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TWO DIMENSIONAL INTERFACE PROBLEMS FOR ELLIPTIC EQUATIONS*

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Dedicated to the 80th birthday of Professor Zhou Yulin

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Abstract We study the structure of solutions to the interface problems for second order quasi-linear elliptic partial differential equations in two dimensional space. We prove that each weak solution can be decomposed into two parts near singular points, a finite sum of functions in the form of $cr^\alpha \log^m r\varphi(\theta)$ and a regular one w. The coefficients c and the $C^{1,\alpha}$ norm of w depend on the H^1-norm and the $C^{0,\alpha}$-norm of the solution, and the equation only.

Key Words Quasilinear elliptic equations; interface problems; weak solutions; singular points.

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1. Introduction

We study the structure of the solutions to the equation

$$\frac{\partial}{\partial x_j}\left(a_{ij}(x,u)\frac{\partial u}{\partial x_i}\right) = \frac{\partial f_i}{\partial x_i}, \quad x \in \Omega_0, \tag{1}$$

where $\Omega_0 \subset \mathbb{R}^2$ and a_{ij}, f_i are discontinuous functions, $i, j = 1, 2$. The summation convention is assumed here. It is known that if u is a weak solution in $H^1(\Omega)$ then $u \in C^{0,\alpha}(\Omega)$ with a certain $\alpha \in (0,1)$. Moreover, if a_{ij}, f_i are piecewise smooth, then the solutions possess some structure near the discontinuous points of the coefficients. This kind of interface problems has been studied by a number of authors [1–8]. In [8] we proved that each weak solution to (1) can be decomposed into two parts near a singular point, a singular part and a regular part. The singular part is a finite sum of particular solutions with the form of $r^\alpha\varphi(\theta)$, or $r^\alpha \log^m r\varphi(\theta)$, where r is the distance to the singular point, and θ is the polar angle, and the regular part is bounded with respect to a norm which is slightly weaker than the H^2 norm, multiplied by a factor $\frac{1}{(|\log r|+1)^M}$.

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The result in [8] does not imply the boundedness of the derivatives of the regular part. The aim of this paper is to study the $C^{1,\alpha}$ norm estimate of the regular part. Our result is optimal here, that is, the regularity of the regular part of a weak solution is the same as the regularity of those solutions for the equations with smooth coefficients a_{ij}.

Let us present a statement of the problem and the main result. Let Ω_0 be a polygonal domain. We assume that Ω_0 is decomposed into a finite number of polygonal sub-domains $\Omega^{(k)}$, such that $\cup\overline{\Omega^{(k)}} = \overline{\Omega_0}$, and a_{ij} are sufficiently smooth on $\overline{\Omega^{(k)}} \times \mathbb{R}$. Moreover, we assume that a_{ij} satisfy the following elliptic condition:

$$a_{ij}(x,u)\xi_i\xi_j \geq \kappa|\xi|^2, \forall \xi \in \mathbb{R}^2,$$

for all $(x,u) \in (\Omega_0 \times \mathbb{R})$, where κ is a positive number. We also assume that $f_i \in C^{0,\alpha}(\Omega^{(k)})$ with $\alpha \in (0,1)$. For simplicity we impose the Dirichlet boundary condition,

$$u|_{x\in\partial\Omega_0} = 0 \tag{2}$$

on (1), where $\partial\Omega_0$ is the boundary.

The following points will be generally known as singular points: the cross points of interfaces, the turning points of interfaces, the cross points of interfaces with the boundary $\partial\Omega_0$, and the points on $\partial\Omega_0$ with interior angles greater than π. Let Σ be the set of singular points. We assume that Σ is a finite set. The problem (1) (2) admits a solution $u \in H_0^1(\Omega_0)$ (see [9–11]), and it is easy to prove that for each sub-domain $\Omega^{(k)}$, $u \in C^{1,\alpha}_{\text{loc}}(\Omega^{(k)}\backslash\Sigma)$. Thus the problem is the behavior of u near the singular points.

Let x_0 be a singular point. We construct local polar coordinates (r,θ) with the origin x_0. Let $s(x_0,\rho) \subset \Omega_0$ be a disc with center x_0 and radius ρ, such that x_0 is the only singular point on the disc. The subsets $s(x_0,\rho) \cap \Omega^{(k)}$ are thus some sectors, denoted by S_m. The main result of this paper is the following:

Theorem 1.1 *Let u be a weak solution to (1) (2) and $u \in H^1(\Omega_0)\bigcap C^{0,\bar\delta}(\Omega_0)$, $\bar\delta \in (0,1)$. Then there is an integer N and a constant $\alpha_0 \in (0,\bar\delta]$, such that if $0<\alpha<\alpha_0$ then $u = \sum_{n=1}^N u_n + w$ on $s(x_0,\rho)$, where*

$$u_n = c_n r^{\alpha_n} \log^{m_n} r \varphi_n(\theta), \tag{3}$$

$$\sum_m \|Dw\|_{C^{0,\alpha}(S_m)} + \sum_n |c_n| \leq C, \tag{4}$$

where m_n are non-negative integers, and φ_n are continuous, periodic, and piecewise infinitely differentiable functions, which depend only on $a_{ij}(x_0,u(x_0))$ and n; and C depends only on a_{ij}, $\|u\|_{H^1(\Omega_0)}$, $\|u\|_{C^{0,\bar\delta}(\Omega_0)}$, and $\|f_i\|_{C^{0,\alpha}(\Omega^{(k)})}$.

We will study homogeneous equations with constant coefficients in the next section, and nonhomogeneous equations with constant coefficients in Section 3, then prove the main theorem in Section 4. In what follows we assume that the singular point is an interior point. For those singular points on the boundary the argument is analogous. Without loss of generality we assume throughout this paper that the radius $\rho = 1$, the singular point $x_0 = 0$, and C is a generic constant possessing the above property.

2. Homogeneous Equations with Constant Coefficients

Without loss of generality we assume that the domain is $\Omega = s(o, 1)$, a disk with center o and radius 1. Let the point o be the singular point. Then the domain Ω is divided into some sectors S_m, $m = 1, \cdots, m_0$, by some rays starting from the point o. We consider the equation

$$Lu = \frac{\partial}{\partial x_j}\left(a_{ij}\frac{\partial u}{\partial x_i}\right) = 0, \tag{5}$$

where a_{ij} are constants on each sector S_m. Denote by Γ_0 the boundary of Ω. We take a constant $\xi \in (0, 1)$. Then we define sub-domains $\Omega_0, \Omega_1, \cdots, \Omega_k, \cdots$, where $\Omega_k = \{\xi^k > r > \xi^{k+1}\}$. In addition, we denote $\xi^k\Omega = \{0 < r < \xi^k\}$ and $\Gamma_k = \{r = \xi^k\}$. Let H be the space $H^{\frac{1}{2}}(\Gamma_0)$. Define a mapping $T_k : x \to \xi^k x$. We take an arbitrary $g \in H$, and consider the boundary condition $u|_{\Gamma_0} = g$. The equation (5) admits a unique solution $u \in H^1(\Omega)$ satisfying the boundary condition. Let $\tilde{g} = u|_{\Gamma_1}$, then $X : g \to \tilde{g} \circ T_1$ is a bounded operator from H to H. It is proved in [4] that X is a compact operator. By the Riesz-Schauder Theorem, the spectrum of X consists of isolated eigenvalues and the point o. The null spaces $N((X - \lambda I)^p)$ for all eigenvalues are finite dimensional. We arrange the eigenvalues so that $|\lambda_1| \geq |\lambda_2| \geq \cdots$. It is proved that if $\{\lambda, g\}$ is a pair of eigenvalue and eigenfunction, then either $\lambda = 1$, g =constant, or $|\lambda| < 1$. There is a particular solution to the equation (5) in the form of $r^\gamma g$, where

$$\gamma = \frac{\log\lambda}{\log\xi}. \tag{6}$$

If the degree of the elementary divisor of an eigenvalue is higher than 1, then there are particular solutions in the form of

$$u = \sum_{n=0}^{N} c_n r^\gamma \log^n r \varphi_n(\theta), \tag{7}$$

where $\varphi_N = g$.

We define a weighted Hölder norm as follows. For $b \in [0, 1]$ and $\alpha \in (0, 1)$ let

$$[u]_{\alpha,b,S_m} = \sup_{x,y\in S_m} \frac{r^b|u(x) - u(y)|}{|x - y|^\alpha},$$

where $r = \min(|x|, |y|)$, and

$$\|u\|_{\alpha,b,S_m} = [u]_{\alpha,b,S_m} + \sup_{x\in S_m} |x|^{b-\alpha}|u(x)|.$$

If $b = 0$, then the norm is abbreviated to $\|\cdot\|_{\alpha,S_m}$.

Lemma 2.1 *If $u \in C(S_m)$, then the norm $\|u\|_{\alpha,b,S_m}$ is equivalent to $\sup_k \xi^{bk}\|u\|_{\alpha,S_m\bigcap\Omega_k}$.*

Proof We have for each k that

$$\begin{aligned}\|u\|_{\alpha,b,S_m} &\geq \sup_{x,y\in S_m\bigcap\Omega_k}\frac{r^b|u(x)-u(y)|}{|x-y|^\alpha}+\sup_{x\in S_m\bigcap\Omega_k}|x|^{b-\alpha}|u(x)|\\ &\geq \sup_{x,y\in S_m\bigcap\Omega_k}\frac{\xi^{b(k+1)}|u(x)-u(y)|}{|x-y|^\alpha}+\sup_{x\in S_m\bigcap\Omega_k}\xi^{b(k+1)}|x|^{-\alpha}|u(x)|\\ &= \xi^{b(k+1)}\|u\|_{\alpha,S_m\bigcap\Omega_k}.\end{aligned}$$

On the other hand, for any $x,y\in S_m$, let $x^{(1)},\cdots,x^{(n)}$ be the cross points of the line segment xy with $\{\Gamma_k\}$. Then we have

$$\begin{aligned}|u(x)-u(y)| &\leq |u(x)-u(x^{(1)})|+\cdots+|u(x^{(n)})-u(y)|\\ &\leq (\xi r)^{-b}\sup_k \xi^{bk}\|u\|_{\alpha,S_m\bigcap\Omega_k}(|x-x^{(1)}|^\alpha+\cdots+|x^{(n)}-y|^\alpha).\end{aligned}$$

We may assume that $x\in\Omega_k$, $y\in\Omega_l$, and $k\leq l$. If $|x-x^{(1)}|=\max(|x-x^{(1)}|,\cdots,|x^{(n)}-y|)$, then

$$\begin{aligned}|x-x^{(1)}|^\alpha+\cdots+|x^{(n)}-y|^\alpha &\leq |x-x^{(1)}|^\alpha+|x-x^{(1)}|^\alpha+\xi|x-x^{(1)}|^\alpha+\cdots\\ &\leq \frac{2-\xi}{1-\xi}|x-x^{(1)}|^\alpha\leq\frac{2-\xi}{1-\xi}|x-y|^\alpha.\end{aligned}$$

Therefore

$$r^b\frac{|u(x)-u(y)|}{|x-y|^\alpha}\leq C\sup_k\xi^{bk}\|u\|_{\alpha,S_m\bigcap\Omega_k}.$$

The other cases can be considered in the same way, and the estimate for the maximum norm is obvious. The lemma is proved.

For simplicity, in what follows we will always omit the domain S_m in the Hölder norm, that is, $\|\cdot\|_{\alpha,b,\Omega_l}$ for $\|\cdot\|_{\alpha,b,S_m\bigcap\Omega_l}$.

We have the following decomposition result:

Lemma 2.2 *The solution u to (5) can be decomposed into $u=v+w$, where v is a finite sum of the above particular solutions (7), and $\|Dw\|_{\alpha,\xi\Omega}<C\|u\|_{H^1(\Omega)}$ with $0<\alpha<1$.*

Proof We define two spectrum sets:$\{\lambda_1,\cdots,\lambda_N\}$, $\{\lambda_{N+1},\cdots,0\}$,where $|\lambda_N|>|\lambda_{N+1}|$ and $|\lambda_{N+1}|<\xi^{1+\alpha}$. The space H is decomposed to two subspaces such that $H=H_1\oplus H_2$ and the spectrum of X_{H_1} in H_1 is just $\{\lambda_1,\cdots,\lambda_N\}$, the spectrum of X_{H_2} in H_2 is $\{\lambda_{N+1},\cdots,0\}$. Since $\lim_{k\to\infty}\|X_{H_2}^k\|^{\frac{1}{k}}=|\lambda_{N+1}|$, where $\|\cdot\|$ stands for the spectrum norm, we have $\|X_{H_2}^k\|\leq(|\lambda_{N+1}|+\varepsilon)^k$ for any $\varepsilon>0$ and sufficiently large k. We require that $|\lambda_{N+1}|+\varepsilon<\xi^{1+\alpha}$.

For any $g\in H$, we have a unique decomposition $g=g_1+g_2$, $g_1\in H_1$, and $g_2\in H_2$. Let v,w be the solutions corresponding to g_1,g_2 respectively. If k is large enough, then $\|X^kg_2\|_H\leq C\xi^{k(1+\alpha)}\|g\|_H$.

Let $\tilde{w}=w\circ T_{k-2}$, then $\tilde{w}$ satisfies the same equation. The standard interior $C^{1,\alpha}$ estimate is valid for this case. See [9] [10] [7] for details.

$$\|D\tilde{w}\|_{\alpha,\Omega_2}\leq C|\tilde{w}|_{H^1(\Omega\setminus\overline{\xi^3\Omega})}\leq C\|X^kg_2\|_H\leq C\xi^{k(1+\alpha)}\|g\|_H.$$

Therefore

$$\|Dw\|_{\alpha,\Omega_k} \le C\|g\|_H \le C\|u\|_{H^1(\Omega)}$$

for sufficiently large k, say $k > K_0$. The estimate of $\|Dw\|_{\alpha,\Omega_k}$ for $k = 1, \cdots, K_0$ is standard since K_0 is a fixed number and there is no singular point. The lemma is proved.

3. Nonhomogeneous Equations with Constant Coefficients

For the nonhomogeneous equation

$$\frac{\partial}{\partial x_j}\left(a_{ij}\frac{\partial u}{\partial x_i}\right) = \frac{\partial f_i}{\partial x_i}, \tag{8}$$

we recall a result in [8] first.

Lemma 3.1 *There is a particular solution to the equation*

$$Lu \equiv \frac{\partial}{\partial x_j}\left(a_{ij}\frac{\partial u}{\partial x_i}\right) = \frac{\partial}{\partial x_{j_0}}\left(r^{\alpha_1}\log^{m_1} r q_1(\theta)\frac{\partial}{\partial x_{i_0}}\left(r^{\alpha_2}\log^{m_2} r q_2(\theta)\right)\right), \tag{9}$$

in the form of

$$u = \sum_n c_n r^{\gamma}\log^{m_n} r\varphi_n(\theta), \tag{10}$$

where i_0 and j_0 are equal to 1 or 2, $Re\alpha_1 > 0$, $Re\alpha_2 > 0$, q_1, q_2 are continuous, periodic, and piecewise infinitely differentiable functions, and $\gamma = \alpha_1 + \alpha_2$.

To study the nonhomogeneous equation (8) we consider the equation on the space $\mathbb{R}^2$ first. The sectors S_m are extended to $|x| = \infty$, and then $\mathbb{R}^2$ is divided into m_0 sectors. We define a space

$$Z^1(\mathbb{R}^2) = \{u \in H^1_{\text{loc}}(\mathbb{R}^2); \nabla u \in L^2(\mathbb{R}^2), \int_{|x|<1} u\,dx = 0\}.$$

Then equipped with the norm $\|\nabla u\|_{L^2(\mathbb{R}^2)}$ it is a Hilbert space. We assume that supp$f_i \subset s(o,1)$, and $f_i \in C^{0,\alpha}(S_m)$. Consider the equation (8) and define the corresponding sesquilinear form

$$a(u,v) = \int_{\mathbb{R}^2} a_{ij}\frac{\partial u}{\partial x_i}\overline{\frac{\partial v}{\partial x_j}}\,dx.$$

The weak formulation of (8) is: find $u \in Z^1(\mathbb{R}^2)$ such that

$$a(u,v) = \int_{|x|<1} f_i\frac{\partial v}{\partial x_i}\,dx, \qquad \forall v \in Z^1(\mathbb{R}^2). \tag{11}$$

By the Lax-Milgram theorem there exists a unique solution u,

$$\|u\|_{Z^1(\mathbb{R}^2)} \le C\sum_i \|f_i\|_{L^2(\mathbb{R}^2)}.$$

Moreover, on any bounded domain $\Omega' \subset\subset \overline{S_m}$, $u \in C^{1,\alpha}(\Omega')$, and

$$\|Du\|_{\alpha,\Omega'} \le \sum_i \sum_m \|f_i\|_{\alpha,S_m}.$$

We return to the domain $s(o,1)$ and construct a particular solution u to the equation (8), so that u possesses the desired regularity. Let

$$\zeta(r,\theta) = \begin{cases} 1, & 1 > r > \xi, \\ 0, & r > \xi^{-1}, \text{ or } r < \xi^2, \end{cases}$$

and $\zeta \in C^\infty$, $0 \le \zeta \le 1$. Then we define $\zeta_k = \zeta \circ T_k / \sum_{l=1}^\infty \zeta \circ T_l$, and $F = (f_1, f_2)$, $F_k = \zeta_k F$. Let u_k be the solution to (11) with F replaced by F_k. u_k satisfies the homogeneous equation (5) on $\xi^{k+2}\Omega$. Analogous to the previous section we have the decomposition $u_k = u_k^{(1)} + u_k^{(2)}$, with $u_k^{(1)}|_{\Gamma_{k+2}} \in H_1$ and $u_k^{(2)}|_{\Gamma_{k+2}} \in H_2$, where H_1 and H_2 will be specified later on. We extend $u_k^{(1)}$ analytically to Ω, which is still denoted by $u_k^{(1)}$. Let $u = \sum_{k=1}^\infty (u_k - u_k^{(1)})$.

Lemma 3.2 *We assume that $\|F\|_{\alpha,b} < \infty$, $\alpha \in (0,1)$, $b \in [0,1]$. The subspaces H_1 and H_2 are defined according to the spectrum sets $\{\lambda_1, \cdots, \lambda_N\}$, $\{\lambda_{N+1}, \cdots, 0\}$, where $|\lambda_{N+1}| + \varepsilon < \xi^{1+\alpha-b} < |\lambda_N| - \varepsilon$, $\varepsilon > 0$. Then*

$$\|D(u_k - u_k^{(1)})\|_{\alpha,b,\Omega_l} \le \begin{cases} C\xi^{lb}\left\{\xi^{(k-l)(1+\alpha)} + \left(\dfrac{\xi^{1+\alpha}}{|\lambda_N| - \varepsilon}\right)^{k-l-1}\right\}\|F_k\|_\alpha, & \forall l \le k-2, \\ C\xi^{lb}\|F_k\|_\alpha, & \forall k+4 > l > k-2, \\ C\xi^{(k-l)(1+\alpha)+lb}\|F_k\|_\alpha, & \forall k+4 \le l < k+K_0, \\ C\xi^{lb}\left(\dfrac{|\lambda_{N+1}| + \varepsilon}{\xi^{1+\alpha}}\right)^{l-k}\|F_k\|_\alpha, & \forall l \ge k+K_0, \end{cases} \tag{12}$$

where K_0 is a fixed positive number.

Proof If $l \le k-3$, let $\tilde{u} = u_k \circ T_{k-1}$, then $\tilde{u}$ satisfies

$$L\tilde{u} = \xi^{k-1}\nabla \cdot (F_k \circ T_{k-1}).$$

Hence

$$\|\tilde{u}\|_{H,\Gamma_0} \le C\xi^{k-1}\|F_k \circ T_{k-1}\|_\alpha \le C\xi^{(k-1)(1+\alpha)}\|F_k\|_\alpha. \tag{13}$$

We consider the exterior problem and let $\tilde{u}|_{\Gamma_{l-k+2}} = X_1^{k-l-2}\tilde{u}|_{\Gamma_0}$, then being the same as X, X_1 is a bounded operator.

$$\|\tilde{u}\|_{H,\Gamma_{l-k+2}} \le C\xi^{(k-1)(1+\alpha)}\|F_k\|_\alpha.$$

Let $u^* = u_k \circ T_{l-1}$, then

$$\|u^*\|_{H,\Gamma_2} \le C\xi^{(k-1)(1+\alpha)}\|F_k\|_\alpha.$$

Applying the $C^{1,\alpha}$ estimate result we get

$$\|Du^*\|_{\alpha,\Omega_1} \leq C\xi^{(k-1)(1+\alpha)}\|F_k\|_\alpha.$$

Returning to the domain Ω_l, we get

$$\|Du_k\|_{\alpha,\Omega_l} \leq C\xi^{(k-l)(1+\alpha)}\|F_k\|_\alpha.$$

There are only a finite number of terms in $u_k^{(1)}$. We consider one of them, $w_{j,k}$, corresponding to an eigenvalue λ_j. By (13) we have

$$\|w_{j,k} \circ T_{k-1}\|_{H,\Gamma_0} \leq C\xi^{(k-1)(1+\alpha)}\|F_k\|_\alpha.$$

We note that

$$w_{j,k} = cr^{\frac{\log\lambda_j}{\log\xi}}\log^m r\varphi(\theta).$$

Consequently, we have

$$\|D(w_{j,k} \circ T_{k-1})\|_{\alpha,\Omega_{l-k+1}} \leq C(|\lambda_j| - \varepsilon)^{l-k+1}\xi^{-(l-k+1)(1+\alpha)}\xi^{(k-1)(1+\alpha)}\|F_k\|_\alpha,$$

which yields

$$\|Dw_{j,k}\|_{\alpha,\Omega_l} \leq C\left(\frac{\xi^{1+\alpha}}{|\lambda_N| - \varepsilon}\right)^{k-l-1}\|F_k\|_\alpha.$$

If $k+4 > l > k-2$, then

$$\|D\tilde{u}\|_{\alpha,\Omega\setminus\overline{\xi^4\Omega}} \leq C\xi^{(k-1)(1+\alpha)}\|F_k\|_\alpha,$$

which yields

$$\|Du_k\|_{\alpha,\xi^{k-1}\Omega\setminus\overline{\xi^{k+3}\Omega}} \leq C\|F_k\|_\alpha.$$

Analogously, we have

$$\|Du_k^{(1)}\|_{\alpha,\xi^{k-1}\Omega\setminus\overline{\xi^{k+3}\Omega}} \leq C\|F_k\|_\alpha.$$

If $l \geq k+4$, then $u_k - u_k^{(1)} = u_k^{(2)}$ on Ω_l, and by (13) we get

$$\|u_k^{(2)} \circ T_{k-1}\|_{H,\Gamma_3} \leq C\xi^{(k-1)(1+\alpha)}\|F_k\|_\alpha.$$

If $l-k \geq K_0$ and K_0 is sufficiently large, then $X_{H_2}^{l-k-4} \leq (|\lambda_{N+1}| + \varepsilon)^{l-k-4}$. Consequently, we have

$$\|u_k^{(2)} \circ T_{k-1}\|_{H,\Gamma_{l-k-1}} \leq C\xi^{(k-1)(1+\alpha)}(|\lambda_{N+1}| + \varepsilon)^{l-k-4}\|F_k\|_\alpha.$$

Let $u^* = u_k^{(2)} \circ T_{l-1}$, then

$$\|u^*\|_{H,\Gamma_{-1}} \leq C\xi^{(k-1)(1+\alpha)}(|\lambda_{N+1}| + \varepsilon)^{l-k-4}\|F_k\|_\alpha.$$

Then we get the $C^{1,\alpha}$ norm estimate

$$\|Du^*\|_{\alpha,\Omega_1} \leq C\xi^{(k-1)(1+\alpha)}(|\lambda_{N+1}| + \varepsilon)^{l-k-4}\|F_k\|_\alpha.$$

Consequently, we have

$$\|Du_k^{(2)}\|_{\alpha,\Omega_l} \le C\left(\frac{|\lambda_{N+1}|+\varepsilon}{\xi^{1+\alpha}}\right)^{l-k}\|F_k\|_\alpha.$$

If $l \ge k+3$ but $l < k+K_0$, then it is easy to see that

$$\|Du_k^{(2)}\|_{\alpha,\Omega_l} \le C\xi^{(k-l)(1+\alpha)}\|F_k\|_\alpha.$$

We multiply each inequality by a factor ξ^{lb} then the conclusion follows. The lemma is proved.

Lemma 3.3 *Under the assumptions of Lemma 3.2 it holds that*

$$\|Du\|_{\alpha,b,\xi\Omega} \le C\|F\|_{\alpha,b,\Omega}. \tag{14}$$

Proof Let $l \ge 1$. By Lemma 3.2 we have

$$\begin{aligned}
\|Du\|_{\alpha,b,\Omega_l} \le & C\sum_{k=1}^{l-K_0} C\xi^{lb}\left(\frac{|\lambda_{N+1}|+\varepsilon}{\xi^{1+\alpha}}\right)^{l-k}\|F_k\|_\alpha + \sum_{k=l-K_0+1}^{l-4} C\xi^{(k-l)(1+\alpha)+lb}\|F_k\|_\alpha \\
& + \sum_{k=l-3}^{l+1} C\xi^{lb}\|F_k\|_\alpha + \sum_{k=l+2}^{\infty} C\xi^{lb}\left\{\xi^{(k-l)(1+\alpha)} + \left(\frac{\xi^{1+\alpha}}{|\lambda_N|-\varepsilon}\right)^{k-l-1}\right\}\|F_k\|_\alpha \\
\le & C\|F\|_{\alpha,b}\left\{\sum_{k=1}^{l-K_0}\xi^{(l-k)b}\left(\frac{|\lambda_{N+1}|+\varepsilon}{\xi^{1+\alpha}}\right)^{l-k} + \sum_{k=l-K_0+1}^{l-4}\xi^{(k-l)(1+\alpha-b)}\right. \\
& \left. + \sum_{k=l-3}^{l+1} 1 + \sum_{k=l+2}^{\infty}\xi^{(l-k)b}\left(\xi^{(k-l)(1+\alpha)} + \left(\frac{\xi^{1+\alpha}}{|\lambda_N|-\varepsilon}\right)^{k-l-1}\right)\right\} \\
\le & C\|F\|_{\alpha,b}.
\end{aligned}$$

The lemma is proved.

4. Nonlinear Equations

We recall some results in [7] for the boundary value problem (1) (2) first. The solution u belongs to $C^{0,\bar\delta}(\Omega)$, and in the neighborhood of a singular point it holds that $\|r^{1-\delta_1}Du\|_{L^\infty} \le C$ with $\bar\delta > 0$ and $\delta_1 > 0$. Following the same argument we can prove the following lemma.

Lemma 4.1 *The weak solution u to (1), (2) satisfies*

$$\|Du\|_{\alpha,1} \le C, \tag{15}$$

provided $0 < \alpha \le \bar\delta$.

Proof Let $v = u \circ T_{k-1}$, then v satisfies

$$\frac{\partial}{\partial x_j}\left(a_{ij}(\xi^{k-1}x, v \circ T_{-k+1})\frac{\partial v}{\partial x_i}\right) = \xi^{k-1}\frac{\partial}{\partial x_i}(f_i \circ T_{k-1}).$$

It was shown in [7] that

$$\|Dv\|_{\alpha,\Omega_1} \leq C(|v|_{H^1(\Omega\backslash\overline{\xi^3\Omega})} + \xi^{k-1}\|F \circ T_k\|_{\alpha,\Omega\backslash\overline{\xi^3\Omega}}).$$

Then

$$\xi^{k-1}\|Du\|_{\alpha,\Omega_k} \leq C(\xi^{-(k-1)\alpha}|u|_{H^1(\xi^{k-1}\Omega\backslash\overline{\xi^{k+2}\Omega})} + \|F\|_{\alpha,\xi^{k-1}\Omega\backslash\overline{\xi^{k+2}\Omega}}). \tag{16}$$

By the Caccioppoli inequality we have the estimate

$$\|Dv\|_{L^2(\Omega_1)} \leq C(\|v - v(0)\|_{L^2(\Omega\backslash\overline{\xi^3\Omega})} + \|\xi^{k-1}F \circ T_{k-1}\|_{L^2(\Omega\backslash\overline{\xi^3\Omega})}).$$

Then

$$\|Du\|_{L^2(\Omega_k)} \leq C(\xi^{-k+1}\|u - u(0)\|_{L^2(\xi^{k-1}\Omega\backslash\overline{\xi^{k+2}\Omega})} + \|F\|_{L^2(\xi^{k-1}\Omega\backslash\overline{\xi^{k+2}\Omega})}). \tag{17}$$

Since $|u - u(0)| \leq Cr^{\bar{\delta}}$, we have

$$\|u - u(0)\|_{L^2(\xi^{k-1}\Omega\backslash\overline{\xi^{k+2}\Omega})} \leq C\xi^{k(1+\bar{\delta})}. \tag{18}$$

We substitute (18) into (17), then (17) into (16). Then (15) follows. The lemma is proved.

To prove the main theorem we need the following lemma. We will assume that $u(0)=0$, otherwise $u(x)$ can be replaced by $u(x)-u(0)$. We denote $L=\frac{\partial}{\partial x_j}\left(a_{ij}(0,0)\frac{\partial}{\partial x_i}\right)$, where $a_{ij}(0,0)$ are piecewise constant functions, which are equal to $a_{ij}(0,0)$ on each sector S_m.

Lemma 4.2 *If a solution u of (1) can be decomposed as*

$$u = \sum_{n=1}^{N} u_n + w, \tag{19}$$

where u_n are in the form of (4), and

$$\|Dw\|_{\alpha,b} + r^{-1}\|w\|_{\alpha,b,s(o,r)} \leq C, \tag{20}$$

where $b \in [0,1]$, $0 < \operatorname{Re}\alpha_1 \leq \operatorname{Re}\alpha_2 \leq \cdots \leq \operatorname{Re}\alpha_N$, and $\alpha < \operatorname{Re}\alpha_1$, then there is a new decomposition of u, still given by (19), such that

$$\|Dw\|_{\alpha,b_1} + r^{-1}\|w\|_{\alpha,b_1,s(o,r)} \leq C, \tag{21}$$

where $b_1 > b - \operatorname{Re}\alpha_1$, $b_1 \geq \max(b - \bar{\delta}, 0)$, and there exists an integer N such that $|\lambda_{N+1}| < \xi^{1+\alpha-b_1} < |\lambda_N|$.

Proof We rewrite the equation (1) as

$$Lu = -\frac{\partial}{\partial x_j}\left((a_{ij}(x,u) - a_{ij}(0,0))\frac{\partial u}{\partial x_i}\right) + \frac{\partial f_i}{\partial x_i}.$$

By Taylor's expansion we have

$$\begin{aligned} a_{ij}(x,u) - a_{ij}(0,0) &= \{a_{ij}(0,u) - a_{ij}(0,0)\} + \{a_{ij}(x,u) - a_{ij}(0,u)\} \\ &= a_{ij}^{(1)}(0,u) + a_{ij}^{(2)}(x,u), \end{aligned}$$

where

$$\begin{aligned} a_{ij}^{(1)}(0,u) &= b_1 u + \cdots + b_{N_1} u^{N_1}, \\ a_{ij}^{(2)}(x,u) &= b(u) u^{N_1+1} + \{a_{ij}(x,u) - a_{ij}(0,u)\}. \end{aligned}$$

Then

$$\begin{aligned} Lu &= -\frac{\partial}{\partial x_j}\left((a_{ij}(x,u) - a_{ij}(0,0))\sum_{n=1}^{N}\frac{\partial u_n}{\partial x_i}\right) \\ &\quad -\frac{\partial}{\partial x_j}\left((a_{ij}(x,u) - a_{ij}(0,0))\frac{\partial w}{\partial x_i}\right) + \frac{\partial f_i}{\partial x_i} \\ &= -\frac{\partial}{\partial x_j}\left(a_{ij}^{(1)}(0,\sum_{n=1}^{N} u_n)\sum_{n=1}^{N}\frac{\partial u_n}{\partial x_i}\right) - \frac{\partial f_j'}{\partial x_j} + \frac{\partial f_i}{\partial x_i}, \end{aligned} \tag{22}$$

where

$$\begin{aligned} f_j' = &\left(a_{ij}^{(1)}(0,u) - a_{ij}^{(1)}(0,\sum_{n=1}^{N} u_n)\right)\sum_{n=1}^{N}\frac{\partial u_n}{\partial x_i} \\ &+ a_{ij}^{(2)}(x,u)\sum_{n=1}^{N}\frac{\partial u_n}{\partial x_i} + ((a_{ij}(x,u) - a_{ij}(0,0))\frac{\partial w}{\partial x_i} \end{aligned}$$

We consider the first term on the right of (22). By Lemma 3.1 there is a particular solution $w_1 = \sum_{n=1}^{N'} w_{1n}$, $w_{1n} = c_n r^{\gamma_n}\log^{m_n} r\varphi_n(\theta)$, such that $|c_n| \le C$, and $\mathrm{Re}\,\gamma_n \ge 2\mathrm{Re}\,\alpha_1$, to the equation

$$Lw_1 = -\frac{\partial}{\partial x_j}\left(a_{ij}^{(1)}(0,\sum_{n=1}^{N} u_n)\sum_{n=1}^{N}\frac{\partial u_n}{\partial x_i}\right).$$

We estimate $\|f_j'\|_{\alpha,\Omega_k}$ next. We have the following estimates on $\Omega_k \bigcap S_m$:

$$\left|a_{ij}^{(1)}(0,u) - a_{ij}^{(1)}(0,\sum_{n=1}^{N} u_n)\right| \le C|w| \le C\xi^{k(1-b+\alpha)},$$

$$\left[a_{ij}^{(1)}(0,u) - a_{ij}^{(1)}(0,\sum_{n=1}^{N} u_n)\right]_\alpha \le C[w]_\alpha + C([u]_\alpha + \sum_{n=1}^{N}[u_n]_\alpha)\max|w| \le C\xi^{k(1-b)};$$

$$\left|\frac{\partial u_n}{\partial x_i}\right| \le C\xi^{k(\delta-1)},$$

where $\delta = \mathrm{Re}\,\alpha_1 - \varepsilon > 0$ with ε positive and small,

$$\left[\frac{\partial u_n}{\partial x_i}\right]_\alpha \le C\xi^{k(\delta-1-\alpha)};$$
$$|a_{ij}^{(2)}| \le C\xi^{k(\min(N_1+1)\delta,1)} = C\xi^k,$$
$$[a_{ij}^{(2)}]_\alpha \le C\xi^{kN_1\delta}[u]_\alpha + C\xi^{k(1-\alpha)} \le C\xi^{k(1-\alpha)}$$

for N_1 large enough;

$$|a_{ij}(x,u) - a_{ij}(0,0)| \le C\xi^{k\bar\delta},$$
$$[a_{ij}(\cdot,u) - a_{ij}(0,0)]_\alpha \le C\xi^{k(\bar\delta-\alpha)}.$$

By interpolation we get

$$|Dw| \le C\xi^{k(-b+\alpha)}.$$

Therefore, we have

$$\|f_j'\|_\alpha \le C\xi^{k(\delta-b)} + C\xi^{k(\delta-\alpha)} + C\xi^{k(\bar\delta-b)}. \tag{23}$$

Let ε be small enough, then $b-\delta = b - \mathrm{Re}\,\alpha_1 + \varepsilon \le b_1$, $\delta - \alpha = \mathrm{Re}\,\alpha_1 - \varepsilon - \alpha \ge 0$. By Lemma 3.3 there is a solution w_2 to the equation $Lw_2 = -\frac{\partial f_j'}{\partial x_j}$ such that

$$\|Dw_2\|_{\alpha,b_1,\xi\Omega} \le C\sum_j \|f_j'\|_{\alpha,b_1,\Omega}. \tag{24}$$

Finally, by Lemma 3.3 there is a solution w_3 to the equation $Lw_3 = \frac{\partial f_i}{\partial x_i}$ such that

$$\|Dw_3\|_{\alpha,b_1,\xi\Omega} \le C\sum_i \|f_i\|_{\alpha,b_1,\Omega}. \tag{25}$$

By (24) (25) we have

$$\|w_1 + w_2 + w_3\|_{H,\Gamma_1} \le C.$$

Since $\|u\|_1 \le C$, we get $\|u\|_{H,\Gamma_1} \le C$. $u - w_1 - w_2 - w_3$ is a solution to the homogeneous equation $Lu = 0$, then using the results for homogeneous equations [4], we get a decomposition on $\xi^2\Omega$, $u - w_1 - w_2 - w_3 = v + w_4$, such that

$$\|Dw_4\|_{\alpha,\xi^2\Omega} < C, \tag{26}$$

and $v = \sum_n u_n$. Let $w = w_2 + w_3 + w_4$ and combine the terms of w_1 and v, still denoted by $\sum_n u_n$. If there are some terms in $\sum_n u_n$ satisfying the conditions for w, then we put them in w. It has no harm in assuming $w(0) = 0$, otherwise we can plus a constant on it, then there is no constant term in $\sum_n u_n$. Combining the estimates (24)-(26), the inequality (21) is verified on the domain $\xi^2\Omega$. Then applying the condition (20), we see that (21) is in fact satisfied on Ω. The lemma is proved.

Proof of Theorem 1.1 We use Lemma 4.2 to prove by induction. We fix a positive constant Δb such that $\Delta b < \mathrm{Re}\,\alpha_1$ and $\Delta b \leq \bar{\delta}$. By Lemma 4.1 we take $w = u$, $b = 1$ first. If $\xi^{1+\alpha-b+\Delta b} = |\lambda_k|$ for some k, we reduce Δb slightly, denoted by $\Delta b'$, such that $\Delta b' > \Delta b/2$ and there exists an integer N such that $|\lambda_{N+1}| < \xi^{1+\alpha-b+\Delta b'} < |\lambda_N|$. Then we get (21) with $b_1 = b - \Delta b$ or $b_1 = b - \Delta b'$. Again we take $b = b_1$, and so on. In each step the exponents decrease by a positive constant Δb or $\Delta b'$. We notice that the spectrum set is discrete, so there is no eigenvalue λ_k satisfying $\xi^{1+\alpha} = |\lambda_k|$ for small α. After a finite number of steps we get $b = 0$, then (3) follows and the proof is complete.

References

[1] Kellogg R. B., Sigularities in Interface Problems, SYNSPADE II, ed. B.Hubbard, 351-400, 1971.

[2] Kellogg R. B., High order regularity for interface problems, in The Mathematical Foundation of the Finite Element Method, ed. A.K.Aziz, Academic Press, 1972.

[3] Blumenfeld M., The regularity of interface problems on corner regions, Lecture Notes in Mathematics 1121, Springer-Verlag, 38-54, 1985.

[4] Ying L.-a., Interface problems for elliptic differential equations, *Chin. Ann. of Math.*, **18B**(1997), 139-152.

[5] Ying L.-a., High order regularity for interface problems, *Northeast. Math. J.*,**13**(1997), 459-476.

[6] Wu J., Interface problems for elliptic systems, *J.Partial Diff. Eqs.*, **12**(1999), 313-323.

[7] Wu J., Interface problems for quasilinear elliptic equations, *J. Diff. Eqs.*, **157**(1999), 102-119.

[8] Ying L.-a., A decomposition theorem for the solutions to the interface problems of quasilinear elliptic equations, Research Report No.31, School of Mathematical Sciences and Institute of Marthematics, Peking University, 2001.

[9] Chen Y. Z. and Wu L. C., Second Order Elliptic Equations and Elliptic Systems of Equations (in Chinese), Science Press, Beijing, 1991.

[10] Gilbarg D. and Trudinger N. S., Elliptic Partial Differential Equations of Second Order, Springer-Verlag, 1977.

[11] Lions J. L. and Magenes E., Nonhomogeneous Boundary Value Problems and Applications, Springer-Verlag, 1972.

[12] Adams R. A., Sobolev Spaces, Academic Press, 1975.