C\|(w_0,\ E_0)\|_{L^p(\mathbb{R}^n)}, \tag{14}$$

$$\int_0^\infty\int_{\mathbb{R}^n}\left[|w|^{p-2}|\nabla w|^2+|E|^{p-2}|\nabla E|^2\right]dxdt \le C\|(w_0,\ E_0)\|^p_{L^p(\mathbb{R}^n)}, \tag{15}$$

$$\int_0^\infty\int_{\mathbb{R}^n}\left[\left|\nabla\left(|w|^{p/2}\right)\right|^2+\left|\nabla\left(|E|^{p/2}\right)\right|^2\right]dxdt \le C\|(w_0,\ E_0)\|^p_{L^p(\mathbb{R}^n)}, \tag{16}$$

where C *depends on* $\mathcal{A}$.

Proof The starting point is the following equations

$$\begin{aligned}\frac{d}{dt}\int_{\mathbb{R}^n}|w|^p dx+p\int_{\mathbb{R}^n}|w|^{p-2}|\nabla w|^2dx+\frac{4(p-2)}{p}\int_{\mathbb{R}^n}\left|\nabla\left(|w|^{p/2}\right)\right|^2dx\\ =-p\int_{\mathbb{R}^n}((w\cdot\nabla)u,|w|^{p-2}w)dx-p\int_{\mathbb{R}^n}((v\cdot\nabla)w,|w|^{p-2}w)dx\\ +p\int_{\mathbb{R}^n}((E\cdot\nabla)A,|w|^{p-2}w)dx+p\int_{\mathbb{R}^n}((B\cdot\nabla)E,|w|^{p-2}w)dx\\ -p\int_{\mathbb{R}^n}(\nabla\pi,|w|^{p-2}w)dx,\end{aligned}$$

and

$$\begin{aligned}\frac{d}{dt}\int_{\mathbb{R}^n}|E|^p dx+p\int_{\mathbb{R}^n}|E|^{p-2}|\nabla E|^2dx+\frac{4(p-2)}{p}\int_{\mathbb{R}^n}\left|\nabla\left(|E|^{p/2}\right)\right|^2dx\\ =-p\int_{\mathbb{R}^n}((w\cdot\nabla)A,|E|^{p-2}E)dx-p\int_{\mathbb{R}^n}((v\cdot\nabla)E,|E|^{p-2}E)dx\\ +p\int_{\mathbb{R}^n}((E\cdot\nabla)u,|E|^{p-2}E)dx+p\int_{\mathbb{R}^n}((B\cdot\nabla)w,|E|^{p-2}E)dx.\end{aligned}$$

If these equations are combined together, we have

$$\begin{aligned}\frac{d}{dt}\int_{\mathbb{R}^n}\left[|w|^p+|E|^p\right]dx+p\int_{\mathbb{R}^n}\left[|w|^{p-2}|\nabla w|^2+|E|^{p-2}|\nabla E|^2\right]dx\\ +\frac{4(p-2)}{p}\int_{\mathbb{R}^n}\left[\left|\nabla\left(|w|^{p/2}\right)\right|^2+\left|\nabla\left(|E|^{p/2}\right)\right|^2\right]dx\end{aligned}$$

$$
\begin{aligned}
&= - p\int_{\mathbb{R}^n}((w\cdot\nabla)u,|w|^{p-2}w)dx - p\int_{\mathbb{R}^n}((w\cdot\nabla)A,|E|^{p-2}E)dx\\
&\quad + p\int_{\mathbb{R}^n}((E\cdot\nabla)A,|w|^{p-2}w)dx + p\int_{\mathbb{R}^n}((B\cdot\nabla)E,|w|^{p-2}w)dx\\
&\quad + p\int_{\mathbb{R}^n}((E\cdot\nabla)u,|E|^{p-2}E)dx + p\int_{\mathbb{R}^n}((B\cdot\nabla)w,|E|^{p-2}E)dx\\
&\quad - p\int_{\mathbb{R}^n}(\nabla\pi,|w|^{p-2}w)dx,
\end{aligned}
$$

where

$$
\int_{\mathbb{R}^n}((v\cdot\nabla)w,|w|^{p-2}w)dx = 0, \qquad \int_{\mathbb{R}^n}((v\cdot\nabla)E,|E|^{p-2}E)dx = 0.
$$

Let us control the integrals on the right hand side one by one. First

$$
\begin{aligned}
-p\int_{\mathbb{R}^n}(\nabla\pi,|w|^{p-2}w)dx &= p\sum_{i=1}^n\int_{\mathbb{R}^n}\pi\frac{\partial}{\partial x_i}\left(|w|^{p-2}w_i\right)dx\\
&= p(p-2)\sum_{i=1}^n\int_{\mathbb{R}^n}\pi w_i|w|^{p-4}\left(w,\frac{\partial w}{\partial x_i}\right)dx\\
&\le p(p-2)\int_{\mathbb{R}^n}|\pi||w|^{p-2}|\nabla w|dx\\
&\le \frac{p}{8}\int_{\mathbb{R}^n}|w|^{p-2}|\nabla w|^2dx + 2p(p-2)^2\int_{\mathbb{R}^n}|\pi|^2|w|^{p-2}dx.
\end{aligned}
$$

Because

$$
\begin{aligned}
\triangle\pi &= \nabla\cdot\{(E\cdot\nabla)A + (B\cdot\nabla)E - (w\cdot\nabla)u - (v\cdot\nabla)w\}\\
&= \sum_{i=1}^n\sum_{j=1}^n\frac{\partial^2}{\partial x_i\partial x_j}\left(A_iE_j + B_iE_j - u_iw_j - v_iw_j\right),
\end{aligned}
$$

applying the Fourier transform yields

$$
|\xi|^2\widehat{\pi}(\xi,t) = \sum_{i=1}^n\sum_{j=1}^n\xi_i\xi_jF\left[A_iE_j + B_iE_j - u_iw_j - v_iw_j\right](\xi,t).
$$

By Calderon-Zygmund's inequality, for $1<\nu<\infty$, one has

$$
\|\pi(\cdot,t)\|^2_{L^\nu(R^n)} \le C(n,\nu)\sum_{i=1}^n\sum_{j=1}^n\left\|\left(A_iE_j + B_iE_j - u_iw_j - v_iw_j\right)(\cdot,t)\right\|^2_{L^\nu(R^n)},
$$

thus by Gagliardo-Nirenberg and Hölder's inequalities, letting $f = |w|^{p/2}$ and $g = |E|^{p/2}$, one obtains the estimates

$$
\begin{aligned}
&2p(p-2)^2\int_{\mathbb{R}^n}|\pi|^2|w|^{p-2}dx\\
&\quad\le 2p(p-2)^2\|\pi(\cdot,t)\|^2_{L^{p^2/2(p-1)}(\mathbb{R}^n)}\|w(\cdot,t)\|^{p-2}_{L^{p^2/(p-2)}(\mathbb{R}^n)}
\end{aligned}
$$

$$
\begin{aligned}
\leq & C\left[\|u(\cdot,t)\|^2_{L^p(\mathbb{R}^n)}+\|v(\cdot,t)\|^2_{L^p(\mathbb{R}^n)}\right]\|w(\cdot,t)\|^2_{L^{p^2/(p-2)}(\mathbb{R}^n)}\|w(\cdot,t)\|^{p-2}_{L^{p^2/(p-2)}(\mathbb{R}^n)}\\
&+C\left[\|A(\cdot,t)\|^2_{L^p(\mathbb{R}^n)}+\|B(\cdot,t)\|^2_{L^p(\mathbb{R}^n)}\right]\|E(\cdot,t)\|^2_{L^{p^2/(p-2)}(\mathbb{R}^n)}\|w(\cdot,t)\|^{p-2}_{L^{p^2/(p-2)}(\mathbb{R}^n)}\\
\leq & C\left[\|u(\cdot,t)\|^2_{L^p(\mathbb{R}^n)}+\|v(\cdot,t)\|^2_{L^p(\mathbb{R}^n)}+\|A(\cdot,t)\|^2_{L^p(\mathbb{R}^n)}\right.\\
&\qquad\left.+\|B(\cdot,t)\|^2_{L^p(\mathbb{R}^n)}\right]\|f(\cdot,t)\|^2_{L^{2p/(p-2)}(\mathbb{R}^n)}\\
&+C\left[\|A(\cdot,t)\|^2_{L^p(\mathbb{R}^n)}+\|B(\cdot,t)\|^2_{L^p(\mathbb{R}^n)}\right]\|g(\cdot,t)\|^2_{L^{2p/(p-2)}(\mathbb{R}^n)}\\
\leq & C\left[\|u(\cdot,t)\|^2_{L^p(\mathbb{R}^n)}+\|v(\cdot,t)\|^2_{L^p(\mathbb{R}^n)}+\|A(\cdot,t)\|^2_{L^p(\mathbb{R}^n)}\right.\\
&\qquad\left.+\|B(\cdot,t)\|^2_{L^p(\mathbb{R}^n)}\right]\|f(\cdot,t)\|^{2(1-n/p)}\|\nabla f(\cdot,t)\|^{2n/p}\\
&+C\left[\|A(\cdot,t)\|^2_{L^p(\mathbb{R}^n)}+\|B(\cdot,t)\|^2_{L^p(\mathbb{R}^n)}\right]\|g(\cdot,t)\|^{2(1-n/p)}\|\nabla g(\cdot,t)\|^{2n/p}\\
\leq & C\left[\|(u,\ A)(\cdot,t)\|^{2p/(p-n)}_{L^p(\mathbb{R}^n)}+\|(v,\ B)(\cdot,t)\|^{2p/(p-n)}_{L^p(\mathbb{R}^n)}\right]\|f(\cdot,t)\|^2+\frac{p-2}{2p}\|\nabla f(\cdot,t)\|^2\\
&+C\left[\|A(\cdot,t)\|^{2p/(p-n)}_{L^p(\mathbb{R}^n)}+\|B(\cdot,t)\|^{2p/(p-n)}_{L^p(\mathbb{R}^n)}\right]\|g(\cdot,t)\|^2+\frac{p-2}{2p}\|\nabla g(\cdot,t)\|^2\\
\leq & C\left[\|(u,\ A)(\cdot,t)\|^q_{L^p(\mathbb{R}^n)}+\|(v,\ B)(\cdot,t)\|^q_{L^p(\mathbb{R}^n)}\right]\int_{\mathbb{R}^n}\left[|w|^p+|E|^p\right]dx\\
&+\frac{p-2}{2p}\int_{\mathbb{R}^n}\left[\left|\nabla\left(|w|^{p/2}\right)\right|^2+\left|\nabla\left(|E|^{p/2}\right)\right|^2\right]dx.
\end{aligned}
$$

Secondly, because

$$
\begin{aligned}
-p\int_{\mathbb{R}^n}((E\cdot\nabla)A,|w|^{p-2}w)dx=&-p\sum_{i=1}^n\sum_{j=1}^n\int_{\mathbb{R}^n}|w|^{p-2}w_i\frac{\partial}{\partial x_j}(A_iE_j)dx\\
=&p\sum_{i=1}^n\sum_{j=1}^n\int_{\mathbb{R}^n}A_iE_j\frac{\partial}{\partial x_j}\left(|w|^{p-2}w_i\right)dx\\
=&p\sum_{i=1}^n\sum_{j=1}^n\int_{\mathbb{R}^n}|w|^{p-2}A_iE_j\frac{\partial w_i}{\partial x_j}dx\\
&+p(p-2)\sum_{i=1}^n\sum_{j=1}^n\int_{\mathbb{R}^n}|w|^{p-4}A_iw_iE_j\left(w,\frac{\partial w}{\partial x_j}\right)dx,
\end{aligned}
$$

we obtain

$$
\begin{aligned}
&\left|p\int_{\mathbb{R}^n}((E\cdot\nabla)A,|w|^{p-2}w)dx\right|\\
&\qquad\leq p\int_{\mathbb{R}^n}|w|^{p-2}\left(\sum_{i=1}^n\sum_{j=1}^n|A_iE_j|^2\right)^{1/2}\left(\sum_{i=1}^n\sum_{j=1}^n\left|\frac{\partial w_i}{\partial x_j}\right|^2\right)^{1/2}dx\\
&\qquad\quad+p(p-2)\int_{\mathbb{R}^n}|w|^{p-3}\left(\sum_{i=1}^n|A_iw_i|\right)\left(\sum_{j=1}^n|E_j|\left|\frac{\partial w}{\partial x_j}\right|\right)dx
\end{aligned}
$$

$$\begin{aligned}
&\le p(p-1)\int_{\mathbb{R}^n}|A||E||w|^{p-2}|\nabla w|dx\\
&\le \frac{p}{8}\int_{\mathbb{R}^n}|w|^{p-2}|\nabla w|^2dx+2p(p-1)^2\int_{\mathbb{R}^n}|A|^2|E|^2|w|^{p-2}dx\\
&\le \frac{p}{8}\int_{\mathbb{R}^n}|w|^{p-2}|\nabla w|^2dx+2(p-2)(p-1)^2\int_{\mathbb{R}^n}|A|^2|w|^pdx\\
&\quad+4(p-1)^2\int_{\mathbb{R}^n}|A|^2|E|^pdx,
\end{aligned}$$

and

$$\begin{aligned}
&2(p-2)(p-1)^2\int_{\mathbb{R}^n}|A|^2|w|^pdx\\
&\quad\le 2(p-2)(p-1)^2\|A(\cdot,t)\|^2_{L^p(\mathbb{R}^n)}\|w(\cdot,t)\|^p_{L^{p^2/(p-2)}(\mathbb{R}^n)}\\
&\quad= 2(p-2)(p-1)^2\|A(\cdot,t)\|^2_{L^p(\mathbb{R}^n)}\|f(\cdot,t)\|^2_{L^{2p/(p-2)}(\mathbb{R}^n)}\\
&\quad\le C\|A(\cdot,t)\|^2_{L^p(\mathbb{R}^n)}\|f(\cdot,t)\|^{2(1-n/p)}\|\nabla f(\cdot,t)\|^{2n/p}\\
&\quad\le C\|A(\cdot,t)\|^{2p/(p-n)}_{L^p(\mathbb{R}^n)}\|f(\cdot,t)\|^2+\frac{p-2}{2p}\|\nabla f(\cdot,t)\|^2\\
&\quad= C\|A(\cdot,t)\|^q_{L^p(\mathbb{R}^n)}\|f(\cdot,t)\|^2+\frac{p-2}{2p}\|\nabla f(\cdot,t)\|^2\\
&\quad= C\|A(\cdot,t)\|^q_{L^p(\mathbb{R}^n)}\|w(\cdot,t)\|^p_{L^p(\mathbb{R}^n)}+\frac{p-2}{2p}\int_{\mathbb{R}^n}\left|\nabla\left(|w|^{p/2}\right)\right|^2dx,\\
&4(p-1)^2\int_{\mathbb{R}^n}|A|^2|E|^pdx\\
&\quad\le C\|A(\cdot,t)\|^q_{L^p(\mathbb{R}^n)}\|E(\cdot,t)\|^p_{L^p(\mathbb{R}^n)}+\frac{p-2}{2p}\int_{\mathbb{R}^n}\left|\nabla\left(|E|^{p/2}\right)\right|^2dx.
\end{aligned}$$

Similar bounds for other nonlinear terms hold. Now we have the inequality

$$\begin{aligned}
&\frac{d}{dt}\int_{\mathbb{R}^n}\left[|w|^p+|E|^p\right]dx\ +\ \frac{p}{2}\int_{\mathbb{R}^n}\left[|w|^{p-2}|\nabla w|^2+|E|^{p-2}|\nabla E|^2\right]dx\\
&\qquad+\ \frac{p-2}{p}\int_{\mathbb{R}^n}\left[\left|\nabla\left(|w|^{p/2}\right)\right|^2+\left|\nabla\left(|E|^{p/2}\right)\right|^2\right]dx\\
&\qquad\le C\left[\|(u,\ A)(\cdot,t)\|^q_{L^p(\mathbb{R}^n)}+\|(v,\ B)(\cdot,t)\|^q_{L^p(\mathbb{R}^n)}\right]\int_{\mathbb{R}^n}\left[|w|^p+|E|^p\right]dx.
\end{aligned}$$

Rewrite this inequality as

$$\begin{aligned}
&\frac{d}{dt}\int_{\mathbb{R}^n}\left[|w|^p+|E|^p\right]dx\ +\ \frac{p}{2}\int_{\mathbb{R}^n}\left[|w|^{p-2}|\nabla w|^2+|E|^{p-2}|\nabla E|^2\right]dx\\
&\qquad+\ \frac{p-2}{p}\int_{\mathbb{R}^n}\left[\left|\nabla\left(|w|^{p/2}\right)\right|^2+\left|\nabla\left(|E|^{p/2}\right)\right|^2\right]dx\\
&\le C\|(u,\ A)(\cdot,t)\|^q_{L^p(\mathbb{R}^n)}\int_{\mathbb{R}^n}\left[|w|^p+|E|^p\right]dx+C\left\{\int_{\mathbb{R}^n}\left[|w|^p+|E|^p\right]dx\right\}^{1+2/(p-n)}.
\end{aligned}$$

This is an inequality of Bernoulli type. It is known that for small initial data, i.e. there is a constant $\delta > 0$, such that if $\int_{\mathbb{R}^n} [|w_0|^p + |E_0|^p]\,dx < \delta$, then the solution exists globally.

Let us go back to the above inequality. Gronwall's inequality yields the estimate

$$\int_{\mathbb{R}^n} [|w|^p + |E|^p]\,dx + \frac{p}{2}\int_0^t \int_{\mathbb{R}^n} \left[|w|^{p-2}|\nabla w|^2 + |E|^{p-2}|\nabla E|^2\right] dxds + \frac{p-2}{p}\int_0^t \int_{\mathbb{R}^n} \left[\left|\nabla\left(|w|^{p/2}\right)\right|^2 + \left|\nabla\left(|E|^{p/2}\right)\right|^2\right] dxds \le \exp\left\{C\int_0^\infty \left[\|(u,\ A)(\cdot,t)\|^q_{L^p(\mathbb{R}^n)} + \|(v,\ B)(\cdot,t)\|^q_{L^p(\mathbb{R}^n)}\right] dt\right\} \|(w_0,\ E_0)\|^p_{L^p(\mathbb{R}^n)}.$$

3. Regularity of Global Weak Solutions of Problems (1)-(4)

In this section we show that the weak solutions of the problem (1)-(4) are indeed strong solutions if they satisfy the Assumption (H).

Theorem 3 *Suppose that (H) holds. Then*

$$(u,\ A) \in \left(\bigcap_{p\le s<\infty} L^{2s/(s-n)}(\mathbb{R}^+; L^s(\mathbb{R}^n))\right) \bigcap \left(\bigcap_{2\le s<\infty} L^\infty(\mathbb{R}^+; L^s(\mathbb{R}^n))\right). \tag{17}$$

Proof Let $(v_0,\ B_0) = (0,0)$, then $(v(x,t),\ B(x,t)) \equiv (0,0)$. Therefore

$$\sup_{t\in\mathbb{R}^+} \|(u,\ A)(\cdot,t)\|_{L^p(\mathbb{R}^n)} \le C\|(u_0,\ A_0)\|_{L^p(\mathbb{R}^n)},$$

$$\int_0^\infty \int_{\mathbb{R}^n} \left[\left|\nabla\left(|u|^{p/2}\right)\right|^2 + \left|\nabla\left(|A|^{p/2}\right)\right|^2\right] dxdt \le C\|(u_0,\ A_0)\|^p_{L^p(\mathbb{R}^n)},$$

where C depends on $\mathcal{A}$. Let

$$m = \frac{n^2}{n^2 - 2n + 4}.$$

Then

$$1 < m < \frac{n}{n-2} \qquad \text{and} \qquad \frac{(p-2)(2m - mn + n)n}{4m(p-n)} = \frac{p-2}{p-n} > 1.$$

Let $q_1 = 2mp/(mp - n)$, then $n/mp + 2/q_1 = 1$. Let $f = |u|^{p/2}$, then one obtains the estimates

$$\begin{aligned}
\|u(\cdot,t)\|_{L^{mp}(\mathbb{R}^n)} &= \|f(\cdot,t)\|^{2/p}_{L^{2m}(\mathbb{R}^n)} \\
&\le C\|f(\cdot,t)\|^{(2m-mn+n)/mp}\|\nabla f(\cdot,t)\|^{(mn-n)/mp}, \\
\|u(\cdot,t)\|^{q_1}_{L^{mp}(\mathbb{R}^n)} &\le C\|f(\cdot,t)\|^{2(2m-mn+n)/(mp-n)}\|\nabla f(\cdot,t)\|^{2(mn-n)/(mp-n)} \\
&\le C\|f(\cdot,t)\|^{2(2m-mn+n)/m(p-n)} + \|\nabla f(\cdot,t)\|^2 \\
&= C\|u(\cdot,t)\|^{p(2m-mn+n)/m(p-n)}_{L^p(\mathbb{R}^n)} + \int_{\mathbb{R}^n} \left|\nabla\left(|u|^{p/2}\right)\right|^2 dx,
\end{aligned}$$

by [2]

$$\|u(\cdot,t)\|_{L^p(\mathbb{R}^n)}^{p(2m-mn+n)/m(p-n)}+\|A(\cdot,t)\|_{L^p(\mathbb{R}^n)}^{p(2m-mn+n)/m(p-n)}$$
$$\le C(1+t)^{-(p-2)(2m-mn+n)n/4m(p-n)}.$$

Therefore

$$\|u(\cdot,t)\|_{L^p(\mathbb{R}^n)}^{p(2m-mn+n)/m(p-n)}\in L^1(\mathbb{R}^+)$$

and

$$\int_{\mathbb{R}^n}\left|\nabla\left(|u|^{p/2}\right)\right|^2dx\in L^1(\mathbb{R}^+)$$

implies that

$$\|u(\cdot,t)\|_{L^{mp}(\mathbb{R}^n)}^{q_1}\in L^1(\mathbb{R}^+),$$

namely

$$u\in L^{q_1}(\mathbb{R}^+;L^{mp}(\mathbb{R}^n))\text{ and }u\in L^\infty(\mathbb{R}^+;L^{mp}(\mathbb{R}^n)).$$

Similarly

$$A\in L^{q_1}(\mathbb{R}^+;L^{mp}(\mathbb{R}^n))\text{ and }A\in L^\infty(\mathbb{R}^+;L^{mp}(\mathbb{R}^n)).$$

Since $mp>n\ge 3$ and $n/mp+2/q_1=1$, the initial velocity $(u_0,A_0)\in L^2(\mathbb{R}^n)\bigcap L^{mp}(\mathbb{R}^n)$ and the solution $(u,\ A)\in L^{q_1}(\mathbb{R}^+;L^{mp}(\mathbb{R}^n))\bigcap L^\infty(\mathbb{R}^+;L^{mp}(\mathbb{R}^n))$. Repeating the same procedure, we obtain

$$(u,\ A)\in L^{q_2}\left(\mathbb{R}^+;L^{m^2p}(\mathbb{R}^n)\right)\bigcap L^\infty\left(\mathbb{R}^+;L^{m^2p}(\mathbb{R}^n)\right).$$

Therefore, if iterate this procedure for infinitely many times, we obtain

$$(u,\ A)\in L^{q_k}\left(\mathbb{R}^+;L^{m^kp}(\mathbb{R}^n)\right)\bigcap L^\infty\left(\mathbb{R}^+;L^{m^kp}(\mathbb{R}^n)\right),$$

for all $k\ge 1$, where $n/m^kp+2/q_k=1$. Notice that $m>1$, hence $m^k\to\infty$, as $k\to\infty$.

Proposition 1 *Let α,β and γ be real numbers such that $1\le\alpha\le\beta\le\gamma<\infty$ and $\alpha<\gamma$. Let $f\in L^{2\alpha/(\alpha-n)}(\mathbb{R}^+;L^\alpha(\mathbb{R}^n))\bigcap L^{2\gamma/(\gamma-n)}(\mathbb{R}^+;L^\gamma(\mathbb{R}^n))$. Then*

$$\|f(\cdot,t)\|_{L^\beta(\mathbb{R}^n)}^\beta\le\left[\|f(\cdot,t)\|_{L^\alpha(\mathbb{R}^n)}^\alpha\right]^{(\gamma-\beta)/(\gamma-\alpha)}\left[\|f(\cdot,t)\|_{L^\gamma(\mathbb{R}^n)}^\gamma\right]^{(\beta-\alpha)/(\gamma-\alpha)},\tag{18}$$

and

$$\int_0^\infty\|f(\cdot,t)\|_{L^\beta(\mathbb{R}^n)}^{2\beta/(\beta-n)}dt$$
$$\le\int_0^\infty\|f(\cdot,t)\|_{L^\alpha(\mathbb{R}^n)}^{2\alpha(\gamma-\beta)/(\gamma-\alpha)(\beta-n)}\|f(\cdot,t)\|_{L^\gamma(\mathbb{R}^n)}^{2\gamma(\beta-\alpha)/(\gamma-\alpha)(\beta-n)}dt$$
$$\le\left(\int_0^\infty\|f(\cdot,t)\|_{L^\alpha(\mathbb{R}^n)}^{2\alpha/(\alpha-n)}dt\right)^{(\gamma-\beta)(\alpha-n)/(\gamma-\alpha)(\beta-n)}$$
$$\times\left(\int_0^\infty\|f(\cdot,t)\|_{L^\gamma(\mathbb{R}^n)}^{2\gamma/(\gamma-n)}dt\right)^{(\beta-\alpha)(\gamma-n)/(\gamma-\alpha)(\beta-n)}.$$

Proof. Applying the Hölder's inequality yields the desired estimates.
Let us now apply the above proposition. For all $s : m^k p \le s \le m^{k+1}p$, let $\alpha = m^k p,\ \beta = s,\ \gamma = m^{k+1}p$ and $f = u$. Then

$$\|u(\cdot,t)\|^s_{L^s(\mathbb{R}^n)}=\|u(\cdot,t)\|^\beta_{L^\beta(\mathbb{R}^n)} \le \Big[\|u(\cdot,t)\|^\alpha_{L^\alpha(\mathbb{R}^n)}\Big]^{(\gamma-\beta)/(\gamma-\alpha)}\Big[\|u(\cdot,t)\|^\gamma_{L^\gamma(\mathbb{R}^n)}\Big]^{(\beta-\alpha)/\beta(\gamma-\alpha)}$$

$$= \|u(\cdot,t)\|^{m^k p\left(m^{k+1}p-s\right)/\left(m^{k+1}p-m^k p\right)}_{L^{m^k p}(\mathbb{R}^n)}\|u(\cdot,t)\|^{m^{k+1}p\left(s-m^k p\right)/\left(m^{k+1}p-m^k p\right)}_{L^{m^{k+1}p}(\mathbb{R}^n)} < \infty,$$

and

$$\begin{aligned}
&\int_0^\infty \|u(\cdot,t)\|^{2s/(s-n)}_{L^s(\mathbb{R}^n)}dt\\
&= \int_0^\infty \|u(\cdot,t)\|^{2\beta/(\beta-n)}_{L^\beta(\mathbb{R}^n)}dt\\
&\le \left(\int_0^\infty \|u(\cdot,t)\|^{2\alpha/(\alpha-n)}_{L^\alpha(\mathbb{R}^n)}dt\right)^{(\gamma-\beta)(\alpha-n)/(\gamma-\alpha)(\beta-n)}\\
&\quad\times\left(\int_0^\infty \|u(\cdot,t)\|^{2\gamma/(\gamma-n)}_{L^\gamma(\mathbb{R}^n)}dt\right)^{(\beta-\alpha)(\gamma-n)/(\gamma-\alpha)(\beta-n)}\\
&= \left(\int_0^\infty \|u(\cdot,t)\|^{2m^k p/\left(m^k p-n\right)}_{L^{m^k p}(\mathbb{R}^n)}dt\right)^{\left(m^{k+1}p-s\right)\left(m^k p-n\right)/\left(m^{k+1}p-m^k p\right)(s-n)}\\
&\quad\times\left(\int_0^\infty \|u(\cdot,t)\|^{2m^{k+1}p/\left(m^{k+1}p-n\right)}_{L^{m^{k+1}p}(\mathbb{R}^n)}dt\right)^{\left(s-m^k p\right)\left(m^{k+1}p-n\right)/\left(m^{k+1}p-m^k p\right)(s-n)} < \infty.
\end{aligned}$$

Therefore $u \in L^\infty(\mathbb{R}^+; L^r(\mathbb{R}^n))$, for all $r : p \le r < \infty$. By Theorem 1, $u \in L^\infty(\mathbb{R}^+; L^2(\mathbb{R}^n))$. As before, applying the Hölder's inequality yields $u \in L^\infty(\mathbb{R}^+; L^s(\mathbb{R}^n))$, for all $2 \le s < \infty$. The same estimates for A hold. This completes the proofs.

4. Appendix: Elementary Estimates of the Global Solutions

We first present some well-known differential inequalities in partial differential equations with dissipation.

Lemma A(Generalized Gronwall's inequality) *Let $g(t) \ge 0$ and $h(t) \ge 0$ satisfy the inequality*

$$g(t) \le C + \int_0^t g(s)h(s)ds,$$

where $C \ge 0$ is a constant, and $h(t) \in L^1(\mathbb{R}^+)$. Then we have the estimate

$$g(t) \le C\exp\left[\int_0^\infty h(t)dt\right].$$

Lemma B(Gagliardo-Nirenberg's inequality) *For all real numbers* $p, q, r \geq 1$ *and for all integers* m, k *with* $m > k$, *there exist constants* $\alpha : k/m \leq \alpha \leq 1$ *and* $C(m, n, k, p, q, r) > 0$ *such that for all* $u(x) \in C_0^\infty(\mathbb{R}^n)$,

$$\|D^k u\|_{L^p(\mathbb{R}^n)} \leq C\|D^m u\|^{\alpha}_{L^r(\mathbb{R}^n)}\|u\|^{1-\alpha}_{L^q(\mathbb{R}^n)},$$

where

$$\begin{aligned} n/p - k &= \alpha(n/r - m) + (1-\alpha)n/q, \\ \left\|D^k u\right\|^p_{L^p(\mathbb{R}^n)} &= \sum_{\beta_1+\cdots+\beta_n=k}\left\|\frac{\partial^{\beta_1+\cdots+\beta_n}u}{\partial x_1^{\beta_1}\cdots\partial x_n^{\beta_n}}\right\|^p_{L^p(\mathbb{R}^n)}. \end{aligned}$$

The only exception is that $\alpha \neq 1$ *if* $m - n/r = k$ *and* $1 < p < \infty$.

We then prove the basic estimates regarding the global weak solutions of the problem (1)-(4).

Lemma C *Let* $(u,\ A,\ p)$ *be the solutions of the problem* (1)-(4) *corresponding to* $(u_0,\ A_0) \in L^2(\mathbb{R}^n)$. *Then*

$$\sup_{t\in\mathbb{R}^+}\|(u,\ A)(\cdot,t)\|^2 \leq \|(u_0,\ A_0)\|^2, \quad 2\int_0^\infty \|\nabla(u,\ A)(\cdot,t)\|^2 dt \leq \|(u_0,\ A_0)\|^2, \tag{19}$$

$$\sup_{t\in\mathbb{R}^+}\|\widehat{p}(\cdot,t)\|_{L^\infty(\mathbb{R}^n)} \leq \|(u_0,\ A_0)\|^2, \quad 2\int_0^\infty \|\widehat{\triangle p}(\cdot,t)\|_{L^\infty(\mathbb{R}^n)}dt \leq \|(u_0,\ A_0)\|^2, \tag{20}$$

$$|\widehat{(u\cdot\nabla)u}(\xi,t)| \leq |\xi|\|u(\cdot,t)\|^2, \qquad |\widehat{(A\cdot\nabla)A}(\xi,t)| \leq |\xi|\|A(\cdot,t)\|^2, \tag{21}$$

$$|\widehat{(u\cdot\nabla)A}(\xi,t)| \leq |\xi|\|u(\cdot,t)\|\|A(\cdot,t)\|, \quad |\widehat{(A\cdot\nabla)u}(\xi,t)| \leq |\xi|\|u(\cdot,t)\|\|A(\cdot,t)\|. \tag{22}$$

Proof By using the equations (1)-(4), one obtains

$$\begin{aligned} &2\int_{\mathbb{R}^n} u\cdot\{u_t + (u\cdot\nabla)u - (A\cdot\nabla)A - \triangle u + \nabla p\}dx \\ &\quad +2\int_{\mathbb{R}^n} A\cdot\{A_t + (u\cdot\nabla)A - (A\cdot\nabla)u - \triangle A\}dx = 0, \end{aligned}$$

note that $\nabla\cdot u = \nabla\cdot A = 0$, equivalently we have

$$\frac{d}{dt}\|(u,\ A)(\cdot,t)\|^2 + 2\|\nabla(u,\ A)(\cdot,t)\|^2 = 0.$$

Thus the estimate (19) follows. The following identity follows by taking the divergence of the equation (1)

$$\sum_{i=1}^n\sum_{j=1}^n \frac{\partial^2}{\partial x_i\partial x_j}(u_iu_j - A_iA_j) + \triangle p = 0. \tag{23}$$

The Fourier transform yields

$$\sum_{i=1}^{n}\sum_{j=1}^{n}\xi_i\xi_j\left(\widehat{u_iu_j}(\xi,t)-\widehat{A_iA_j}(\xi,t)\right)+|\xi|^2\widehat{p}(\xi,t)=0.$$

Applying triangle inequality and Cauchy-Schwartz inequality gives the estimates

$$\begin{aligned}
|\xi|^2|\widehat{p}(\xi,t)| &\leq \sum_{i=1}^{n}\sum_{j=1}^{n}|\xi_i||\xi_j|\left[\|u_i(\cdot,t)\|\|u_j(\cdot,t)\|+\|A_i(\cdot,t)\|\|A_j(\cdot,t)\|\right]\\
&= \sum_{i=1}^{n}|\xi_i|\|u_i(\cdot,t)\|\sum_{j=1}^{n}|\xi_j|\|u_j(\cdot,t)\|+\sum_{i=1}^{n}|\xi_i|\|A_i(\cdot,t)\|\sum_{j=1}^{n}|\xi_j|\|A_j(\cdot,t)\|\\
&\leq |\xi|^2\left[\|u(\cdot,t)\|^2+\|A(\cdot,t)\|^2\right]=|\xi|^2\|(u,\ A)(\cdot,t)\|^2.
\end{aligned}$$

Therefore, we obtain

$$\|\widehat{p}(\cdot,t)\|_{L^\infty(\mathbb{R}^n)}\leq\|(u,\ A)(\cdot,t)\|^2\leq\|(u_0,\ A_0)\|^2.$$

On the other hand, we have also the relation from the equation (1)

$$\sum_{i=1}^{n}\sum_{j=1}^{n}\left(\frac{\partial u_j}{\partial x_i}\frac{\partial u_i}{\partial x_j}-\frac{\partial A_j}{\partial x_i}\frac{\partial A_i}{\partial x_j}\right)+\triangle p=0,$$

which leads to

$$\sum_{i=1}^{n}\sum_{j=1}^{n}F\left[\frac{\partial u_j}{\partial x_i}\frac{\partial u_i}{\partial x_j}-\frac{\partial A_j}{\partial x_i}\frac{\partial A_i}{\partial x_j}\right](\xi,t)+\widehat{\triangle p}(\xi,t)=0.$$

Therefore, we get

$$\begin{aligned}
\|\widehat{\triangle p}(\cdot,t)\|_{L^\infty(\mathbb{R}^n)} &\leq \sum_{i=1}^{n}\sum_{j=1}^{n}\left\|\frac{\partial u_j}{\partial x_i}(\cdot,t)\right\|\left\|\frac{\partial u_i}{\partial x_j}(\cdot,t)\right\|+\sum_{i=1}^{n}\sum_{j=1}^{n}\left\|\frac{\partial A_j}{\partial x_i}(\cdot,t)\right\|\left\|\frac{\partial A_i}{\partial x_j}(\cdot,t)\right\|\\
&\leq \left[\sum_{i=1}^{n}\sum_{j=1}^{n}\left\|\frac{\partial u_j}{\partial x_i}(\cdot,t)\right\|^2\right]^{1/2}\left[\sum_{i=1}^{n}\sum_{j=1}^{n}\left\|\frac{\partial u_i}{\partial x_j}(\cdot,t)\right\|^2\right]^{1/2}\\
&\quad+\left[\sum_{i=1}^{n}\sum_{j=1}^{n}\left\|\frac{\partial A_j}{\partial x_i}(\cdot,t)\right\|^2\right]^{1/2}\left[\sum_{i=1}^{n}\sum_{j=1}^{n}\left\|\frac{\partial A_i}{\partial x_j}(\cdot,t)\right\|^2\right]^{1/2}\\
&= \|\nabla u(\cdot,t)\|^2+\|\nabla v(\cdot,t)\|^2=\|\nabla(u,\ A)(\cdot,t)\|^2.
\end{aligned}$$

By these estimates, the estimate (20) follow. Furthermore, by $\nabla\cdot u=0$ in $\mathbb{R}^n$, we obtain

$$(u\cdot\nabla)A=\sum_{j=1}^{n}u_j\frac{\partial A}{\partial x_j}=\sum_{j=1}^{n}\frac{\partial}{\partial x_j}(u_jA).$$

Applying the Fourier transform gives

$$\widehat{(u\cdot\nabla)A}(\xi,t)=\ \mathrm{i}\ \sum_{j=1}^{n}\xi_j\widehat{u_jA}(\xi,t).$$

If we apply the Cauchy-Schwartz inequality, we get

$$\left|\widehat{(u\cdot\nabla)A}(\xi,t)\right|\le|\xi|\|u(\cdot,t)\|\|A(\cdot,t)\|.$$

Lemma D *For the solutions of the problem* (1)-(4)*, there holds the estimates*

$$\left|\left(\widehat{u}(\xi,t),\widehat{A}(\xi,t)\right)\right|\le\|(u_0,\ A_0)\|_{L^1(\mathbb{R}^n)}+3|\xi|\int_0^t\|(u,\ A)(s)\|^2ds. \tag{24}$$

Proof Applying the Fourier transform to the Magnetohydrodynamics equations (1)-(4) gives

$$\begin{aligned}&\widehat{u}_t(\xi,t)+|\xi|^2\widehat{u}(\xi,t)+\left[\widehat{(u\cdot\nabla)u}-\widehat{(A\cdot\nabla)A}+\widehat{\nabla p}\right](\xi,t)=0,\\&\widehat{A}_t(\xi,t)+|\xi|^2\widehat{A}(\xi,t)+\left[\widehat{(u\cdot\nabla)A}-\widehat{(A\cdot\nabla)u}\right](\xi,t)=0.\end{aligned}$$

Multiplying these equations by the integrating factor $\exp\left(|\xi|^2t\right)$ yields

$$\begin{aligned}&\frac{d}{dt}\left[\exp\left(|\xi|^2t\right)\widehat{u}(\xi,t)\right]+\exp\left(|\xi|^2t\right)\left[\widehat{(u\cdot\nabla)u}-\widehat{(A\cdot\nabla)A}+\widehat{\nabla p}\right](\xi,t)=0,\\&\frac{d}{dt}\left[\exp\left(|\xi|^2t\right)\widehat{A}(\xi,t)\right]+\exp\left(|\xi|^2t\right)\left[\widehat{(u\cdot\nabla)A}-\widehat{(A\cdot\nabla)u}\right](\xi,t)=0.\end{aligned}$$

Integrating in time yields

$$\begin{aligned}&\exp\left(|\xi|^2t\right)\widehat{u}(\xi,t)+\int_0^t\exp\left(|\xi|^2s\right)\left[\widehat{(u\cdot\nabla)u}-\widehat{(A\cdot\nabla)A}+\widehat{\nabla p}\right](\xi,s)ds=\widehat{u_0}(\xi),\\&\exp\left(|\xi|^2t\right)\widehat{A}(\xi,t)+\int_0^t\exp\left(|\xi|^2s\right)\left[\widehat{(u\cdot\nabla)A}-\widehat{(A\cdot\nabla)u}\right](\xi,s)ds=\widehat{A_0}(\xi),\end{aligned}$$

or we have

$$\begin{aligned}&\widehat{u}(\xi,t)=\exp\left(-|\xi|^2t\right)\widehat{u_0}(\xi)-\int_0^t\exp\left[-|\xi|^2(t-s)\right]\left[\widehat{(u\cdot\nabla)u}-\widehat{(A\cdot\nabla)A}+\widehat{\nabla p}\right](\xi,s)ds,\\&\widehat{A}(\xi,t)=\exp\left(-|\xi|^2t\right)\widehat{A_0}(\xi)-\int_0^t\exp\left[-|\xi|^2(t-s)\right]\left[\widehat{(u\cdot\nabla)A}-\widehat{(A\cdot\nabla)u}\right](\xi,s)ds.\end{aligned}$$

Therefore

$$\begin{aligned}&\left(\widehat{u}(\xi,t),\widehat{A}(\xi,t)\right)=\exp\left(-|\xi|^2t\right)\left(\widehat{u_0}(\xi),\widehat{A_0}(\xi)\right)\\&\quad-\int_0^t\exp\left[-|\xi|^2(t-s)\right]\left(\widehat{(u\cdot\nabla)u}-\widehat{(A\cdot\nabla)A}+\widehat{\nabla p},\ \widehat{(u\cdot\nabla)A}-\widehat{(A\cdot\nabla)u}\right)(\xi,s)ds,\end{aligned}$$

and then

$$\begin{aligned}
\left|\left(\widehat{u}(\xi,t),\widehat{A}(\xi,t)\right)\right| &\leq \left|\left(\widehat{u_0}(\xi),\ \widehat{A_0}(\xi)\right)\right| \\
&\quad + \int_0^t \left\{\left|\left[\widehat{(u\cdot\nabla)u} - \widehat{(A\cdot\nabla)A} + \widehat{\nabla p}\right](\xi,s)\right| + \left|\left[\widehat{(u\cdot\nabla)A} - \widehat{(A\cdot\nabla)u}\right](\xi,s)\right|\right\} ds \\
&\leq \|(u_0,\ A_0)\|_{L^1(\mathbb{R}^n)} + 3|\xi|\int_0^t \|(u,\ A)(s)\|^2 ds \\
&\leq \|(u_0,\ A_0)\|_{L^1(\mathbb{R}^n)} + 3|\xi|\|(u_0,\ A_0)\|^2.
\end{aligned}$$

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