J. Partial Diff. Eqs. **16**(2003), 49–61
©International Academic Publishers Vol.16, No.1

INITIAL BOUNDARY VALUE PROBLEM FOR A DAMPED NONLINEAR HYPERBOLIC EQUATION *

Chen Guowang
(Department of Mathematics, Zhengzhou University, Zhengzhou 450052, China
E-mail : chen_gw@371.net)

Dedicated to the 80th birthday of Professor Zhou Yulin

(Received May 20, 2002)

Abstract In the paper, the existence and uniqueness of the generalized global solution and the classical global solution of the initial boundary value problems for the nonlinear hyperbolic equation

$$u_{tt} + k_1 u_{xxxx} + k_2 u_{xxxxt} + g(u_{xx})_{xx} = f(x,t)$$

are proved by Galerkin method and the sufficient conditions of blow-up of solution in finite time are given.

Key Words Nonlinear hyperbolic equation, initial boundary value problem, global solution, blow-up of solution

2000 MR Subject Classification 35L35, 35G30.

Chinese Library Classification O175.27, O175.29, O175.4.

1. Introduction

In this work we devote to the following damped nonlinear hyperbolic equation

$$u_{tt} + k_1 u_{x^4} + k_2 u_{x^4 t} + g(u_{xx})_{xx} = f(x,t), \qquad x \in \Omega,\ t > 0 \tag{1.1}$$

with the initial boundary value conditions

$$u(0,t) = u(1,t) = 0, \qquad u_{xx}(0,t) = u_{xx}(1,t) = 0, \qquad t > 0, \tag{1.2}$$

$$u(x,0) = \varphi(x), \qquad u_t(x,0) = \psi(x), \qquad x \in \overline{\Omega} \tag{1.3}$$

or with

$$u_x(0,t) = u_x(1,t) = 0, \qquad u_{x^3}(0,t) = u_{x^3}(1,t) = 0, \qquad t > 0, \tag{1.4}$$

$$u(x,0) = \varphi(x), \qquad u_t(x,0) = \psi(x), \qquad x \in \overline{\Omega} \tag{1.5}$$

or with

$$u(0,t) = u(1,t) = 0, \qquad u_x(0,t) = u_x(1,t) = 0, \qquad t > 0, \tag{1.6}$$

$$u(x,0) = \varphi(x), \qquad u_t(x,0) = \psi(x), \qquad x \in \overline{\Omega}, \tag{1.7}$$

*Supported by the National Natural Science Foundation of China (No.10071074) and by the Natural Science Foundation of Henan Province (No.00450100).

where $u(x,t)$ denotes an unknown function, k_1 and k_2 are two positive constants, $g(s)$ is a given nonlinear function, $f(x,t)$ is a given function, $\varphi(x)$ and $\psi(x)$ are given initial value functions which satisfy the continuous conditions:

$$\varphi_{x^{2k}}(0)=\varphi_{x^{2k}}(1)=\psi_{x^{2k}}(0)=\psi_{x^{2k}}(1)=0, \qquad (k=0,1) \ in \ (1.3);$$
$$\varphi_{x^{2k+1}}(0)=\varphi_{x^{2k+1}}(1)=\psi_{x^{2k+1}}(0)=\psi_{x^{2k+1}}(1)=0, \qquad (k=0,1) \ in \ (1.5)$$

and $\Omega=(0,1)$.

The equation (1.1) describes the motion for a class of nonlinear beam models with linear damping and general external time dependent forcing; for more physical interpretation of the equation (1.1) we refer to [1, 2].

The equation (1.1) and its multidimensional case have attracted much attention in recent years; for the well-posedness we refer to [3–5]. In [2] the authors have proved that the problem (1.1), (1.6), (1.7) has a unique global weak solution. In [1] the authors have been successful in proving the global existence of weak solutions for the multidimensional problem (1.1), (1.6), (1.7) by using a variational approach and the semigroup formulation. The energy decay of the mutidimensional problem (1.1), (1.6), (1.7) was given in [6].

In this paper, we are going to prove that the problem (1.1)-(1.3) or the problem (1.1), (1.4),(1.5) has a unique generalized global solution and a unique classical global solution by Galerkin method. We shall also show that the problem (1.1), (1.6), (1.7) has a unique generalized local solution. Finally, some sufficient conditions of blow-up of the solution for the problem (1.1), (1.6), (1.7) are given.

Throughout this paper, we use the following notations:$\|\cdot\|$, $\|\cdot\|_{Q_t}$, $\|\cdot\|_\infty$, $\|\cdot\|_{p(\Omega)}$ and $\|\cdot\|_{p(Q_t)}$ denote the norm of spaces $L^2(\Omega)$, $L^2(Q_t)$, $L^\infty(\Omega)$, $H^p(\Omega)$ and $H^p(Q_t)$, where $Q_t=\Omega\times(0,t)$ and $1\le p<\infty$.

2. Global existence and uniqueness of solutions

In order to prove that the problem (1.1)-(1.3) has the generalized global solution and the classical global solution, we now introduce an orthonormal base in $L^2(\Omega)$. Let $\{y_i(x)\}$ be the orthonormal base in $L^2(\Omega)$ composed of the eigenvalue problem

$$y''+\lambda y=0, \qquad x\in\Omega,$$
$$y(0)=y(1)=0$$

corresponding to eigenvalue $\lambda_i (i=1,2,\cdots)$, where "$'$" denotes the derivative. Let

$$u_N(x,t)=\sum_{i=1}^{N}\alpha_{Ni}(t)y_i(x) \tag{2.1}$$

be Galerkin approximate solution of the problem (1.1)-(1.3), where $\alpha_{Ni}(t)$ $(i=1,2,\cdots,N)$ are the undetermined functions, N is a natural number. Suppose that the initial value functions $\varphi(x)$ and $\psi(x)$ may be expressed

$$\varphi(x)=\sum_{i=1}^{\infty}a_iy_i(x),\qquad \psi(x)=\sum_{i=1}^{\infty}b_iy_i(x),$$

where $a_i,\ b_i(i=1,2,\cdots)$ are constants. Substituting the approximate solution $u_N(x,t)$ into (1.1), multiplying both sides by $y_s(x)$ and integrating on Ω, we obtain

$$(u_{Ntt}+k_1u_{Nx^4}+k_2u_{Nx^4t}+g(u_{Nxx})_{xx},y_s)=(f,y_s),\qquad s=1,2,\cdots,N, \tag{2.2}$$

where $(\cdot,\cdot)$ denotes the inner product of $L^2(\Omega)$. Substituting the approximate solution $u_N(x,t)$ and the approximations

$$\varphi_N(x)=\sum_{i=1}^{N}a_iy_i(x),\qquad \psi_N(x)=\sum_{i=1}^{N}b_iy_i(x)$$

of the initial value functions into (1.3), we have

$$\alpha_{Ns}(0)=a_s,\qquad \alpha_{Nst}(0)=b_s,\qquad s=1,2,\cdots,N. \tag{2.3}$$

Lemma 2.1 *Suppose that* $g\in C^2(R)$, $G(s)=\int_0^s g(y)dy\ge 0$, $\forall s\in R$, $g(0)=0$; $f\in L^2(Q_T)$; $\varphi\in H^3(\Omega)$ *and* $\psi\in L^2(\Omega)$. *Then for every* N, *the Cauchy problem* (2.2), (2.3) *for the system of the ordinary differential equations has a classical solution* $\alpha_{Ns}\in C^2[0,T](s=1,2,\cdots,N)$ *and the following estimation holds*

$$\|u_N\|^2_{2(\Omega)}+\|u_{Nt}\|^2+\|u_{Nx^2t}\|^2_{Q_t}+\int_\Omega\int_0^{u_{Nx^2}}g(y)dydx\le C_1(T),\qquad t\in[0,T], \tag{2.4}$$

where and in the sequel $C_1(T)$ *and* $C_i(T)(i=2,3,\cdots)$ *are constants which depend on* T, *but do not depend on* N.

Proof Multiplying both sides of (2.2) by $2\alpha_{Nst}$, summing up the products for $s=1,2,\cdots,N$, adding $2(u_N,u_{Nt})$ to the above both sides and integrating by parts with respect to x, we get

$$\begin{aligned}&\frac{d}{dt}(\|u_N\|^2+\|u_{Nt}\|^2+k_1\|u_{Nx^2}\|^2+2\int_\Omega\int_0^{u_{Nx^2}}g(y)dydx)\\&\qquad+2k_2\|u_{Nx^2t}\|^2\le\|f\|^2+2\|u_{Nt}\|^2+\|u_N\|^2.\end{aligned} \tag{2.5}$$

Observe that the following properties of the orthonormal base $\{y_i(x)\}$ on the boundary points of Ω have been used in (2.5):

$$y_i^{(2m)}(0)=y_i^{(2m)}(1),m=0,1,2,\cdots;\ i=1,2,\cdots,$$

where $(2m)$ denotes the order of the derivatives of the function $y_i(x)$. Gronwall inequality yields from (2.5)

$$\begin{aligned}&\|u_N\|^2+\|u_{Nt}\|^2+k_1\|u_{Nx^2}\|^2+2k_2\|u_{Nx^2t}\|^2_{Q_t}+2\int_\Omega\int_0^{u_{Nx^2}}g(y)dydx\\&\le e^{2T}\{(1+k_1)\|\varphi\|^2_{2(\Omega)}+\|\psi\|^2+2\int_\Omega\int_0^{\varphi_{x^2}}g(y)dy\\&\quad+\|f\|^2_{Q_t}+1\},\qquad t\in[0,T].\end{aligned} \tag{2.6}$$

It follows from (2.6) that the estimation (2.4) holds.

Similarly in [7], we can prove from (2.6) by Leray-Schauder fixed point theorem that the Cauchy problem (2.2),(2.3) has a solution $\alpha_{Ns} \in C^2[0,T](s = 1, 2, \cdots, N)$. The lemma is proved.

Lemma 2.2 *Suppose that the conditions of Lemma 2.1 and the following conditions hold:* $g \in C^3(R)$, $\forall s \in R$, $g'(s) \geq 0$, $g''(0) = 0$; $\varphi \in H^5(\Omega)$; $\psi \in H^3(\Omega)$; $f_x \in L^2(Q_T)$ *and* $f(0,t) = f(1,t) = 0$. *Then the approximate solution* $u_N(x,t)$ *has the estimation*

$$\|u_{Nt^2}\|^2_{Q_t} + \|u_N\|^2_{5(\Omega)} + \|u_{Nt}\|^2_{3(\Omega)} + \|u_{Nt}\|^2_{5(Q_t)} \leq C_2(T), \qquad t \in [0,T]. \tag{2.7}$$

Proof Multiplying both sides of (2.2) by $\lambda_s \alpha_{Ns}(t)$, summing up the products for $s = 1, 2, \cdots, N$, integrating with respect to t and integrating by parts with respect to x, we have

$$-2\int_0^t \int_\Omega u_{Nt^2} u_{Nx^2} dx d\tau + 2k_1 \int_0^t \int_\Omega u^2_{Nx^3} dx d\tau + k_2 \int_0^t \frac{d}{d\tau} \|u_{Nx^3}\|^2 d\tau$$
$$+ 2\int_0^t \int_\Omega g'(u_{Nx^2}) u^2_{Nx^3} dx d\tau = -2\int_0^t \int_\Omega f u_{Nx^2} dx d\tau. \tag{2.8}$$

Integrating by parts with respect to t, we get

$$-2\int_0^t \int_\Omega u_{Nx^2} u_{Nt^2} dx d\tau = -2\int_\Omega u_{Nt} u_{Nx^2} dx + 2\int_\Omega \psi_N \varphi_{Nx^2} dx$$
$$+ 2\int_\Omega \int_0^t u_{Nt} u_{Nx^2 t} dx d\tau. \tag{2.9}$$

Substituting (2.9) into (2.8), using Hölder inequality, assumptions and (2.4), we obtain

$$\|u_{Nx^3}\|^2 + \|u_{Nx^3}\|^2_{Q_t} \leq C_3(T), \qquad t \in [0,T]. \tag{2.10}$$

Multiplying both sides of (2.2) by $2\lambda_s^2 \alpha_{Nst}(t)$, summing up the products for $s = 1, 2, \cdots, N$, we have

$$\frac{d}{dt}(\|u_{Nx^2 t}\|^2 + k_1 \|u_{Nx^4}\|^2) + 2k_2 \|u_{Nx^4 t}\|^2 = -2\int_\Omega (g''(u_{Nx^2}) u^2_{Nx^3} u_{Nx^4 t}$$
$$+ g'(u_{Nx^2}) u_{Nx^4} u_{Nx^4 t}) dx + 2\int_\Omega f u_{Nx^4 t} dx. \tag{2.11}$$

It follows from (2.4), (2.10) and Sobolev embedding theorem, that

$$\|u_N\|_{C^2(\Omega)} \leq C_4(T), \qquad t \in [0,T]. \tag{2.12}$$

Using Gagliardo-Nirenberg interpolation theorem, we have

$$\|u_{Nx^3}\|_{L^4(\Omega)} \leq C_5 \|u_{Nx^3}\|^{\frac{3}{4}} \|u_{Nx^3}\|^{\frac{1}{4}}_{1(\Omega)}. \tag{2.13}$$

By use of Young inequality, (2.4), (2.10), (2.12) and (2.13), it follows from (2.11) that

$$\begin{aligned}&\frac{d}{dt}(\|u_{Nx^2t}\|^2+k_1\|u_{Nx^4}\|^2)+k_2\|u_{Nx^4t}\|^2\\&\qquad\le C_6(T)\|u_{Nx^4}\|^2+C_7\|f\|^2+C_8(T),\qquad t\in[0,T].\end{aligned}\tag{2.14}$$

Gronwall inequality yields from (2.14)

$$\|u_{Nx^2t}\|^2+\|u_{Nx^4}\|^2+\|u_{Nx^4t}\|^2_{Q_t}\le C_9(T),\qquad t\in[0,T].\tag{2.15}$$

Multiplying both sides of (2.2) by $\alpha_{Nst^2}(t)$, summing up the products for $s=1,2,\cdots,N$, integrating over $(0,t)$ with respect to t, observing (2.4), (2.15) and Sobolev embedding theorem, we obtain

$$\|u_{Ntt}\|_{Q_t}\le C_{10}(T),\qquad t\in[0,T].\tag{2.16}$$

Multiplying both sides of (2.2) by $-2\lambda_s^3\alpha_{Nst}(t)$, summing up the products for $s=1,2,\cdots,N$ and integrating by parts with respect to x, we have

$$\begin{aligned}&\frac{d}{dt}(\|u_{Nx^3t}\|^2+k_1\|u_{Nx^5}\|^2)+2k_2\|u_{Nx^5t}\|^2+2\int_\Omega g(u_{Nx^2})_{x^3}u_{Nx^5t}dx\\&\qquad=2(f_x,u_{Nx^5t}).\end{aligned}\tag{2.17}$$

By use of Hölder inequality, (2.4), (2.15) and Sobolev embedding theorem, it follows from (2.17) that

$$\|u_{Nx^3t}\|^2+\|u_{Nx^5}\|^2+\|u_{Nx^5t}\|_{Q_t}\le C_{11}(T),\qquad t\in[0,T].\tag{2.18}$$

From (2.4), (2.16) and (2.18) we see that (2.7) holds. This completes the proof of the lemma.

Theorem 2.1 *Under the conditions of Lemma 2.2, the problem* (1.1)-(1.3) *has a unique generalized global solution* $u(x,t)$, *i.e.* $u(x,t)$ *satisfies the identity*

$$\int_0^T\int_\Omega\{u_{tt}+k_1u_{x^4}+k_2u_{x^4t}+g(u_{xx})_{xx}-f(x,t)\}h(x,t)dxdt=0,\qquad\forall h\in L^2(Q_T)$$

and the initial boundary value conditions (1.2), (1.3) *in the classical sense. The solution* $u(x,t)$ *has the continuous derivatives* $u_{x^i}(x,t)(i=1,2)$ *and the generalized derivatives* $u_{x^i}(x,t)$, $u_{x^it}(x,t)(i=3,4,5)$ *and* $u_{tt}(x,t)$.

Proof From Lemma 2.2 and Sobolev embedding theorem we know that

$$\|u_N\|_{C^{4,\lambda}(\overline{\Omega})}\le C_{12}(T),\qquad\|u_{Nt}\|_{C^{2,\lambda}(\overline{\Omega})}\le C_{13}(T),\qquad t\in[0,T],\tag{2.19}$$

where $0<\lambda\le\frac{1}{2}$. It follows from (2.19) and Ascoli-Arzelá theorem that there exist a function $u(x,t)$ and a subsequence of $\{u_N(x,t)\}$ still denoted by $\{u_N(x,t)\}$ such that when $N\to\infty$, $\{u_N(x,t)\}$, $\{u_{Nx}(x,t)\}$ and $\{u_{Nx^2}(x,t)\}$ uniformly converge to $u(x,t)$, $u_x(x,t)$ and $u_{x^2}(x,t)$ on $\overline{Q}_T$ respectively. We also know from the estimation (2.7) that subsequences $\{u_{Nx^i}(x,t)\}$, $\{u_{Nx^it}(x,t)\}(i=3,4,5)$, $\{u^2_{Nx^3}(x,t)\}$ and $\{u_{Nt^2}(x,t)\}$ weakly converge to $u_{x^i}(x,t)$, $u_{x^it}(x,t)$ $(i=3,4,5)$, $u^2_{x^3}(x,t)$ and $u_{x^2}(x,t)$ in $L^2(Q_T)$

respectively. Thus we can prove by weakly compact theorem of the space $L^2(Q_T)$ that the problem (1.1)-(1.3) has a generalized global solution.

Now, we prove the uniqueness of the generalized solution $u(x,t)$. Suppose that $u_1(x,t)$ and $u_2(x,t)$ are two generalized solutions of the problem (1.1)-(1.3). Let $w(x,t) = u_1(x,t) - u_2(x,t)$. Then $w(x,t)$ satisfies the initial boundary value problem

$$w_{tt} + k_1 w_{x^4} + k_2 w_{x^4 t} + g(u_{1xx})_{xx} - g(u_{2xx})_{xx} = 0, \qquad x \in \Omega, t > 0, \tag{2.20}$$

$$w(0,t) = w(1,t) = 0, \qquad w_{xx}(0,t) = w_{xx}(1,t) = 0, t > 0, \tag{2.21}$$

$$w(x,0) = 0, \qquad w_t(x,0) = 0, \qquad x \in \Omega. \tag{2.22}$$

Multiplying both sides of the equation (2.20) by $2w_t(x,t)$, adding $2ww_t$ to the both sides and integrating over Ω, we get by calculation

$$\begin{aligned}\frac{d}{dt}(\|w\|^2 + \|w_t\|^2 + k_1\|w_{x^2}\|) + 2k_2\|w_{x^2 t}\|^2 \\ = -2\int_\Omega g'(u_{1xx} + \theta(u_{2xx} - u_{1xx})w_{xx}w_t dx + 2\int_\Omega ww_t dx,\end{aligned} \tag{2.23}$$

where $0 < \theta < 1$. Since $g'(u_{1xx} + \theta(u_{2xx} - u_{1xx}))$ is bounded, it follows from (2.23) that

$$\|w\|^2 + \|w_t\|^2 + \|w_{xx}\|^2 + \|w_{xxt}\|^2_{Q_t} \le \overline{C}\int_o^t (\|w\|^2 + \|w_t\|^2 + \|w_{xx}\|^2)d\tau,$$

where $\overline{C}$ is a constant, Gronwall inequality yields

$$\|w\|^2 + \|w_t\|^2 + \|w_{xx}\|^2 = 0.$$

Therefore, $u_1(x,t) = u_2(x,t)$.

The theorem is proved.

In order to prove that the problem (1.1)-(1.3) has a classical global solution, we make further estimations for the approximate solution $u_N(x,t)$.

Lemma 2.3 *Suppose that the conditions of Lemma 2.2 and the following conditions hold:* $g \in C^7(R)$, $g^{(2m)}(0) = 0 (m = 2,3)$; $\varphi \in H^9(\Omega)$; $\psi \in H^9(\Omega)$; $f \in H^1((0,T); H^3(\Omega)) \cap C^1([0,T]; H^1(\Omega))$, $f(x,0) \in H^5(\Omega)$ *and* $f_{x^{2m}}(0,t) = f_{x^{2m}}(1,t) = 0 (m = 1,2)$. *Then the approximate solution* $u_N(x,t)$ *has the estimation*

$$\|u_N\|^2_{7(\Omega)} + \|u_{Nt}\|^2_{7(\Omega)} + \|u_{Nt^2}\|^2_{5(\Omega)} + \|u_{Nt^3}\|^2_{1(\Omega)} \le C_{14}(T), \qquad t \in [0,T]. \tag{2.24}$$

Proof Multiplying both sides of (2.2) by $-2\lambda_s^5 \alpha_{Nst}(t)$, summing up the products for $s = 1, 2, \cdots, N$ and integrating by parts with respect to x, we obtain

$$\begin{aligned}\frac{d}{dt}(\|u_{Nx^5 t}\|^2 + k_1\|u_{Nx^7}\|^2) + 2k_2\|u_{Nx^7 t}\|^2 + 2\int_\Omega g(u_{Nx^2})_{x^5} u_{Nx^7 t} dx \\ = 2\int_\Omega f_{x^3} u_{Nx^7 t} dx.\end{aligned} \tag{2.25}$$

By use of straightforward calculation, it follows from (2.25) that

$$\frac{d}{dt}(\|u_{Nx^5t}\|^2+k_1\|u_{Nx^7}\|^2)+k_2\|u_{Nx^7t}\|^2 \le C_{15}(T)\|u_{Nx^7}\|^2+C_{16}\|f_{x^3}\|^2+C_{17}(T). \tag{2.26}$$

Gronwalll inequality from (2.26) yields

$$\|u_{Nx^5t}\|^2+\|u_{Nx^7}\|^2+\|u_{Nx^7t}\|^2_{Q_t}\le C_{18}(T), \qquad t\in[0,T]. \tag{2.27}$$

Differentiating (2.2) with respect to t, we have

$$(u_{Nt^3}+k_1u_{Nx^4t}+k_2u_{Nx^4t^2}+g(u_{Nx^2})_{x^2t},y_s)=(f_t,y_s). \tag{2.28}$$

Multiplying both sides of (2.28) by $-\lambda_s^5\alpha_{Nst^2}(t)$, summing up the products for $s=1,2,\cdots,N$, integrating by parts with respect to x, using (2.27) and Sobolev embedding theorem, we obtain

$$\frac{d}{dt}(\|u_{Nx^5t^2}\|^2+k_1\|u_{Nx^7t}\|^2)+2k_2\|u_{Nx^7t^2}\|^2 \le C_{19}(T)\|u_{Nx^7t}\|^2+\|f_{x^3t}\|^2+C_{20}(T). \tag{2.29}$$

Multiplying both sides of (2.2) by $-\lambda_s^5\,\alpha_{Nst^2}\,(t)$, summing up the products for $s=1,2,\cdots,N$, integrating by parts with respect to x and taking $t=0$, we have $\|u_{Nx^5t^2}(\cdot,0)\|^2\le C_{21}$. By use of Gronwall inequality, it follows from (2.29) that

$$\|u_{Nx^5t^2}\|^2+\|u_{Nx^7t}\|^2+\|u_{Nx^7t^2}\|^2_{Q_t}\le C_{22}(T), \qquad t\in[0,T]. \tag{2.30}$$

Multiplying both sides of (2.28) by $\alpha_{Nst^3}(t)$ and summing up the products for $s=1,2,\cdots,N$, we obtain

$$\|u_{Nt^3}\|^2\le C_{23}, \qquad t\in[0,T]. \tag{2.31}$$

Multiplying both sides of (2.28) by $-\lambda_s\alpha_{Nst^3}(t)$, summing up the products for $s=1,2,\cdots,N$ and integrating by parts with respect to x, we have

$$\|u_{Nxt^3}\|^2\le C_{24}(T), \qquad t\in[0,T]. \tag{2.32}$$

It follows from (2.7), (2.27), (2.30), (2.31) and (2.32) that (2.24) holds. The lemma is proved.

Theorem 2.2 *Under the conditions of Lemma 2.3, the problem* (1.1)-(1.3) *has a unique classical global solution* $u(x,t)$.

Proof We know from (2.24) and Sobolev embedding theorem that

$$\begin{aligned}&\|u_N\|_{C^6(\overline{\Omega})}\le C_{25}(T), \qquad \|u_{Nt}\|_{C^6(\overline{\Omega})}\le C_{26}(T),\\ &\|u_{Nt^2}\|_{C^4(\overline{\Omega})}\le C_{27}(T), \qquad \|u_{Nt^3}\|_{C(\overline{\Omega})}\le C_{28}(T), \qquad t\in[0,T].\end{aligned} \tag{2.33}$$

Using the estimation (2.33) and Ascoli-Arzelá theorem, we can prove that the problem (1.1)-(1.3) has a unique classical global solution $u(x,t)$. Since the generalized solution is unique, the classical solution also is unique. The theorem is proved.

Similarly, we can prove the following theorem.

Theorem 2.3 *Suppose that* $g \in C^3(R)$, $\forall s \in R$, $G(s) = \int_0^s g(y)dy \geq 0$, $g'(s) \geq 0$; $\varphi \in H^5(\Omega)$; $\psi \in H^3(\Omega)$ *and* $f_x \in L^2(Q_T)$. *Then the problem* (1.1), (1.4), (1.5) *has a unique generalized global solution* $u(x,t)$, *i.e.* $u(x,t)$ *satisfies the identity*

$$\int_0^T \int_\Omega \{u_{tt} + k_1 u_{x^4} + k_2 u_{x^4 t} + g(u_{xx})_{xx} - f(x,t)\} h(x,t) dx dt = 0, \qquad \forall h \in L^2(Q_T)$$

and the initial boundary value conditions (1.4), (1.5) *in the classical sense. The solution* $u(x,t)$ *has the continuous derivatives* $u_{x^i}(x,t)$ $(i=1,2)$ *and the generalized derivatives* $u_{x^i}(x,t)$, $u_{x^i t}(x,t)$ $(i=3,4,5)$ *and* $u_{tt}(x,t)$.

Except the above assumptions if $g \in C^7(R)$; $\varphi \in H^9(\Omega)$; $\psi \in H^9(\Omega)$; $f \in H^1((0,T); H^3(\Omega)) \cap C^1([0,T]; H^1(\Omega))$, $f(x,0) \in H^5(\Omega)$ *and* $f_x(0,t) = f_x(1,t) = 0$, *then the problem* (1.1), (1.4), (1.5) *has a unique classical global solution* $u(x,t)$.

3. Blow-up of solution

In this section we are going to consider the blow-up of solution. First of all, we can prove the existence and uniqueness of the generalized local solution for the equation (1.1) ($f(x,t) = 0$) with (1.6), (1.7) by the contraction mapping principle as in [8]. Thus we obtain the following theorem.

Theorem 3.1 *Suppose that* $\varphi \in H^4(\Omega)$, $\psi \in H^2(\Omega)$ *and* $g \in C^3(R)$. *Then the problem* (1.1), (1.6), (1.7) *has a unique generalized local solution* $u \in C([0,T_0); H^4(\Omega))$, $u_t \in C([0,T_0); H^2(\Omega)) \cap L^2([0,T_0); H^4(\Omega))$, $u_{tt} \in L^2(Q_{T_0})$, *where* $[0,T_0)$ *is a maximal time interval.*

In order to give the sufficient conditions of blow-up of the solution, we introduce the following lemma.

Lemma 3.1[9] *Suppose that* $\dot{u} = F(t,u)$, $\dot{v} \geq F(t,v)$, $F \in C$, $t_0 \leq t < \infty$, $-\infty < u < \infty$ *and* $u(t_0) = v(t_0)$, *then when* $t \geq t_0$, $v(t) \geq u(t)$.

Theorem 3.2 *Suppose that (1)* $sg(s) \leq KG(s)$, $G(s) \leq -\alpha|s|^{p+1}$, *where* $G(s) = \int_0^s g(\tau)d\tau$, $K > 2$, $\alpha > 0$ *and* $p > 1$ *are constants,*
(2) $k_2 = 1$, $\varphi \in H^2(\Omega)$, $\psi \in L^2(\Omega)$,

$$\begin{aligned} E(0) &= \|\psi\|^2 + k_1\|\varphi_{xx}\|^2 + 2\int_\Omega G(\varphi_{xx})dx \\ &\leq \frac{-4}{[(K-2)\alpha/(p+3)]^{\frac{2}{p-1}}(1 - e^{-\frac{p-1}{4}})^{\frac{4}{p-1}}} < 0, \end{aligned}$$

then the generalized solution of the problem (1.1) ($f(x,t) = 0$), (1.6), (1.7) *blows-up in finite time* $\tilde{T}$, *i.e.*

$$\|u(\cdot,t)\|^2 + \int_0^t \int_\Omega u_{xx}^2(x,\tau)dxd\tau + \int_0^t \int_0^\tau \int_\Omega u_{xx}^2(x,s)dxdsd\tau \to \infty, \qquad as\ t \to \tilde{T}^- .$$

Proof Multiplying both sides of the equation (1.1) by $2u_t$, integrating the product over Ω, we obtain

$$\dot{E}(t) = 0, \qquad t > 0, \tag{3.1}$$

where $\cdot = \frac{d}{dt}$,

$$E(t) = \|u_t(\cdot,t)\|^2 + k_1\|u_{xx}(\cdot,t)\|^2 + 2\int_\Omega G(u_{xx}(x,t))dx + 2k_2\int_0^t \|u_{xxt}\|^2 d\tau.$$

Hence

$$E(t) = E(0), \qquad t > 0. \tag{3.2}$$

Let

$$M(t) = \|u(\cdot,t)\|^2 + \int_0^t\int_\Omega u_{xx}^2(x,\tau)dxd\tau + \int_0^t\int_0^\tau\int_\Omega u_{xx}^2(x,s)dxdsd\tau. \tag{3.3}$$

We have

$$\dot{M}(t) = 2\int_\Omega u(x,t)u_t(x,t)dx + \int_\Omega u_{xx}^2(x,t)dx + \int_0^t\int_\Omega u_{xx}^2(x,\tau)dxd\tau. \tag{3.4}$$

Using the condition (1) of Theorem 3.2 and observing

$$\begin{aligned} K\int_\Omega G(u_{xx})dx =& E(0) - \|u_t(\cdot,t)\|^2 - 2k_2\int_0^t \|u_{xxt}(\cdot,\tau)\|^2 d\tau - k_1\|u_{xx}(\cdot,t)\|^2 \\ &+ (K-2)\int_\Omega G(u_{xx}(x,t))dx, \end{aligned} \tag{3.5}$$

we get

$$\begin{aligned} \ddot{M}(t) =& 2\int_\Omega \{u_t^2(x,t) + u(x,t)u_{tt}(x,t) + u_{xx}(x,t)u_{xxt}(x,t) + \frac{1}{2}u_{xx}^2(x,t)\}dx \\ =& 2\int_\Omega \{u_t^2(x,t) - k_1u_{xx}^2(x,t) - k_2u_{xx}(x,t)u_{xxt}(x,t) - u_{xx}(x,t)g(u_{xx}(x,t)) \\ &+ u_{xx}(x,t)u_{xxt}(x,t) + \frac{1}{2}u_{xx}^2(x,t)\}dx \\ \geq& 2\int_\Omega \{u_t^2(x,t) - k_1u_{xx}^2(x,t) - KG(u_{xx}(x,t)) + \frac{1}{2}u_{xx}^2(x,t)\}dx \\ \geq& 4\|u_t(\cdot,t)\|^2 - 2E(0) + 2(K-2)\alpha\int_\Omega |u_{xx}(x,t)|^{p+1}dx \\ &+ \|u_{xx}(\cdot,t)\|^2 > 0, \qquad t > 0. \end{aligned} \tag{3.6}$$

It follows from (3.6) that

$$\dot{M}(t) \geq -2E(0)t + 2(K-2)\alpha\int_0^t\int_\Omega |u_{xx}(x,\tau)|^{p+1}dxd\tau + \dot{M}(0), \tag{3.7}$$

$$M(t) \geq -E(0)t^2 + 2(K-2)\alpha\int_0^t\int_0^\tau\int_\Omega |u_{xx}(x,s)|^{p+1}dxdsd\tau + \dot{M}(0)t + M(0), \tag{3.8}$$

where

$$\dot{M}(0) = 2\int_\Omega \varphi(x)\psi(x)dx + \int_\Omega \psi_{xx}^2(x)dx, \qquad M(0) = \|\varphi\|^2.$$

From (3.6)-(3.8) we have

$$\begin{aligned}\ddot{M}(t) + \dot{M}(t) + M(t) \geq & 4\|u_t(\cdot,t)\|^2 + 2(K-2)\alpha\{\int_\Omega |u_{xx}(x,t)|^{p+1}dx \\ & + \int_0^t\int_\Omega |u_{xx}(x,\tau)|^{p+1}dxd\tau + \int_0^t\int_0^\tau\int_\Omega |u_{xx}(x,s)|^{p+1}dxdsd\tau\} \\ & + \|u_{xx}(\cdot,t)\|^2 - 2E(0)(\frac{t^2}{2}+t+1) \\ & + \dot{M}(0)(t+1) + M(0). \qquad (3.9)\end{aligned}$$

Substituting (3.4) into the left side of (3.9) we obtain

$$\begin{aligned}& \ddot{M}(t) + 2\int_\Omega u(x,t)u_t(x,t)dt + \int_\Omega u_{xx}^2(x,t)dx + \int_0^t\int_\Omega u_{xx}^2(x,\tau)dxd\tau + M(t) \\ & \geq 4\|u_t(\cdot,t)\|^2 + 2(K-2)\alpha\{\int_\Omega |u_{xx}(x,t)|^{p+1}dx \\ & + \int_0^t\int_\Omega |u_{xx}(x,\tau)|^{p+1}dxd\tau + \int_0^t\int_0^\tau\int_\Omega |u_{xx}(x,s)|^{p+1}dxdsd\tau\} \\ & + \|u_{xx}(\cdot,t)\|^2 - 2E(0)(\frac{t^2}{2}+t+1) + \dot{M}(0)(t+1) + M(0). \qquad (3.10)\end{aligned}$$

Since $\ddot{M}(t) > 0$, $M(t) \geq 0$ and

$$2\int_\Omega u(x,t)u_t(x,t)dx \leq \|u(\cdot,t\|^2 + \|u_t(\cdot,t)\|^2,$$

from (3.10) we have

$$\begin{aligned}\ddot{M}(t) + M(t) \geq & (K-2)\alpha\{\int_\Omega |u_{xx}(x,t)|^{p+1}dx + \int_0^t\int_\Omega |u_{xx}(x,\tau)|^{p+1}dxd\tau \\ & + \int_0^t\int_0^\tau\int_\Omega |u_{xx}(x,s)|^{p+1}dxdsd\tau\} - 2E(0)(\frac{t^2}{2}+t+1) \\ & + \frac{1}{2}\dot{M}(0)(t+1) + \frac{1}{2}M(0). \qquad (3.11)\end{aligned}$$

Using Hölder inequality and Poincaré inequality, we can obtain

$$\int_\Omega |u_{xx}(x,t)|^{p+1}dx \geq \|u_{xx}\|^{p+1} \geq \|u_x\|^{p+1} \geq \|u\|^{p+1}, \qquad (3.12)$$

$$\int_0^t\int_\Omega |u_{xx}(x,\tau)|^2dxd\tau \leq t^{\frac{p-1}{p+1}}(\int_0^t\int_\Omega |u_{xx}(x,\tau)|^{p+1}dxd\tau)^{\frac{2}{p+1}}, \qquad (3.13)$$

$$\int_0^t\int_0^\tau\int_\Omega |u_{xx}(x,s)|^2dxdsd\tau \leq (\int_0^t\int_0^\tau\int_\Omega |u_{xx}(x,s)|^{p+1}dxdsd\tau)^{\frac{2}{p+2}}(\frac{t^2}{2})^{\frac{p-1}{p+1}}. \quad (3.14)$$

It follows from (3.13) and (3.14) respectively that

$$\int_0^t \int_\Omega |u_{xx}(x,\tau)|^{p+1} dx d\tau \geq t^{\frac{1-p}{2}} \{ \int_0^t \int_\Omega |u_{xx}(x,\tau)|^2 dx d\tau \}^{\frac{p+1}{2}}, \tag{3.15}$$

$$\int_0^t \int_0^\tau \int_\Omega |u_{xx}(x,s)|^{p+1} dx ds d\tau \geq 2^{\frac{p-1}{2}} t^{1-p} \{ \int_0^t \int_0^\tau \int_\Omega |u_{xx}(x,s)|^2 dx ds d\tau \}^{\frac{p+1}{2}}. \tag{3.16}$$

Substituting (3.12), (3.15) and (3.16) into (3.11) and using the inequality

$$(a+b+c)^n \leq 2^{2(n-1)}(a^n + b^n + c^n), \qquad a, b, c, > 0, \qquad n > 1$$

we obtain

$$\begin{aligned}
\ddot{M}(t) + M(t) \geq & (K-2)\alpha \{ \|u\|^{p+1} + t^{\frac{1-p}{2}} [\int_0^t \int_\Omega |u_{xx}(x,\tau)|^2 dx d\tau]^{\frac{p+1}{2}} \\
& + 2^{\frac{p-1}{2}} t^{1-p} [\int_0^t \int_0^\tau \int_\Omega |u_{xx}(x,s)|^2 dx ds d\tau]^{\frac{p+1}{2}} \} \\
& - E(0)(\frac{t^2}{2} + t + 1) + \frac{1}{2}\dot{M}(0)(t+1) + \frac{1}{2}M(0) \\
\geq & 2^{1-p}(K-2)\alpha t^{1-p} M^{\frac{p+1}{2}}(t) - E(0)(\frac{t^2}{2} + t + 1) \\
& + \frac{1}{2}\dot{M}(0)(t+1) + \frac{1}{2}M(0), \qquad t \geq 1.
\end{aligned} \tag{3.17}$$

We see from (3.7) and (3.8) that $\dot{M}(t) \to \infty$ and $M(t) \to \infty$ as $t \to \infty$. Therefore, there is a $t_0 \geq 1$ such that when $t \geq t_0$, $\dot{M}(t) > 0$ and $M(t) > 0$. Multiplying both sides of (3.17) by $2\dot{M}(t)$ and using (3.7), we obtain

$$\frac{d}{dt}[\dot{M}^2(t) + M^2(t)] \geq C_4 t^{1-p} \frac{d}{dt} M^{\frac{p+3}{2}}(t) + Q(t), \qquad t \geq t_0, \tag{3.18}$$

where $C_4 = \frac{2(K-2)\alpha}{2^{p-2}(p+3)}$, $Q(t) = [-4E(0)t + 2\dot{M}(0)][-E(0)(\frac{t^2}{2} + t + 1) + \frac{1}{2}\dot{M}(0)(t+1) + \frac{1}{2}M(0)]$.

From (3.18) we get

$$\frac{d}{dt}[t^{p-1}(\dot{M}^2(t) + M^2(t) - C_4 M^{\frac{p+3}{2}}(t)] \geq t^{p-1} Q(t), \qquad t \geq t_0. \tag{3.19}$$

Integrating (3.19) over (t_0, t), we have

$$\begin{aligned}
t^{p-1}(\dot{M}^2(t) + M^2(t) - C_4 M^{\frac{p+3}{2}}(t)) \geq & \int_{t_0}^t \tau^{p-1} Q(\tau) d\tau + t_0^{p-1}(\dot{M}^2(0) + M^2(0)) \\
& - C_4 M^{\frac{p+3}{2}}(t_0), \qquad t \geq t_0.
\end{aligned} \tag{3.20}$$

Observe that when $t \to \infty$, the right-hand side of (3.20) approaches to positive infinity, hence there is a $t_1 \geq t_0$ such that when $t \geq t_1$, the right side of (3.20) is larger than or equal to zero. We thus have

$$t^{p-1}(\dot{M}(t) + M(t))^2 \geq t^{p-1}(\dot{M}^2(t) + M^2(t)) \geq C_4 M^{\frac{p+3}{2}}(t), \qquad t \geq t_1. \tag{3.21}$$

Extracting the square root of both sides of (3.21), we obtain

$$\dot{M}(t) + M(t) \geq t^{\frac{1-p}{2}} C_4^{\frac{1}{2}} M^{\frac{p+3}{4}}(t), \qquad t \geq t_1. \tag{3.22}$$

We consider the following initial value problem of the Bernoulli equation

$$\begin{aligned} &\dot{Z} + Z = C_4^{\frac{1}{2}} t^{\frac{1-p}{2}} Z^{\frac{p+3}{4}}, \qquad t > t_1 \\ &Z(t_1) = M(t_1). \end{aligned} \tag{3.23}$$

Solving the problem (3.23), we obtain the solution

$$\begin{aligned} Z(t) &= e^{-(t-t_1)} \left[M^{\frac{1-p}{4}}(t_1) - \frac{C_4^{\frac{1}{2}}(p-1)}{4} \int_{t_1}^{t} \tau^{\frac{1-p}{2}} e^{-\frac{p-1}{4}(\tau - t_1)} d\tau \right]^{\frac{4}{1-p}} \\ &= e^{-(t-t_1)} M(t_1) H^{\frac{4}{1-p}}(t), \qquad t \geq t_1, \end{aligned} \tag{3.24}$$

where $H(t) = 1 - \frac{p-1}{4} C_4^{\frac{1}{2}} M^{\frac{p-1}{4}}(t_1) \int_{t_1}^{t} \tau^{\frac{1-p}{2}} e^{-\frac{p-1}{4}(\tau - t_1)} d\tau$. Obviously, $H(t_1) = 1$ and

$$\begin{aligned} \sigma(t) &= \frac{p-1}{4} M^{\frac{p-1}{4}}(t_1) C_4^{\frac{1}{2}} \int_{t_1}^{t} \tau^{\frac{1-p}{2}} e^{-\frac{p-1}{4}(\tau - t_1)} d\tau \\ &\geq \frac{p-1}{4} M^{\frac{p-1}{4}}(t_1) C_4^{\frac{1}{2}} (t_1+1)^{\frac{1-p}{2}} \int_{t_1}^{t_1+1} e^{-\frac{p-1}{4}(\tau - t_1)} d\tau \\ &= M^{\frac{p-1}{4}}(t_1) C_4^{\frac{1}{2}} (t_1+1)^{\frac{1-p}{2}} (1 - e^{-\frac{p-1}{4}}), \qquad t \geq t_1 + 1. \end{aligned} \tag{3.25}$$

From (3.8) we see that

$$M^{\frac{p-1}{4}}(t)(t+1)^{\frac{1-p}{2}} \geq \left[\frac{-E(0)t^2 + \dot{M}(0)t + M(0)}{(t+1)^2} \right]^{\frac{p-1}{4}} \to (-E(0))^{\frac{p-1}{4}}$$

as $t \to \infty$. Take t_1 sufficiently large such that $M^{\frac{p-1}{4}}(t_1)(t_1+1)^{\frac{p-1}{2}} \geq \frac{1}{2}(-E(0))^{\frac{p-1}{4}}$. It follows from (3.25) and the condition of Theorem 3.2 that

$$\sigma(t) \geq \frac{C_4^{\frac{1}{2}}}{2} (-E(0))^{\frac{p-1}{4}} (1 - e^{-\frac{p-1}{4}}) \geq 1, \qquad t \geq t_1 + 1. \tag{3.26}$$

Therefore

$$H(t) = 1 - \sigma(t) \leq 0, \qquad t \geq t_1 + 1.$$

By virtue of the continuity of $H(t)$ and the theorem of intermediate values there is a constant $t_1 < \widetilde{T} \leq t_1 + 1$ such that $H(\widetilde{T}) = 0$. Hence $Z(t) \to \infty$ as $t \to \widetilde{T}^-$. It follows from Lemma 3.1 that when $t \geq t_1$, $M(t) \geq Z(t)$. Thus $M(t) \to \infty$ as $t \to \widetilde{T}^-$. Theorem 3.2 is proved.

References

[1] Banks H. T., Gilliam D. S. and Shubov V. I., Global solvability for damped abstract nonlinear hyperbolic systems, *Differential and Integral Equations*, **10**(2)(1997), 309-332.

[2] Banks H. T., Gilliam D. S. and Shubov V. I., Well-posedness for a one dimensional nonlinear beam, in "Computation and Control IV", Progess in systems and control theory 20, Birkhäuser, Boston, 1995.

[3] Banks H. T., Ito K. and Wang Y., Well-posedness for damped second order system with unbounded input operators, *Differential and Integral Equations*, **8**(1995), 587-606.

[4] Banchau A. and Hong C. H., Nonlinear composite beam theory, *ASME J. Applied Mechanics*, **55**(1988), 156-163.

[5] Luo S. Y. and Chou T. S., Finite deformation and nonlinear elastic behavior of flexible composite, *ASME J. Applied Mechanics*, **55**(1988), 149-155.

[6] Aassila M. and Guesmia A., Energy decay for a damped nonlinear hyperbolic equation, *Applied Mathematics Letters*, **12**(3)(1999), 49-52.

[7] Chen Guowang, Xing Jiasheng and Yang Zhijian, Cauchy Problem for genralized IMBq equation with several variables, *Nonlinear analysis TMA*, **26**(7)(1996), 1255-1270.

[8] Chen Guowang and Lü Shengguan, Initial boundary value problem for three dimensional Ginzburg-Landau model equation in population problems (in Chinese) *Acta Mathematicae Applicatae Sinica*, **23**(4)(2000), 507-517.

[9] Li Yuesheng, Basic inequality and uniqueness of the solution for differential equations(I), *Acta Sci. Natur. Univ. Jilin*, **1**(1960), 7-22.