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A REMARK ON THE REGULARITY OF SOLUTIONS TO THE NAVIER–STOKES EQUATIONS*

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Abstract In this note we shall give a simple proof of a result in [1] which gives a sufficient condition for the regularity of solutions to the Navier–Stokes equation in $\mathbb{R}^n$ based on estimates on the vorticity.

Key Words Regularity, Cauchy problem, Navier–Stokes equation, admissible triplet, time–space estimates.

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1. Introduction and the Main Results

In this note we are concerned with the following Cauchy problem in $\mathbb{R}^n \times (0,T)$

$$\partial_t v - \triangle v + (v\cdot\nabla)v + \nabla P = 0, \tag{1.1}$$

$$\text{div } v = 0, \tag{1.2}$$

$$v(0) = v_0(x), \tag{1.3}$$

where $v(t) = v(t,x) = (v_1(t,x), v_2(t,x), \cdots, v_n(t,x))$, is the velocity field, P is the pressure.

Definition 1.1 *A vector field $v \in L^\infty((0,T);L^2(\mathbb{R}^n)) \cap L^2((0,T);\dot{H}^1(\mathbb{R}^n))$ is called the Leray–Hopf weak solution if*

$$\int_0^T\int_{\mathbb{R}^n}[v\cdot\varphi_t + (v\cdot\nabla)\varphi\cdot v + v\cdot\triangle\varphi]dxdt = 0,$$
$$\textit{for}\quad \forall\varphi\in[\mathcal{C}_0^\infty(\mathbb{R}^n\times(0,T)]^n, \quad \textit{with}\quad \textit{div}\,\varphi = 0, \tag{1.4}$$

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and

$$div\ v = 0, \tag{1.5}$$

in the distributional sense.

For $v_0 \in L^2(\mathbb{R}^n)$ with div $v_0 = 0$, the global existence of weak solution was established by Leray and Hopf in [2] and [3]. It is still unknown whether the Leray–Hopf weak solution to the Navier–Stokes equations is unique. As for the strong solution or $L^q(I; L^p)$–solutions, it is well known that for $v_0 \in H^1(\mathbb{R}^n)$ $(n \leq 4)$ with div $v_0 = 0$ or $v_0 \in L^r(\mathbb{R}^n)$ $(r \geq n)$ with div $v_0 = 0$ in distributional sense, then there exists a local unique strong solution $v \in \mathcal{C}([0,T); H^1(\mathbb{R}^n))$ $(n \leq 4)$ or $L^q(I; L^p)$–solution for any space dimensions, where the maximal time existence T_* depends on the initial data $\|v_0 :\ H^1(\mathbb{R}^n)\|$ $(n \leq 4)$ or $\|v_0\|_r$ in the subcritical case $r > n$ and depends on v_0 itself in the critical case $r = n$, for details see [4–13] and [14]. As an immediate consequence of regularity of analytic semigroup which is generated by the Stokes operator, one easily sees that the strong solution $(n \leq 4)$ and the $L^q(I; L^p)$–solution belong to the $\mathcal{C}((0,T); \mathcal{C}^\infty(\mathbb{R}^n))$, see [9] and [13]. The global in time existence of strong solution or $L^q(I; L^p)$–solution is an outstanding open problem. Many authors have deduced the sufficient conditions under which the Leray–Hopf weak solution agrees with the smooth solution. In this direction, there is a classical result due to Serrin [12], which states that if a Leray–Hopf weak solution belongs to $L^q(I; L^p(\mathbb{R}^3))$, $\frac{2}{q} + \frac{3}{p} < 1$ and $q < \infty$, then v becomes the smooth solution. Later, Fabes, Jone and Riviere in [4] extend the above criterion to the case $\frac{2}{q} + \frac{3}{p} = 1$. The case $q = \infty$, $p = 3$ in Serrin's conditions, regularity and uniqueness of the solution to the Navier–Stokes equations was established in [13]. For general space dimension case $(\frac{2}{q} + \frac{n}{p} \leq 1)$ has been studied by many authors, see [8,9] and [13] and references therein.

Recently, Beirão da Veiga [1] obtained a sufficient condition for regularity using the vorticity $w =$ curl v, rather than the velocity v, his results can be stated as follows:

Theorem 1.1 *Let $v_0 \in L^2(\mathbb{R}^3)$ with div v_0=0 and $w_0 =$ curl $v_0 \in L^2(\mathbb{R}^3)$. If the Leray–Hopf weak solution v satisfies $w =$ curl $v \in L^q(I; L^p(\mathbb{R}^3))$ with $\frac{2}{q} + \frac{3}{p} \leq 2$, $1 < q < \infty$, then v becomes the classical solution on $I = (0,T)$.*

In [15] Dongho Chae & Hi–Jun Choe extended the results of [1] as:

Theorem 1.2 *Let $v_0 \in L^2(\mathbb{R}^3)$ with div v_0=0 and $\omega_0 =$ curl $v_0 \in L^2(\mathbb{R}^3)$. Let v be the Leray–Hopf weak solution to* (1.1)*, $w =$ curl v. Assume that $\tilde{\omega} \in L^q(I; L^p(\mathbb{R}^3))$ with $\frac{2}{q} + \frac{3}{p} \leq 2$, $1 < q < \infty$, where*

$$\tilde{\omega} = \omega_1 e_1 + \omega_2 e_2, \quad e_1 = (1,0,0), \quad e_2 = (0,1,0). \tag{1.6}$$

Then v becomes the classical solution on $I = (0,T)$.

In (1.6) $\omega_1 e_1$ or $\omega_2 e_2$ can be replaced by $\omega_3 e_3$, which means that the regularity of the solution of (1.1) depends on two components of the vorticity field.

In this note, we shall give a simple proof of Theorem 1.1 and its generalization in higher dimensions.

Theorem 1.3 *Let $r \geq n$, $v_0 \in L^2(\mathbb{R}^n) \cap L^r(\mathbb{R}^n)$ with div v_0=0 and $\omega_0 =$ curl $v_0 \in L^2(\mathbb{R}^n)$. If the Leray–Hopf weak solution v satisfies $\omega =$ curlv $\in L^{\tilde{q}}(I; L^{\tilde{p}}(\mathbb{R}^n))$ with $\frac{2}{\tilde{q}} + \frac{n}{\tilde{p}} \leq 2$, $1 < \tilde{q} < \infty$, then v becomes the classical solution on $I = (0, T)$.*

Remark 1.1 When $n \leq 4$, $v_0 \in L^2(\mathbb{R}^n)$ and curl $v_0 \in L^2(\mathbb{R}^n)$ imply $\nabla v_0(x) \in L^2(\mathbb{R}^n)$. In fact,

$$\begin{aligned} \int_{\mathbb{R}^n} |\nabla v_0|^2 dx &= \int_{\mathbb{R}^n} |\text{dvi } v_0|^2 dx + \int_{\mathbb{R}^n} |\text{curl } v_0|^2 dx \\ &= \int_{\mathbb{R}^n} |\text{curl } v_0|^2 dx, \end{aligned} \tag{1.7}$$

by div $v_0(x) = 0$. One finds that $v_0 \in H^1(\mathbb{R}^n)$, so we have that $v_0 \in L^n(\mathbb{R}^n)$ if $2 \leq n \leq 4$.

2. The Proof of Theorem 1.3

The IVP (1.1) can be written as the following abstract Cauchy problem [7]

$$\begin{cases} \partial_t v + Av + \mathcal{P}\partial \cdot (v \otimes v) = 0, & x \in \mathbb{R}^n, n \geq 2, t > 0 \\ v(0) = v_0(x), \end{cases} \tag{2.1}$$

where $v = v(t)$: $\mathbb{R}^n \to \mathbb{R}^n$ is the velocity field, $A = \mathcal{P}\triangle$, $\mathcal{P}$ is the projection operator into divergence-free vectors along gradients, $v \otimes v$ is the tensor with jk- components $v_j v_k$ and $\partial \cdot (v \otimes v)$ is the vector with j-component $\partial_k(v_j v_k)$. Of course, A generates an analytic operator semigroup in L^p with divergence-free vectors, where $1 < p < \infty$.

For convenience, we introduce the admissible triplet and generalized admissible triplet before we prove our main results.

Definition 2.1 *We call (p, q, r) an admissible triplet if*

$$\frac{1}{q} = \frac{n}{2}(\frac{1}{r} - \frac{1}{p}), \tag{2.2}$$

where

$$1 < r \leq p < \begin{cases} \dfrac{nr}{n-2}, & n > 2, \\ \infty, & n \leq 2. \end{cases} \tag{2.3}$$

Definition 2.2 *We call (p, q, r) a generalized admissible triplet if*

$$\frac{1}{q} = \frac{n}{2}(\frac{1}{r} - \frac{1}{p}), \tag{2.4}$$

where

$$1 < r \leq p < \begin{cases} \dfrac{nr}{n-2r}, & n > 2r, \\ \infty, & n \leq 2r. \end{cases} \tag{2.5}$$

Remark 2.1 (i) The admissible triplet was introduced in [10, 16], for its generalied form can be founded in [11].

(ii) It is easy to see that $r < q \leq \infty$ if (p, q, r) is an admissible triplet. Also one has that $1 < q \leq \infty$ if (p, q, r) is a generalized admissible triplet.

(iii) One can deal with the well posedness of the nonlinear parabolic equations and Navier–Stokes equations in the working space $L^q(I; L^p)$ with initial data $v_0 \in L^r$ (with div $v_0(x) = 0$ for Navier– Stokes equations and $r \geq n$), where (p, q, r) is an admissible triplet. For details see [5–7, 10, 11, 16].

(iv) We observe that $\frac{2}{q} + \frac{n}{p} \leq 1$ is equivalent to: there is a $r \geq n$ such that

$$\frac{2}{q} = n(\frac{1}{r} - \frac{1}{p}).$$

Proposition 2.1 *Let (p, q, r) be any admissible triplet with $r \geq n$. Assume that v be a Leray–Hopf weak solution with $v \in L^q(I; L^p)$, then v becomes smooth solution.*

Remark 2.2 (i) Let $v = v(t, x)$ be the solution to the Navier–Stokes equations, then $v_\lambda = \lambda v(\lambda^2 t, \lambda x)$ should be the solution of the Navier–Stokes equations by the scaling technique. It is easily seen that

$$\|v_\lambda\|_{L^q(\mathbb{R}; L^p(\mathbb{R}^n))} = \lambda^{1-\frac{n}{p}-\frac{2}{q}} \|v(\cdot, \cdot)\|_{L^q(\mathbb{R}; L^p(\mathbb{R}^n))}. \tag{2.6}$$

Hence for $1 - \frac{n}{p} - \frac{2}{q} = 0$, i.e. $\frac{2}{q} = n(\frac{1}{n} - \frac{1}{p})$, one gets that the initial critical functional space $L^n(\mathbb{R}^n)$. Also it follows by homogeneity that $v \in L^q(\mathbb{R}; L^p(\mathbb{R}^n))$ provided that $\|\varphi\|_n$ is sufficently small.

When $1 - \frac{n}{p} - \frac{2}{q} \geq 0$, there exists a $r \geq n$ such that $\frac{2}{q} = n(\frac{1}{r} - \frac{1}{p})$, which is the definition of an admissible triplet. In Proposition 2.1, $r \geq n$ implies that $1 - \frac{n}{p} - \frac{2}{q} \geq 0$.

(ii) Let $\omega = \text{curl } v = (\frac{\partial v_k}{\partial x_j} - \frac{\partial v_j}{\partial x_k})_{1 \leq j,k \leq n}$, where v is the solution of the Navier–Stokes equations, then $\omega_\lambda = \text{curl } v_\lambda$ satisfies

$$\omega_{\lambda t} + (v_\lambda \cdot \nabla)\omega_\lambda = F(\nabla v_\lambda) + \triangle \omega_\lambda \tag{2.7}$$

by the scalling method, where

$$F(\nabla v) = (\sum_{i=1}^{n} \frac{\partial v_i}{\partial x_j} \frac{\partial v_k}{\partial x_i} - \sum_{i=1}^{n} \frac{\partial v_i}{\partial x_k} \frac{\partial v_j}{\partial x_i})_{1 \leq j,k \leq n}.$$

Notice that

$$\|\omega_\lambda\|_{L^{\tilde{q}}(\mathbb{R}; L^{\tilde{p}}(\mathbb{R}^n))} = \lambda^{2-\frac{n}{\tilde{p}}-\frac{2}{\tilde{q}}} \|v\|_{L^{\tilde{q}}(\mathbb{R}; L^{\tilde{p}}(\mathbb{R}^n))}. \tag{2.8}$$

It follows that

$$2 - \frac{n}{\tilde{p}} - \frac{2}{\tilde{q}} \leq 0$$

is an admissible triplet $(\tilde{p}, \tilde{q}, \tilde{r})$ with $\tilde{r} \geq \frac{n}{2}$.

We now consider the following abstract linear parabolic Cauchy problem

$$\begin{cases} u_t + Au = f(x,t), & t \in [0,T), \quad 0 < T \le \infty \\ u(0) = \varphi(x), & \varphi \in D(A), \end{cases} \tag{2.9}$$

where $D(A) = W^{2,p} \cap W_0^{1,p}$. It is well known that

$$u(t) = e^{-At}\varphi + \int_0^t e^{-(t-s)A} f(x,t)ds \triangleq e^{-At}\varphi + Gf(x,t) \tag{2.10}$$

solves (2.1). By interpolation one gets that

$$\|e^{-At}\varphi\|_p \le Ct^{-\frac{n}{2}(\frac{1}{r}-\frac{1}{p})}\|\varphi\|_r, \quad t>0, \\ p \ge r, \quad \varphi \in D(A), \tag{2.11}$$

see [5, 16, 18, 19] for details. From Marcinkiewicz's interpolation theorem in [19] and (2.11), one obtains the following estimate:

Proposition 2.2 *Let (p,q,r) be any admissible triplet. If $\varphi(x) \in L^r$, then $e^{-At}\varphi \in L^q([0,\infty);L^p) \cap \mathcal{C}_b([0,\infty);L^r)$ with*

$$\|e^{-At}\varphi\|_{L^q(I;L^p)} \le C\|\varphi\|_r, \quad I = [0,\infty) \text{ or } I \subset [0,\infty). \tag{2.12}$$

where C is constant independent of $\varphi(x)$.

For proof we refer to [5, 10].

The proof of Theorem 1.3 By using Fourier transformation, ∇v can be written in terms of w as

$$\frac{\partial v_j}{\partial x_l} = -\sum_{k=1}^n R_l R_k \omega_{j,k}, \quad 1 \le j,l \le n, \tag{2.13}$$

by div $v=0$, where R_l and R_k are classical Riesz transformation.

Case 1 Let $(\tilde{q}, \tilde{p}, \frac{n}{2})$ be any admissible triplet.

Notice that $w \in L^{\tilde{q}}((0,T);L^{\tilde{p}})$ and

$$\|R_k f\|_r \le \|f\|_r, \quad 1<r<\infty, \quad 1 \le k \le n, \quad f \in L^r, \tag{2.14}$$

one obtains $\nabla v \in L^{\tilde{q}}((0,T);L^{\tilde{p}})$ $(n \ge 3)$ due to $\infty > \tilde{p} \ge \frac{n}{2} > 1$. We rewrite (2.1) as

$$v(x,t) = e^{-At}v_0 + \int_0^t e^{-(t-\tau)A}\nabla(v\otimes v)d\tau. \tag{2.15}$$

For any admissible triplet (p,q,n), one sees by using (2.10) and (2.12) that

$$\begin{aligned} \|v\|_{L^q((0,T);L^p)} &\le \|e^{-At}v_0\|_{L^q((0,T);L^p)} \\ &\quad + \left\|\int_0^t e^{-A(t-\tau)}\mathcal{P}\nabla(v\otimes v)d\tau\right\|_{L^q((0,T);L^p)} \\ &\le \|v_0\|_{L^n} + \left\|\int_0^t |t-\tau|^{-\frac{n}{2}(\frac{1}{\bar{p}}-\frac{1}{p})}\|\nabla u \otimes u\|_{\bar{p}}\,d\tau\right\|_q \\ &\le \|v_0\|_{L^n} + C\|\nabla v\otimes v\|_{L^{\bar{q}}((0,T);L^{\bar{p}})} \\ &\le \|v_0\|_{L^n} + C\|\nabla v\|_{L^{\tilde{q}}((0,T);L^{\tilde{p}})}\|v\|_{L^q((0,T);L^p)}, \end{aligned} \tag{2.16}$$

with

$$\frac{1}{\bar{p}}=\frac{1}{p}+\frac{1}{\tilde{p}},\quad \frac{1}{\bar{q}}=\frac{1}{q}+\frac{1}{\tilde{q}}, \tag{2.17}$$

and

$$\frac{1}{q}=\frac{1}{\bar{q}}-1+\frac{n}{2}(\frac{1}{\bar{p}}-\frac{1}{p}). \tag{2.18}$$

In fact, for any admissible triplet (p,q,n), it follows that

$$\begin{aligned}\frac{1}{q}&=\frac{1}{\bar{q}}-\frac{1}{\tilde{q}}=\frac{1}{\bar{q}}-\frac{n}{2}(\frac{2}{n}-\frac{1}{\tilde{p}})\\&=\frac{1}{\bar{q}}-1+\frac{n}{2\tilde{p}}=\frac{1}{\bar{q}}-1-\frac{n}{2}(\frac{1}{\bar{p}}-\frac{1}{p}).\end{aligned} \tag{2.19}$$

Notice that since $\|\nabla v\|_{L^{\tilde{q}}((0,T);L^{\tilde{p}})}<\infty$, there exists a $\delta>0$ such that for any $T_0<T-\delta$

$$C\|\nabla v : L^{\tilde{q}}((T_0,T_0+\delta);L^{\tilde{p}})\|<\frac{1}{2}. \tag{2.20}$$

From (2.16) one gets that $v\in L^q(0,T);L^p)$ by the finite number steps.

Case 2 Let $(\tilde{q},\tilde{p},\tilde{r})$ be any admissible triplet with $\tilde{r}\geq\frac{n}{2}$.

Since $w\in L^{\tilde{q}}((0,T);L^{\tilde{p}})$ and (2.14), it follows that $\nabla v\in L^{\tilde{q}}((0,T);L^{\tilde{p}})$ $(n\geq 3)$ by the boundedness of the Calderón-Zygmund singular operator in $L^{\tilde{p}}$ and $\infty>\tilde{p}\geq\frac{n}{2}>1$.

For any admissible triplet (p,q,r) with $r\geq n$, one has using (2.10) and (2.12)

$$\begin{aligned}\|v\|_{L^q((0,T);L^p)}\leq&\|e^{-At}v_0\|_{L^q((0,T);L^p)}+\left\|\int_0^t e^{-A(t-\tau)}\mathcal{P}\nabla(v\otimes v)d\tau\right\|_{L^q((0,T);L^p)}\\\leq&\|v_0\|_r+\left\|\int_0^t|t-\tau|^{-\frac{n}{2\tilde{p}}}\|\nabla u\|_{\tilde{p}}\|u\|_p\,d\tau\right\|_q\\\leq&\|v_0\|_{L^r}+CT^{1-\frac{n}{2\tilde{r}}}\|\nabla v\|_{L^{\tilde{q}}((0,T);L^{\tilde{p}})}\cdot\|v\|_{L^q((0,T);L^p)},\end{aligned} \tag{2.21}$$

with

$$\frac{1}{\bar{p}}=\frac{1}{p}+\frac{1}{\tilde{p}}, \tag{2.22}$$

and

$$\frac{1}{q}+1=\frac{1}{\tilde{q}}+\frac{1}{q}+(1-\frac{1}{\tilde{q}}). \tag{2.23}$$

Notice that $\|\nabla v\|_{L^{\tilde{q}}((0,T);L^{\tilde{p}})}<\infty$, there exists a fixed $\tilde{T}>0$ suitable small such that

$$C\tilde{T}^{1-\frac{n}{2\tilde{r}}}\|\nabla v : L^{\tilde{q}}((0,T);L^{\tilde{p}})\|<\frac{1}{2}, \tag{2.24}$$

using (2.16) one gets that $v\in L^q(0,T);L^p)$ by the finite number steps. Therefore we complete the proof of Theorem 1.2 by Proposition 2.1.

References

[1] Beirão Da Veiga, Concerning the regularity problem for the solutions of Navier–Stokes equations, *C.R. Acad.Sci.Paris, t.321.Sërie I*, (1995), 405–408.

[2] J.Leray, Sur le mouvement d'um liquide visqieux emlissant l'space, *Acta. Math.*, **64**(1934), 193-284.

[3] E.Hopf, Über die anfang swetaufgabe für die hydrodynamischer grundgleichungan, *Math. Nach.*, **4**(1951), 213–231.

[4] E.B.Fabes, B.F.Jones and N.M.Riviere, The initial value problem for the Navier–Stokes equations with data in L^p, *Arch. Rational Mech. Anal.*, **45**(1972), 222-240.

[5] Y.Giga, Solutions for Semilinear parabolic equations in L^p and regularity of weak solutions of the Navier-Stokes System, *J. of Diff. Equ.*, **61**(1986), 186-212.

[6] T.Kato, Strong L^p–solutions of the Navier–Stokes equation in $\mathbb{R}^m$, with applications to weak solutions, *Math. Z.* **187**(1984), 471–480.

[7] T.Kato and G.Ponce, The Navier–Stokes equation with weak initial data, *IMRN international Mathematics Research Notices*, No. 10(1994), 435–444.

[8] H.Kozono and H.Sohr, The regularity criterion on weak solutions to the Navier–Stokes equations, *Adv. Diff.Equations*, **2**(1994) No. 4, 535–554.

[9] O.A.Ladyzhenskaya, On the Theory of Mathematical in the Incompressible Fiuld, Gordon and Breach, New York, 2nd Ed., 1969.

[10] B. Guo and C.X.Miao, Space–time means and solutions to a class of nonlinear parabolic equations, *Science in China*, **41**(1998), 682-693.

[11] C.X.Miao, Time-space estimates of solutions to general semilinear parabolic equations, *Tokyo J. of Mathematics*, **24** (2001), 245-276.

[12] J.Serrin, The initial value problem for Navier–Stokes equations, In: Non-linear problems, Univ. Wisconsin Press, R.E. Langer Ed, 1963, 69–98.

[13] S.Takahashi, On interior regularity criteria for weak solutions of Navier-Stokes equations, *Manuscripta Math.*, **69**(1990), 237-254.

[14] W. Von Wahl, The Equation of the Navier–Stokes and Abstract Parabolic equations, Braunsch-Wiesbaden; Vieweg Verlag, 1985.

[15] D. Chae and H.J. Choe, The regularity problem for the solutions of Navier–Stokes equations, *Electronic Journal of Differential Equations* **5**(1999), 1–7.

[16] C.X.Miao, Harmonic Analysis and Applications to the Partial Differential Equations, Acadmic Press, (in Chinese), 1999.

[17] G.Ponce, On the well posedness of some nonlinear evolution equations, *Nonlinear Wave, Proceedings of the Fourth MSJ International Research Institute*, (1996) 379–409, Edited by Agemi R., Giga Y. and Ozawa T.

[18] S.Mizohata, The Theory of Partial Differential Equations, Cambridge University Press, 1973.

[19] E.M.Stein, Singular Integral and Differential Property of Functions, Princeton University Press, 1970.

[20] A.Pazy, Linear Operator Semigroup Theory and Applications to Partial Differential Equations, Springer-Verlag, 1983.