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# TRAVELING WAVE FRONTS OF A DEGENERATE PARABOLIC EQUATION WITH NON-DIVERGENCE FORM

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Dedicated to the 80th birthday of Professor Zhou Yulin
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**Abstract** We study the traveling wave solutions of a nonlinear degenerate parabolic equation with non-divergence form. Under some conditions on the source, we establish the existence, and then discuss the regularity of such solutions.
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## 1. Introduction

This paper is concerned with the traveling wave fronts of the following nonlinear degenerate equation with non-divergence form

$$\frac{\partial u}{\partial t} = u^m \triangle u + u^n f(u), \qquad x \in \mathbb{R}^N, t \in \mathbb{R}^+, \tag{1.1}$$

where $m \geq 1$, $n > 0$ and $f$ is continuously differentiable. Such an equation is quite different from the well-known porous medium equation with an absorption

$$\frac{\partial u}{\partial t} = \Delta u^p + u^q f(u), \qquad (p > 1, q > 0) \tag{1.2}$$

although it can be transformed into an equation like (1.1), with the exponent $m = \dfrac{p-1}{p}$ which falls into the interval $(0, 1)$. During the past decades, the equations whose principal parts are in divergence form, like (1.2), have been deeply investigated. However, as far as we know, there are only a few works devoted to the equations whose principal parts are not in divergence form like (1.1). Among the earliest works in this respect, it is worthy to mention the work [1] by Allen, who did discuss such kind of equation with $m = 1$ in one dimensional case, modeling the diffusive process for biological species. It was Friedman and McLeod [2] who studied the blow-up properties of solutions for the equation with $m = 2$, $n = 3$ in multi-dimensional case. We may also mention the work

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[3] by Passo, where the basic existence, uniqueness and the properties of solutions are investigated in detail for the case $m = 1$. Recently, Wang, Wang and Xie [4] studied the equation for any $m > 1$ with $n = m + 1$, and discussed the global existence and blow-up properties of solutions. Furthermore, we point out that Bertsch has obtained several important results on the similar equations like (1.1) or (1.2), see [5–7].

In this paper, we are much interested in the discussion of the traveling wave solutions of the equation (1.1) with $m \geq 1$ and $n > 0$. For the same question about the degenerate or non-degenerate diffusion equations whose principal parts are in divergence form, we refer to [8–13]. First, we introduce the following

**Definition** *A function $u(z) \in C(\mathbb{R})$ with $z = \gamma \cdot x + t$ for some $0 \neq \gamma \in \mathbb{R}^N$ is called a traveling wave front of the equation (1.1) if there exist $-\infty \leq z_l < z_r \leq +\infty$ such that*

(i) *$u(z) \in C^2(z_l, z_r)$ and satisfies*

$$u' = |\gamma|^2 u^m u'' + u^n f(u), \qquad \forall z \in (z_l, z_r);$$

(ii) *$u(z_l) = \theta_l$, $u(z_r) = \theta_r$, where $\theta_l$ and $\theta_r$ are zero or the zero points of $f(u)$;*

(iii) *$u(z)$ is strictly monotone in the interval $(z_l, z_r)$, $u(z) = \theta_l$ for $z \in (-\infty, z_l)$ and $u(z) = \theta_r$ for $z \in (z_r, +\infty)$;*

(iv) *If $u(z_l) < u(z_r)$, then $u'(z_r) = 0$, while if $u(z_l) > u(z_r)$, then $u'(z_l) = 0$.*

*Furthermore, if $u'_+(z_l) = u'_-(z_r) = 0$, we call $u(z)$ a smooth traveling wave front, where $u'_+$ and $u'_-$ denote the right and the left derivative of $u$.*

To discuss the traveling wave fronts, let us first change the form of the equation. Let $p = u'$ and $c = \dfrac{1}{|\gamma|^2}$, the wave speed. Then for $z \in \{z \in (z_l, z_r) : u(z) > 0\}$, we get that

$$\begin{cases} u' = p, \\ p' = cu^{-m}p - cu^{n-m}f(u). \end{cases} \tag{1.3}$$

As we did for the equation whose principal part is in divergence form, we consider the following two typical cases

$$f(1) = 0, f'(1) < 0, \text{ and } f(s) > 0 \text{ for } s \in [0, 1), \tag{H1}$$

and

$$f(0) < 0, f(1) = 0, f'(1) < 0, f(u) < 0 \text{ for } s \in (0, a) \text{ and } f(s) > 0 \text{ for } s \in (a, 1), \tag{H2}$$

where $a$ is a given number in $(0, 1)$. First, in Section 2 we discuss the case for $f$ satisfying (H1). Different from the equation (1.2), see [14], there is no minimal wave speed for the solutions of the equation (1.1). In other words, for any $c$, there always exists a traveling wave front with the wave speed $c$ for equation (1.1). Then in Section 3, we study the case with $f$ changing sign, namely, the case for $f$ satisfying (H2). As it was shown in [15], there exists one and only one wave speed $c^*$ such that the equation

(1.2) has smooth traveling wave fronts. However, for the equation (1.1), we will show that the wave speed might not be existent in general. We have found a sufficient and necessary condition on $f$, namely,

$$\int_0^1 s^{n-m} f(s)ds < 0,$$

for the existence of the unique wave speed, see the details in Section 3. After establishing the existence, the last section is devoted to the regularity of the traveling wave solutions, namely, the finiteness of $z_l$ and $z_r$.

## 2. The Case $f > 0$

In this section, we investigate the increasing traveling wave fronts under assumption (H1). In this case, $\theta_l = 0$, $\theta_r = 1$. It is easy to see that $u(z)$ is the increasing and smooth traveling wave front if and only if $u(z)$ satisfies that

$$\begin{cases} \dfrac{dp}{du} = cu^{-m} - \dfrac{cu^{n-m} f(u)}{p}, \\ p(0) = p(1) = 0, \\ p(u) > 0, \quad u \in (0,1). \end{cases} \tag{2.1}$$

**Lemma 2.1** (Comparison) *Let $\beta > \alpha \geq 0$, $c_1, c_2 > 0$ and $p_i(u)(i = 1, 2)$ be the solutions of*

$$\begin{cases} \dfrac{dp_i}{du} + \dfrac{c_i u^{n-m} f(u)}{p_i} = c_i u^{-m}, \quad u \in (\alpha, \beta), \\ p_i(\beta) = \beta_i. \end{cases}$$

*If $p_1(u)p_2(u) > 0$ in $(\alpha, \beta)$ and there exists $\delta > 0$ such that $f(u) \geq 0$ for $u \in (\beta - \delta, \beta)$, then*

(i) *If $c_1 = c_2$ and $\beta_1 = \beta_2$, then $p_1(u) = p_2(u)$ for $u \in (\alpha, \beta)$;*

(ii) *If $c_1 > c_2$ and $\sqrt{c_2}\beta_1 \leq \sqrt{c_1}\beta_2$, then $\sqrt{c_2}p_1(u) < \sqrt{c_1}p_2(u)$ for $u \in (\alpha, \beta)$.*

**Proof** From the ordinary differential equations which $p_1$ and $p_2$ satisfy, we see that

$$\frac{d(\sqrt{c_2}p_1 - \sqrt{c_1}p_2)}{du} - \frac{\sqrt{c_1 c_2} u^{n-m} f(u)}{p_1 p_2} (\sqrt{c_2}p_1 - \sqrt{c_1}p_2) = (c_1\sqrt{c_2} - c_2\sqrt{c_1})\, u^{-m}.$$

Let

$$G(u) = (\sqrt{c_2}p_1 - \sqrt{c_1}p_2) \exp\left\{-\int_{\beta-\delta}^{u} \frac{\sqrt{c_1 c_2} s^{n-m} f(s)}{p_1(s)p_2(s)} ds\right\}, \quad u \in (\alpha, \beta).$$

We get that

$$\frac{dG(u)}{du} = (c_1\sqrt{c_2} - c_2\sqrt{c_1})\, u^{-m} \exp\left\{-\int_{\beta-\delta}^{u} \frac{\sqrt{c_1 c_2} s^{n-m} f(s)}{p_1(s)p_2(s)} ds\right\}.$$

Since $p_1(u)p_2(u) > 0$ in $(\alpha, \beta)$ and $f(u) \geq 0$ for $u \in (\beta - \delta, \beta)$,

$$0 \leq \exp\left\{-\int_{\beta-\delta}^{\beta} \frac{\sqrt{c_1c_2}s^{n-m}f(s)}{p_1(s)p_2(s)}ds\right\} \leq 1.$$

(i) If $c_1 = c_2$ and $\beta_1 = \beta_2$, then

$$\lim_{u\to\beta^-} G(u) = 0, \qquad \frac{dG(u)}{du} = 0, \quad u \in (\alpha, \beta).$$

Thus $G(u) \equiv 0$ for $u \in (\alpha, \beta)$. So $p_1(u) = p_2(u)$ for $u \in (\alpha, \beta)$.

(ii) If $c_1 > c_2 > 0$ and $\sqrt{c_2}\beta_1 \leq \sqrt{c_1}\beta_2$, then

$$\lim_{u\to\beta^-} G(u) \leq 0, \qquad \frac{dG(u)}{du} > 0, \quad u \in (\alpha, \beta).$$

Thus $G(u) < 0$ for $u \in (\alpha, \beta)$. So $\sqrt{c_2}p_1(u) < \sqrt{c_1}p_2(u)$ for $u \in (\alpha, \beta)$. The proof is complete.

**Theorem 2.1** *Under assumption* (H1), *for all* $c > 0$, *the problem* (2.1) *admits at least one solution, namely,* (1.1) *admits at least one increasing and smooth traveling wave front.*

**Proof** From $f'(1) < 0$, we see that $(1, 0)$ is a saddle point of the system (1.3) and the two eigenvalues are $\delta_\pm = \frac{c}{2}(1 \pm \sqrt{1 - 4f'(1)})$. Set $D = \{(u, p) : 0 < u < 1, p > 0\}$. For all $c > 0$, from the Liapunov disturbance theorem, we see that there exists a path curve $\Gamma_c$ of (1.3), exiting from point $(1, 0)$ with slope $\delta_- < 0$ and entering into region $D$. Since $f(u) > 0$ for $u \in (0, 1)$, $\Gamma_c$ does not intersect with the $u$-axis when $u \in (0, 1)$.

Now we show that $\Gamma_c$ crosses the point $(0, 0)$, namely, (2.1) is solvable. Otherwise, there would be some $A > 0$ such that $p(u) > A$ for all $u \in (0, \frac{1}{2}]$. Let $a = \max_{0\leq u\leq 1} f(u)$ and $\delta = \min\left\{\frac{1}{2}, \left(\frac{A}{2a}\right)^{1/n}\right\}$. Then for $u \in (0, \delta]$,

$$\frac{dp}{du} = cu^{-m}\left(1 - \frac{u^n f(u)}{p}\right) > cu^{-m}\left(1 - \frac{\delta^n a}{A}\right) \geq \frac{c}{2}u^{-m},$$

and consequently,

$$p(u) = p(\delta) - \int_u^\delta \frac{dp(s)}{ds}ds < p(\delta) - \frac{c}{2}\int_u^\delta s^{-m}ds, \quad u \in (0, \delta].$$

Noticing that $m \geq 1$, we see that $\lim_{u\to 0^+} p(u) = -\infty$, which contradicts the fact that $\Gamma_c$ does not intersect with the $u$-axis when $u \in (0, 1)$. So $\Gamma_c$ crosses the point $(0, 0)$. The proof is complete.

## 3. The Case With $f$ Changing Sign

In this section, we discuss the case with $f$ changing sign, namely, the case for $f$

satisfying (H2). Under this assumption, we investigate the increasing traveling wave fronts of (1.1) with $\theta_l = 0$, $\theta_r = 1$ or $\theta_l = a$, $\theta_r = 1$ and the decreasing traveling wave fronts of (1.1) with $\theta_l = 1$, $\theta_r = 0$ or $\theta_l = 1$, $\theta_r = a$.

We first investigate the increasing traveling wave fronts of (1.1) with $\theta_l = 0$, $\theta_r = 1$ or $\theta_l = a$, $\theta_r = 1$. It is easy to see that $u(z)$ is the increasing and smooth traveling wave front with $\theta_l = 0$, $\theta_r = 1$ if and only if $u(z)$ satisfies (2.1). The following theorem shows that there is no increasing and smooth traveling wave front of (1.1) with $\theta_l = 0$, $\theta_r = 1$.

**Theorem 3.1** *Under assumption* (H2)*, for any $c > 0$,* (2.1) *admits no solution, namely, there is no increasing and smooth traveling wave front of* (1.1) *with $\theta_l = 0$, $\theta_r = 1$.*

**Proof** We prove the result by contradiction. Let $p(u)$ be a solution of (2.1) for some $c > 0$. Then for $u \in (0, a]$,

$$\frac{dp}{du} = cu^{-m}\left(1 - \frac{u^n f(u)}{p}\right) \geq cu^{-m},$$

which implies by integration

$$p(u) = p(a) - \int_u^a \frac{dp(s)}{ds}ds < p(a) - c\int_u^a s^{-m}ds, \quad u \in (0, a].$$

By virtue of $m \geq 1$, we see that $\lim\limits_{u\to 0^+} p(u) = -\infty$, which contradicts that $p(0) = 0$. So there is no solution of (2.1) for any $c > 0$. The proof is complete.

From the proof of Theorem 3.1 we see that $p(u)$ intersects with the $u$-axis when $u \in (0, 1)$. Since $f(u) > 0$ for $u \in (a, 1)$, $p(u)$ does not intersect with the $u$-axis when $u \in (a, 1)$. Thus $p(u)$ intersects with the $u$-axis when $u \in (0, a]$. So we have the following theorem.

**Theorem 3.2** *Under assumption* (H2)*, for all $c > 0$,* (1.1) *admits at least one increasing traveling wave front $u(z)$ with $\theta_l = a$, $\theta_r = 1$, and $u'_+(z_l) \geq 0$, $u'_-(z_r) = 0$.*

Next we investigate decreasing traveling wave fronts of (1.1) with $\theta_l = 1$, $\theta_r = 0$ or $\theta_l = 1$, $\theta_r = a$. It is easy to see that $u(z)$ is the decreasing and smooth traveling wave front with $\theta_l = 1$, $\theta_r = 0$ if and only if $u(z)$ satisfies that

$$\begin{cases} \dfrac{dp}{du} = cu^{-m} - \dfrac{cu^{n-m}f(u)}{p}, \\ p(0) = p(1) = 0, \\ p(u) < 0, \quad u \in (0, 1). \end{cases} \tag{3.1}$$

From $f'(1) < 0$, we see that $(1, 0)$ is a saddle point of the system (1.3) and the two eigenvalues are $\delta_{\pm} = \frac{c}{2}(1 \pm \sqrt{1 - 4f'(1)})$. For all $c > 0$, from the Liapunov disturbance theorem, we see that there exists a path curve $\Gamma_c$ of (1.3), exiting from the point $(1, 0)$ with slope $\delta_+ > 0$ and entering into the region $E = \{(u, p) : 0 < u < 1, p < 0\}$. We have the following lemmas about $\Gamma_c$.

**Lemma 3.1** *Let $\Gamma_c$ be the path curve of* (1.3) *exiting from* $(1,0)$ *and entering into $E$ and $p(u)$ be the corresponding solution. Then for sufficiently large $c>0$, $\Gamma_c$ does not intersect with the $u$-axis when $u\in[0,1)$ and $\lim\limits_{u\to 0^+}p(u)=-\infty$.*

**Proof** For $u\in[a,1)$,

$$\frac{dp}{du}=cu^{-m}\left(1-\frac{u^nf(u)}{p}\right)\geq cu^{-m}.$$

Thus

$$p(a)=-\int_a^1\frac{dp(s)}{ds}ds\leq -c\int_a^1 s^{-m}ds.$$

Let $A=\max\limits_{0\leq u\leq a}\{-f(u)\}$ and $\delta=\dfrac{2A}{\displaystyle\int_a^1 s^{-m}ds}$. Then for $c>\delta$ and $u\in(0,a]$,

$$1-\frac{u^nf(u)}{-c\displaystyle\int_a^1 s^{-m}ds}>1-\frac{-A}{-\delta\displaystyle\int_a^1 s^{-m}ds}=\frac{1}{2}.$$

Therefore, for $c>\delta$, $p(u)$ is increasing in $(0,1)$ and for $u\in(0,a]$,

$$\frac{dp}{du}=cu^{-m}\left(1-\frac{u^nf(u)}{p}\right)\geq \frac{c}{2}u^{-m}.$$

Thus

$$p(u)=p(a)-\int_u^a\frac{dp(s)}{ds}ds<p(a)-\frac{c}{2}\int_u^a s^{-m}ds,\quad u\in(0,a].$$

Noticing that $m\geq 1$, we see that $\lim\limits_{u\to 0^+}p(u)=-\infty$. The proof is complete.

**Lemma 3.2** *Let $\Gamma_c$ be the path curve of* (1.3) *exiting from* $(1,0)$ *and entering into $E$ and $p(u)$ be the corresponding solution.*

(i) *If $\displaystyle\int_0^1 s^{n-m}f(s)ds\geq 0$, then for any $c>0$, $\Gamma_c$ does not intersect with the $u$-axis when $u\in[0,1)$ and $\lim\limits_{u\to 0^+}p(u)=-\infty$.*

(ii) *If $\displaystyle\int_0^1 s^{n-m}f(s)ds<0$, then for sufficiently small $c>0$, $\Gamma_c$ intersects with the $u$-axis when $u\in[0,1)$.*

**Proof** We prove the two results respectively.

(i) Assume that $\displaystyle\int_0^1 s^{n-m}f(s)ds\geq 0$. Since $p(u)<0$ for $u\in(0,1)$ and

$$\frac{dp}{du}=cu^{-m}\left(1-\frac{u^nf(u)}{p}\right)>-\frac{cu^{n-m}f(u)}{p},\quad u\in(0,1),$$

we get that

$$\frac{dp^2}{du}=2p\frac{dp}{du}<-2cu^{n-m}f(u),\quad u\in(0,1).$$

Noticing that $p(1)=0$, we see that for $u\in[0,1)$,

$$p^2(u)>2c\int_u^1 s^{n-m}f(s)ds.$$

Since $\int_0^1 s^{n-m}f(s)ds\ge 0$ and $f(u)<0$ for $u\in[0,a)$, we see that for all $u\in[0,1)$, $p^2(u)>0$, namely, $p(u)<0$. So $\Gamma_c$ does not intersect with the $u$-axis when $u\in[0,1)$.

Now we show that $\lim\limits_{u\to 0^+}p(u)=-\infty$. From the above discussion, we see that there exists $A>0$ such that $p(u)<-A$ for all $u\in[0,a]$. Let $B=\max\limits_{0\le u\le a/2}\{-f(u)\}$ and $\delta=\min\left\{\left(\frac{A}{2B}\right)^{1/n},\frac{a}{2}\right\}$. Then for $u\in(0,\delta]$,

$$1-\frac{u^nf(u)}{p}>1-\frac{-\delta^nB}{-A}\ge\frac{1}{2}.$$

Thus for $u\in(0,\delta]$,

$$\frac{dp}{du}=cu^{-m}\left(1-\frac{u^nf(u)}{p}\right)\ge\frac{c}{2}u^{-m}.$$

So

$$p(u)=p(\delta)-\int_u^\delta\frac{dp(s)}{ds}ds<p(\delta)-\frac{c}{2}\int_u^a s^{-m}ds,\quad u\in(0,\delta].$$

Noticing that $m\ge 1$, we see that $\lim\limits_{u\to 0^+}p(u)=-\infty$.

(ii) Assume that $\int_0^1 s^{n-m}f(s)ds<0$. Let $p_0(u)$ be the solution of (1.3) with respect to $c=c_0$, where $c_0$ is sufficiently large such that $\Gamma_{c_0}$ does not intersect with the $u$-axis when $u\in[0,1)$. From Lemma 2.1 we see that for $c<c_0$, $\sqrt{c_0}p(u)>\sqrt{c}p_0(u)$. Since $p(u)<0$ for $u\in(0,1)$, we get that

$$\frac{dp^2}{du}=2p\frac{dp}{du}=2cu^{-m}p(u)-2cu^{n-m}f(u)>\frac{2c\sqrt{c}}{\sqrt{c_0}}u^{-m}p_0(u)-2cu^{n-m}f(u),\quad u\in(0,1).$$

Noticing that $p(1)=0$, we see that for $u\in[0,1)$,

$$p^2(u)<2c\int_u^1 s^{n-m}f(s)ds-\frac{2c\sqrt{c}}{\sqrt{c_0}}\int_u^1 s^{-m}p_0(s)ds.$$

Since $\int_0^1 s^{n-m}f(s)ds<0$, it is easy to see that for sufficiently small $c>0$, $\Gamma_c$ must intersect with the $u$-axis when $u\in[0,1)$. The proof is complete.

**Theorem 3.3** *Suppose that $f$ satisfies* (H2).

(i) *If $\int_0^1 s^{n-m}f(s)ds\ge 0$, then for any $c>0$, (3.1) admits no solution, namely, there is no decreasing and smooth traveling wave front of (1.1) with $\theta_l=1$, $\theta_r=0$.*

(ii) *If* $\int_0^1 s^{n-m} f(s)ds < 0$*, then there exists a unique* $c^* > 0$ *such that* (3.1) *admits at least one solution, namely* (1.1) *admits at least one decreasing and smooth traveling wave front with* $\theta_l = 1$*,* $\theta_r = 0$ *if and only if* $c = c^*$*.*

**Proof** Conclusion (i) can be obtained directly from Lemma 3.2 (i). Now, we show the conclusion (ii). Assume that $\int_0^1 s^{n-m} f(s)ds < 0$. Let

$$F = \{c > 0 : \Gamma_c \text{ intersects with the } u\text{-axis when } u \in [0, 1)\},$$

where $\Gamma_c$ is the path curve of (1.3) exiting from $(1, 0)$ and entering into $E = \{(u, p) : 0 < u < 1, p < 0\}$. From Lemma 3.1 and Lemma 3.2 (ii) we see that $F$ is bounded and not empty. Let $c^* = \sup F$. Then $c^* > 0$. Let $\Gamma^*$ be the path curve of (3.2) with $c^*$ exiting from $(1, 0)$ and entering into $E$ and $p^*(u)$ be the corresponding solution of (1.3).

We first show that $c^* \in F$. Otherwise, assume that $\Gamma^*$ did not intersect with the $u$-axis when $u \in [0, 1)$. From the proof of Lemma 3.2 (i) we see that $\lim\limits_{u\to 0^+} p^*(u) = -\infty$. Let $\{c_i\}_{i=1}^{\infty} \subset F$ with $c_i \nearrow c^*$ and $p_i(u)$ be the corresponding solution of (3.2) with respect to $\Gamma_{c_i}$. Let $u_i$ denote the maximal root of $p_i$ in $[0, 1)$. By Lemma 2.1 we see that $u_i$ is monotone decreasing. Thus $u_0 = \lim\limits_{i\to\infty} u_i \geq 0$. If $u_0 > 0$, then we get that $p^*(u_0) = 0$ from $p_i(u_i) = 0$, which contradicts that $\Gamma^*$ did not intersect with the $u$-axis when $u \in [0, 1)$. Thus $u_0 = 0$, namely, $\lim\limits_{i\to\infty} u_i = 0$. From Lemma 2.1 we see that

$$\frac{p_i(u)}{\sqrt{c_i}} > \frac{p_{i+1}(u)}{\sqrt{c_{i+1}}}, \quad \forall u \in (u_i, 1).$$

Thus

$$0 < \frac{-u^{-m} p_i(u)}{\sqrt{c_i}} < \frac{-u^{-m} p_{i+1}(u)}{\sqrt{c_{i+1}}}, \quad \forall u \in (u_i, 1).$$

From Levy's Theorem and noticing that $\lim\limits_{i\to\infty} u_i = 0$, we get that

$$\lim_{i\to\infty} \int_{u_i}^u \frac{-s^{-m} p_i(s)}{\sqrt{c_i}} ds = \int_0^u \frac{-s^{-m} p^*(s)}{\sqrt{c^*}} ds, \quad \forall u \in (0, 1),$$

namely

$$\lim_{i\to\infty} \int_{u_i}^u s^{-m} p_i(s)ds = \int_0^u s^{-m} p^*(s)ds, \quad \forall u \in (0, 1).$$

Noticing that $p_i(u_i) = 0$ and

$$\frac{dp_i^2}{du} = 2p_i \frac{dp_i}{du} = 2c_i u^{-m} p_i(u) - 2c_i u^{n-m} f(u), \quad u \in (0, 1),$$

we see that for $u \in (0, 1)$,

$$p_i^2(u) = 2c_i \left( \int_{u_i}^u s^{-m} p_i(s)ds - \int_{u_i}^u s^{n-m} f(s)ds \right).$$

Letting $i \to \infty$ and noticing that $\lim\limits_{i\to\infty} u_i = 0$, we see that for $u \in (0,1)$,

$$p^{*2}(u) = 2c^*\left(\int_0^u s^{-m}p^*(s)ds - \int_0^u s^{n-m}f(s)ds\right),$$

which contradicts that $\lim\limits_{u\to 0^+} p^*(u) = -\infty$. So $c^* \in F$, namely, $\Gamma^*$ intersects with the $u$-axis when $u \in [0,1)$. Let $(u^*, p^*(u^*))$ be the intersection point.

Next, we show that $u^* = 0$. Otherwise, assume that $u^* > 0$. From the continuity of path curve with respect to parameter, we see that for $c > c^*$ and sufficiently near $c^*$, there exist at least two extreme points $u^1$ and $u^2$ of $p_c(u)$, the corresponding solution of (1.3) with respect to $\Gamma_c$, such that $0 < u^1 < u^* < u^2 < 1$, $p_c(u^2) < p_c(u^1) < 0$, $u^1 \to u^*$ and $p_c(u^1) \to p^*(u^*) = 0$ as $c \searrow c^*$. From $\dfrac{dp_c(u^1)}{du} = \dfrac{dp_c(u^2)}{du} = 0$, we see that

$$p_c(u^1) = (u^1)^n f(u^1), \tag{3.2}$$

$$p_c(u^2) = (u^2)^n f(u^2). \tag{3.3}$$

Let $c \searrow c^*$ in (3.2), we get that $p^*(u^*) = (u^*)^n f(u^*)$. Since $p^*(u^*) = 0$, we see that $f(u^*) = 0$. Therefore, $u^* = a$. Thus, $f(u^2) > 0$. So $p_c(u^2) > 0$ from (3.3), which contradicts that $p_c(u) < 0$ for $u \in (0,1)$. Therefore, $u^* = 0$, namely, $p^*(u)$ is a solution of (3.1) with $c = c^*$.

Finally, the uniqueness of $c^*$ can be obtained from the proof of Lemma 2.1. The proof is complete.

From Lemma 2.1, Lemma 3.1, Lemma 3.2 and Theorem 3.3, we get the following theorem.

**Theorem 3.4** *Let $f$ satisfy* (H2) *and $c^*$ be given in Theorem 3.3.*

(i) *If $\int_0^1 s^{n-m}f(s)ds \geq 0$, then for all $c > 0$, (1.1) admits at least one decreasing and non-smooth traveling wave front $u(z)$ with $\theta_l = 1$, $\theta_r = 0$, and $u'_+(z_l) = 0$ ,$u'_-(z_r) = -\infty$.*

(ii) *If $\int_0^1 s^{n-m}f(s)ds < 0$, then for $c > c^*$, (1.1) admits at least one decreasing and non-smooth wave front $u(z)$ with $\theta_l = 1$, $\theta_r = 0$, and $u'_+(z_l) = 0$, $u'_-(z_r) = -\infty$. While for $0 < c < c^*$, (1.1) admits at least one decreasing wave front $u(z)$ with $\theta_l = 1$, $\theta_r = a$, and $u'_+(z_l) = 0$, $-\infty < u'_-(z_r) \leq 0$.*

## 4. Regularity

Now, we turn to the discussion of the regularity of the traveling wave fronts, namely, for the increasing traveling wave fronts, investigate the finiteness of $z_l$, and for the decreasing traveling wave fronts, investigate the finiteness of $z_r$.

We first have

**Theorem 4.1** *Let $u(z)$ be the increasing and smooth traveling wave front of* (1.1) *with $f$ satisfying* (H1).

(i) *If $0 < n < 1$, then $z_l > -\infty$.*

(ii) *If $n \geq 1$, then $z_l = -\infty$.*

**Proof** We prove the two results respectively.

(i) Assume that $0 < n < 1$. Since $p(0) = 0$ and $p(u) > 0$ for $u \in (0,1)$, there exists $\epsilon \in (0,1)$ such that $\left.\frac{dp}{du}\right|_{u=\epsilon} > 0$. Let $A = \min\limits_{0\leq u\leq\epsilon} f(u)$. For any fixed $u_1 \in (0,\epsilon)$, if $\left.\frac{dp}{du}\right|_{u=u_1} \geq 0$, then from the equation in (2.1),

$$p(u_1) \geq u_1^n f(u_1) \geq A u_1^n.$$

If $\left.\frac{dp}{du}\right|_{u=u_1} < 0$, we set $u_2 = \sup\left\{v \in (u_1,\epsilon) : \left.\frac{dp}{du}\right|_{u=s} < 0, \forall s \in (u_1, v)\right\}$ and deduce that $u_1 < u_2 < \epsilon$, $\left.\frac{dp}{du}\right|_{u=u_2} = 0$ and $\frac{dp}{du} < 0$ for all $u \in (u_1, u_2)$. Thus

$$p(u_1) > p(u_2) = u_2^n f(u_2) \geq A u_2^n > A u_1^n.$$

Therefore

$$\frac{du}{dz} = p(u) \geq A u^n, \qquad \forall u \in (0,\epsilon).$$

Integrating the above inequality from $z_1$ to $z_2$ with $z_l < z_1 < z_2 < z_r$ and noticing that $0 < n < 1$, we see that

$$z_2 - z_1 \leq \int_{u(z_1)}^{u(z_2)} \frac{1}{Au^n} du \leq \int_0^1 \frac{1}{Au^n} du = \frac{1}{A(1-n)}.$$

Due to the arbitrariness of $z_1$, we conclude that $z_l > -\infty$.

(ii) Assume that $n \geq 1$. Let $B = \max\limits_{0\leq u\leq 1} f(u)$. Since $p(0) = 0$, there exists $0 < b < \min\left\{\frac{c}{4B}, 1\right\}$ such that $p(u) < \frac{c}{2}$ for all $u \in (0,b]$. Let $q(u) = \frac{c}{2b}u$. Then $p(b) < \frac{c}{2} = q(b)$. We declare that $p(u) < q(u)$ for all $u \in (0,b]$. Otherwise, let $u_0 = \sup\{u \in (0,b) : p(u) = q(u)\}$. Then $0 < u_0 < b$, $p(u_0) = q(u_0)$ and $p(u) < q(u)$ for all $u \in (u_0, b]$. However,

$$\begin{aligned} \left.\frac{dp}{du}\right|_{u=u_0} &= c u_0^{-m}\left(1 - \frac{u_0^n f(u_0)}{q(u_0)}\right) = c u_0^{-m}\left(1 - \frac{2b u_0^{n-1} f(u_0)}{c}\right) \\ &> c b^{-m}\left(1 - \frac{2bB}{c}\right) > c b^{-1}\left(1 - \frac{2cB}{4cB}\right) = \frac{c}{2b} = \left.\frac{dq}{du}\right|_{u=u_0}, \end{aligned}$$

which contradicts the fact that $p(u_0) = q(u_0)$ and $p(u) < q(u)$ for all $u \in (u_0, b]$. So

$$\frac{du}{dz} = p(u) < q(u) = \frac{c}{2b}u, \qquad \forall u \in (0,b].$$

Integrating the above inequality from $z_1$ to $z_2$ with $z_l < z_1 < z_2 < z_r$, we see that

$$z_2 - z_1 \geq \int_{u(z_1)}^{u(z_2)} \frac{2b}{cu} du.$$

Letting $z_1 \to z_l$, we conclude that $z_l = -\infty$. The proof is complete.

Now, we turn to the discussion of the case with $f$ changing sign. We have

**Theorem 4.2** *Let $u(z)$ be the decreasing and smooth traveling wave front of* (1.1) *with $f$ satisfying* (H2) *corresponding to $c^*$, determined in Theorem 3.3.*

(i) *If $0 < n < 1$, then $z_r < +\infty$.*

(ii) *If $n \geq 1$, then $z_r = +\infty$.*

**Proof** We prove the two results respectively.

(i) Assume that $0 < n < 1$. Let $A = \min\limits_{0 \leq u \leq a/2} \{-f(u)\}$, $\epsilon = \min \left\{ \frac{a}{2}, \left( \frac{2c^*}{nA} \right)^{1/(n+m-1)} \right\}$ and $q(u) = -\frac{A}{2} u^n$. We declare that $p(u) < q(u)$ for all $u \in (0, \epsilon)$. Otherwise, there would exist $u_0 \in (0, \epsilon)$ such that $p(u_0) \geq q(u_0)$. Then

$$\left.\frac{dp}{du}\right|_{u=u_0} = c^* u_0^{-m} \left( 1 - \frac{u_0^n f(u_0)}{p(u_0)} \right) \leq c^* u_0^{-m} \left( 1 - \frac{A u_0^n}{\frac{A}{2} u_0^n} \right) \leq -c^* u_0^{-m}.$$

Since $0 < u_0 < \epsilon$, it follows that

$$\left.\frac{dq}{du}\right|_{u=u_0} = -\frac{nA}{2} u_0^{n-1} > -c^* u_0^{-m} \geq \left.\frac{dp}{du}\right|_{u=u_0}.$$

Let $u_1 = \inf\{u \in [0, u_0] : p(u) \geq q(u)\}$. If $u_1 > 0$, then from the above discuss,

$$\left.\frac{dq}{du}\right|_{u=u_1} > \left.\frac{dp}{du}\right|_{u=u_1},$$

which contradicts that $u_1 = \inf\{u \in [0, u_0) : p(u) \geq q(u)\}$. Therefore $u_1 = 0$, namely for all $u \in (0, u_0]$,

$$p(u) \geq q(u).$$

Thus

$$\frac{dq}{du} > \frac{dp}{du}, \quad \forall u \in (0, u_0].$$

Therefore, $p(0) > q(0) = 0$, which contradicts that $p(0) = 0$. So

$$\frac{du}{dz} = p(u) < q(u) = -\frac{A}{2} u^n, \qquad \forall u \in (0, \epsilon].$$

Let $z_1 \in (z_l, z_r)$ such that $u(z_1) = \epsilon$. Then $u(z) \in (0, \epsilon]$ for all $z \in [z_1, z_r)$. Integrating the above inequality from $z_1$ to $z_2$ with $z_l < z_1 < z_2 < z_r$ and noticing that $0 < n < 1$, we see that

$$z_2 - z_1 \leq \int_{u(z_1)}^{u(z_2)} \frac{-2}{Au^n} du \leq \int_0^1 \frac{2}{Au^n} du = \frac{2}{A(1-n)}.$$

Due to the arbitrariness of $z_2$, we conclude that $z_r < +\infty$.

(ii) Assume that $n \geq 1$. Let $B = \max\limits_{0\leq u\leq a}\{-f(u)\} > 0$ and fix $u_1 \in (0,a)$. If $\left.\dfrac{dq}{du}\right|_{u=u_1} \leq 0$, then

$$p(u_1) \geq u_1^n f(u_1) \geq -Bu_1^n.$$

If $\left.\dfrac{dq}{du}\right|_{u=u_1} > 0$, let $u_2 = \inf\left\{u \in (0,u_1) : \left.\dfrac{dq}{du}\right|_{u=s} > 0, \forall s \in (u,u_1)\right\}$. Then $\left.\dfrac{dq}{du}\right|_{u=u_2} = 0$ and $\dfrac{dp}{du} > 0$ for all $u \in (u_2,u_1)$. Since $p(0) = 0$ and $p(u) < 0$ for $u \in (0,1)$, we see that $0 < u_2 < u_1$. Thus

$$p(u_1) > p(u_2) = u_2^n f(u_2) \geq -Bu_2^n > -Bu_1^n.$$

Therefore

$$\frac{du}{dz} = p(u) \geq -Bu^n, \qquad \forall u \in (0,a).$$

Integrating the above inequality from $z_1$ to $z_2$ with $z_l < z_1 < z_2 < z_r$, we see that

$$z_2 - z_1 \geq \int_{u(z_1)}^{u(z_2)} \frac{-1}{Bu^n} du.$$

Letting $z_2 \to z_r$ and noticing that $u(z_r) = 0$ and $n \geq 1$, we conclude that $z_r = +\infty$. The proof is complete.

## References


[1] Allen, Linda J. S., Persistence and extinction in single-species reaction-diffusion models, *Bull. Math. Biol.*, **45**(2)(1983), 209–227.

[2] Friedman and McLeod, Blow-up of solutions of nonlinear degenerate parabolic equations, *Arch. Rat. Mech. Anal.*, **96**(1)(1986), 55–80.

[3] Dal Passo, R. and Luckhaus, S., A degenerate diffusion problem not in divergence form,*J. Differential Equations,* **69**(1)(1983), 1–14.

[4] Wang M. X., Wang S. and Xie C. H., A nonlinear degenerate diffusion equation not in divergence form, *Z. Angew. Math. Phys.*, **51**(1)(2000), 149–159.

[5] Michiel Bertsch and Maura Ughi, Positivity properties of viscosity solutions of a degenerate parabolic equation, *Nonlinear Anal.*, **14**(7)(1990), 571–592.

[6] Michiel Bertsch and Paolo Bisegna, Blow-up of solutions of a nonlinear parabolic equation in damage mechanics, *European J. Appl. Math.*, **8**(1)(1997), 89–123.

[7] Bertsch M. and Kamin S., The porous media equation with nonconstant coefficients, *Adv. Differential Equations*, **5**(1-3)(2000), 269–292.

[8] Fife P. C. and McLeod J. B., The approach of solutions of nonlinear diffusion equations to traveling front solutions, *Arch. Rat. Mech. Anal.*, **65**(1977), 335–361.

[9] Roquejoffre J. M., Terman D. and Volpert V. A., Global stability of traveling fronts and convergence towards stacked families of waves in monotone parabolic systems, *SIAM J. Math. Anal.*, **27**(1996), 1261–1269.

[10] Esquinas J. and Herrero M. A., Traveling wave solutions to semilinear diffusion system, *SIAM J. Math. Anal.*, **21**(1990), 123–136.

[11] Wang Shu, Finite traveling waves for a nonlinear degenerate reaction-diffusion system, *Nonlinear Anal., Ser. A: Theory Methods*, **41**(1-2)(2000), 15–31.

[12] Wang Mingxin, Traveling wave solutions to degenerate parabolic equations I., *Chinese Ann. Math. Ser. A*, **12**(5)(1991), 627–635.

[13] Baizhanova K. S. and Razzhevaikin V. N., Traveling waves in a population problem with degenerate diffusion, Mathematical modeling and discrete optimization, 71–77, Akad. Nauk SSSR, Vychisl. Tsentr, Moscow, 1988.

[14] Wang M. X., Traveling wave solutions for some degenerate-singular parabolic equations I, *Journal of Systems Science and Mathematical Sciences*, **12**(2)(1992), 127–135.

[15] Wang M. X., Traveling wave solutions for some degenerate-singular parabolic equations II, *Journal of Systems Science and Mathematical Sciences*, **12**(4)(1992), 341–349.