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THE EXISTENCE AND UNIQUENESS OF A POSITIVE SOLUTION OF AN ELLIPTIC SYSTEM

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Abstract The existence and uniqueness of the positive solution for the generalized Lotka-Volterra competition model for several competing species

$$\Delta u^i + u^i(a - g(u^1, ..., u^N)) = 0 \text{ in } \Omega,$$
$$u^i = 0 \text{ on } \partial\Omega,$$

for $i = 1, ..., N$ were investigated. The techniques used in this paper are elliptic theory, upper-lower solutions, maximum principles and spectrum estimates. The arguments also rely on some detailed properties for the solution of logistic equations.
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1. Introduction

A lot of research has been focused on reaction-diffusion equations modeling of various systems in mathematical biology, especially the elliptic steady states of competitive and predator-prey interacting processes with various boundary conditions. In the earlier literature, investigations into mathematical biology models were concerned with studying those with homogeneous Neumann boundary conditions. From here on, the more important Dirichlet problems, which allow flux across the boundary, became the subject of study.(see [1-8])While necessary and sufficient conditions for the existence of positive solutions to the steady states have been established for rather general types of systems(see [7, 8]), our knowledge about the uniqueness of positive solutions is limited to somewhat rather special systems, whose relative growth rates are linear; the results established are only for the following competition models(see [?], [3-6])

$$\begin{cases} \Delta u^i + u^i(a_i - \beta_i u^i - \sum_{j=1, j\neq i}^{N} c_{ij} u^j) = 0 \text{ in } \Omega, \\ u^i|_{\partial\Omega} = 0, \\ u^i > 0 \text{ in } \Omega \end{cases}$$

for $i = 1, ..., N$.

The question in this paper concerns the existence and uniqueness of positive coexistence when the relative growth rates are nonlinear, more precisely, the existence and uniqueness of positive steady state of

$$\Delta u^i + u^i \left(a_i - g_i(u^1, ..., u^N) \right) = 0 \text{ in } \Omega,$$
$$u^i = 0 \text{ on } \partial\Omega$$

for $i = 1, ..., N$. Here, a_i's are positive constants, g_i's are C^1 functions, Ω is a bounded domain in R^n and u^i's are densities of N competitive species.

The followings are the problems which we will discuss in this paper.

Problem 1 What are the sufficient conditions for existence and uniqueness of steady state at a fixed reproduction $(a_1, ..., a_N)$ in R^N? When does either one of the species become extinct, i.e. when is either one of the species excluded by the other?

Problem 2 Assume the uniqueness of coexistence state at a fixed reproduction $(a_1, ..., a_N)$, is it possible to do the perturbation to an open ball around $(a_1, ..., a_N)$ with the uniqueness, strictly speaking, is there a neighborhood V of the fixed reproduction rate $(a_1, ..., a_N)$ such that the uniqueness of coexistence state is guaranteed for any reproduction rates $(a'_1, ..., a'_N)$ in V?

Problem 3 This is the generalization of Problem 2. What is the answer of Problem 2 when we have the uniqueness of positive solution to the above equation on the left or right boundary of a closed, convex region Γ of the reproductions $(a_1, ..., a_N)$? Can we still perturb the region Γ to an open set including Γ with the uniqueness?

In Section 3, some sufficient conditions to guarantee the existence, uniqueness of positive solutions are obtained and we also see that they can not coexist for small self-reproduction rates using upper-lower solutions and spectrum estimates, which solves Problem 1. In Sections 4 and 5, we provide the answers for Problems 2 and 3 using elliptic theory, maximum principles and implicit function theorem.

2. Preliminaries

In this section we state some preliminary results which will be useful for our later arguments.

Definition 2.1 (Upper and Lower solutions) *The vector functions* $(\bar{u}^1, ..., \bar{u}^N)$, $(\underline{u}^1, ..., \underline{u}^N)$ *form an upper/lower solution pair for the system*

$$\begin{cases} \Delta u^i + g^i(u^1, ..., u^N) = 0 & in \;\; \Omega \\ u^i = 0 & on \;\; \partial\Omega \end{cases}$$

if for $i = 1, ..., N$

$$\begin{cases} \Delta \bar{u}^i + g^i(u^1, ..., u^{i-1}, \bar{u}^i, u^{i+1}, ..., u^N) \le 0 \\ \Delta \underline{u}^i + g^i(u^1, ..., u^{i-1}, \underline{u}^i, u^{i+1}, ..., u^N) \ge 0 \\ in \;\; \Omega \;\; for \;\; \underline{u}^j \le u^j \le \bar{u}^j, j \ne i, \end{cases}$$

and

$$\underline{u}^i \le \bar{u}^i \quad on \ \ \Omega$$
$$\underline{u}^i \le 0 \le \bar{u}^i \quad on \ \ \partial\Omega.$$

Lemma 2.1 ([?]) *If g^i in Definition **??** are in C^1 and the system admits an upper/lower solution pair $(\underline{u}^1, ..., \underline{u}^N), (\bar{u}^1, ..., \bar{u}^N)$, then there is a solution of the system in **??** with $\underline{u}^i \le u^i \le \bar{u}^i$ in $\bar{\Omega}$. If*

$$\Delta\bar{u}^i + g^i(\bar{u}^1, ..., \bar{u}^N) \ne 0,$$
$$\Delta\underline{u}^i + g^i(\underline{u}^1, ..., \underline{u}^N) \ne 0$$

in Ω for $i = 1, ..., N$, then $\underline{u}^i < u^i < \bar{u}^i$ in Ω.

Lemma 2.2 (The first eigenvalue)([?])

$$\begin{cases} -\Delta u + q(x)u = \lambda u \ \ in \ \ \Omega, \\ u|_{\partial\Omega} = 0, \end{cases} \tag{1}$$

where $q(x)$ is a smooth function from Ω to R and Ω is a bounded domain in R^n.

(A) *The first eigenvalue $\lambda_1(q)$ of (**??**), denoted by simply λ_1 when $q \equiv 0$, is simple with a positive eigenfunction.*

(B) *If $q_1(x) < q_2(x)$ for all $x \in \Omega$, then $\lambda_1(q_1) < \lambda_1(q_2)$.*

(C)*(Variational Characterization of the first eigenvalue)*

$$\lambda_1(q) = \min_{\phi\in W_0^1(\Omega),\phi\ne 0} \frac{\displaystyle\int_\Omega (|\nabla\phi|^2 + q\phi^2)dx}{\displaystyle\int_\Omega \phi^2 dx}.$$

Lemma 2.3 ([?])

$$Lu = \sum_{i,j=1}^n a_{ij}(x)D_{ij}u + \sum_{i=1}^n a_i(x)D_i u + a(x)u = f(x) \ \ in \ \ \Omega,$$

where Ω is a bounded domain in R^n and

(M1) $\partial\Omega \in C^{2,\alpha}(0 < \alpha < 1)$,

(M2) $|a_{ij}(x)|_\alpha, |a_i(x)|_\alpha, |a(x)|_\alpha \le M(i, j = 1, ..., n)$,

(M3) *L is uniformly elliptic in $\bar{\Omega}$, with ellipticity constant γ, i.e., for every $x \in \bar{\Omega}$ and every real vector $\xi = (\xi_1, ..., \xi_n)$*

$$\sum_{i,j=1}^n a_{ij}(x)\xi_i\xi_j \ge \gamma\sum_{i=1}^n |\xi_i|^2.$$

Under this conditions, we have the following statements: Maximum principles and Schauder's estimates.

1. *Maximum principles*

Let $u \in C^2(\Omega) \cap C(\bar{\Omega})$ *be a solution of* $Lu \geq 0 (Lu \leq 0)$ *in* Ω.

(A) *If* $a(x) \equiv 0$, *then* $\max_{\bar{\Omega}} u = \max_{\partial\Omega} u (\min_{\bar{\Omega}} u = \min_{\partial\Omega} u)$.

(B) *If* $a(x) \leq 0$, *then* $\max_{\bar{\Omega}} u \leq \max_{\partial\Omega} u^+ (\min_{\bar{\Omega}} u \geq -\max_{\partial\Omega} u^-)$, *where* $u^+ = \max(u, 0), u^- = -\min(u, 0)$.

(C) *If* $a(x) \equiv 0$ *and* u *attains its maximum (minimum) at an interior point of* Ω, *then* u *is identically a constant in* Ω.

(D) *If* $a(x) \leq 0$ *and* u *attains a nonnegative maximum (nonpositive minimum) at an interior point of* Ω, *then* u *is identically a constant in* Ω.

2. *Schauder's estimate*

If $u \in C^{2,\alpha}(\bar{\Omega})$ *and* $u|_{\partial\Omega} = \phi \in C^{2,\alpha}(\partial\Omega)$, *then*

$$|u|_{2,\alpha} \leq c(|Lu|_\alpha + |u|_0 + |\phi|_{2,\alpha}^{\partial\Omega}),$$

where the constant $c > 0$ *is independent of* u.

Lemma 2.4 (Implicit Function Theorem)([?]) *Let* X, Y, Z *be Banach spaces. For a given* $(u_0, v_0) \in X \times Y$ *and* $a, b > 0$, *let* $S = \{(u, v) : \| u - u_0 \| \leq a, \| v - v_0 \| \leq b\}$. *Suppose* $F : S \to Z$ *satisfies the following:*

(A) F *is continuous.*

(B) $F_v(\cdot, \cdot)$ *exists and is continuous in* S *(in the operator norm).*

(C) $F(u_0, v_0) = 0$.

(D) $[F_v(u_0, v_0)]^{-1}$ *exists and is a continuous map from* Z *to* Y.

Then there are neighborhoods U *of* u_0 *and* V *of* v_0 *such that the equation* $F(u, v) = 0$ *has exactly one solution* $v \in V$ *for every* $u \in U$ *and the solution* v *depends continuously on* u.

We also need some information on the solutions of the following logistic equations.

Lemma 2.5 ([?])

$$\begin{cases} \Delta u + uf(u) = 0 \ \text{ in } \ \Omega, \\ u|_{\partial\Omega} = 0, u > 0, \end{cases}$$

where Ω *is a bounded domain in* R^n *and*

(A) f *is a strictly decreasing* C^1 *function,*

(B) *there exists* $c_0 > 0$ *such that* $f(u) \leq 0$ *for* $u \geq c_0$.

(1) *If* $f(0) > \lambda_1$, *then the above equation has a unique positive solution.*

(2) *If* $f(0) \leq \lambda_1$, *then* $u \equiv 0$ *is the only nonnegative solution of the above equation.*

In the case (1), we denote this unique positive solution as θ_f. The main property about this positive solution is that θ_f is larger as f is larger, i.e. $\theta_g \leq \theta_f$ if $g \leq f$.

3. Existence, Nonexistence and Uniqueness

We consider the Lotka - Volterra model with general and combined self-limitation and competition rates

$$\begin{cases} \Delta u^i + u^i(a_i - g_i(u^1, ..., u^N)) = 0 \ \text{ in } \ \Omega, \\ u^i = 0 \ \text{ on } \ \partial\Omega, \end{cases} \tag{2}$$

where a_i's are positive constants, Ω is a bounded, smooth domain in R^n and
(U1) $g_i \in C^1$ is a strictly increasing function with respect to u_i with $i = 1, ..., N$,
(U2) there exist $k_1, ..., k_N > 0$ such that $g_i(0, ..., 0, x, 0, ..., 0) > a_i$ for $x \geq k_i$.

Theorem 3.1 (Existence and Nonexistence)

(A) *If $a_i > \lambda_1 + g_i(k_1, ..., k_{i-1}, 0, k_{i+1}, ..., k_N)$ for $i = 1, ..., N$, then the equation (??) has a positive solution $(u^1, ..., u^N)$ with*

$$\theta_{a_i - g_i(k_1,...,k_{i-1},\cdot,k_{i+1},...,k_N)} < u^i < \theta_{a_i - g_i(0,...,0,\cdot,0,...,0)}$$

in Ω for $i = 1, ..., N$.
Conversely, any solution $(u^1, ..., u^N)$ of (??) with $u^i > 0$ for all $i = 1, ..., N$ in Ω must satisfy these inequalities.

(B) *If $a_i \leq \lambda_1$ for some $i = 1, ..., N$, then the equation (??) has no positive solution.*

Proof (A) Let $\bar{u}^i = \theta_{a_i - g_i(0,...,0,\cdot,0,...,0)}$ for $i = 1, ..., N$. Then since g_i is increasing, we have

$$\begin{aligned}\Delta \bar{u}^i + \bar{u}^i(a_i - g_i(\bar{u}^1, ..., \bar{u}^N)) &= \Delta \bar{u}^i + \bar{u}^i(a_i - g_i(0, ..., 0, \bar{u}^i, 0, ..., 0) \\ &\quad + g_i(0, ..., 0, \bar{u}^i, 0, ..., 0) - g_i(\bar{u}^1, ..., \bar{u}^N)) \\ &= \bar{u}^i(g_i(0, ..., 0, \bar{u}^i, 0, ..., 0) - g_i(\bar{u}^1, ..., \bar{u}^N)) < 0.\end{aligned}$$

So, $\bar{u}^i$ is an upper solution of (??).

Let $\underline{u}^i = \theta_{a_i - g_i(k_1,...,k_{i-1},\cdot,k_{i+1},...,k_N)}$ for $i = 1, ..., N$. Then by the Maximum Principle, we obtain

$$\underline{u}^i \leq \theta_{a_i - g_i(0,...,0,\cdot,0,...,0)} \leq k_i \text{ for } i = 1, ..., N.$$

Since g_i is increasing, we get

$$\begin{aligned}\Delta \underline{u}^i + \underline{u}^i(a_i - g_i(\underline{u}^1, ..., \underline{u}^N)) &= \Delta \underline{u}^i + \underline{u}^i(a_i - g_i(k_1, ..., k_{i-1}, \underline{u}^i, k_{i+1}, ..., k_N) \\ &\quad + g_i(k_1, ..., k_{i-1}, \underline{u}^i, k_{i+1}, ..., k_N) - g_i(\underline{u}^1, ..., \underline{u}^N)) \\ &= \underline{u}^i(g_i(k_1, ..., k_{i-1}, \underline{u}^i, k_{i+1}, ..., k_N) - g_i(\underline{u}^1, ..., \underline{u}^N)) \geq 0.\end{aligned}$$

Therefore, $\underline{u}^i$ is a lower solution of (??).

Furthermore, $\underline{u}^i < \bar{u}^i$ in Ω and $\underline{u}^i = \bar{u}^i = 0$ on $\partial\Omega$ for $i = 1, ..., N$. So, by Lemma ??, (??) has a solution $(u^1, ..., u^N)$ with

$$\theta_{a_i - g_i(k_1,...,k_{i-1},\cdot,k_{i+1},...,k_N)} < u^i < \theta_{a_i - g_i(0,...,0,\cdot,0,...,0)}$$

for $i = 1, ..., N$.

Suppose $(u^1, ..., u^N)$ is a coexistence state for (??). Then since

$$\begin{aligned}\Delta u^i + u^i(a_i &- g_i(0, ..., 0, u^i, 0, ..., 0)) \\ &\geq \Delta u^i + u^i(a_i - g_i(u^1, ..., u^N)) = 0,\end{aligned}$$

u^i is a lower solution of

$$\begin{aligned}&\Delta Z + Z(a_i - g_i(0, ..., 0, Z, 0, ..., 0)) = 0 \text{ in } \Omega\\ &Z = 0 \text{ on } \partial\Omega.\end{aligned} \tag{3}$$

But, since any constant larger than k_i is an upper solution of (**??**), by Lemma **??** and the uniqueness of positive solution of (**??**), we have

$$u^i < \theta_{a_i - g_i(0,...,0,\cdot,0,...,0)} \text{ for } i = 1, ..., N. \tag{4}$$

Since g_i is increasing and $u^i < \theta_{a_i - g_i(0,...,0,\cdot,0,...,0)} \leq k_i$ for $i = 1, ..., N$,

$$\begin{aligned}\Delta u^i + u^i(a_i \; - \; &g_i(k_1, ..., k_{i-1}, u^i, k_{i+1}, ..., k_N))\\ &\leq \Delta u^i + u^i(a_i - g_i(u_1, ..., u_N)) = 0.\end{aligned}$$

Therefore, u^i is an upper solution of

$$\begin{aligned}&\Delta Z + Z(a_i - g_i(k_1, ..., k_{i-1}, Z, k_{i+1}, ..., k_N)) = 0 \text{ in } \Omega\\ &Z = 0 \text{ on } \partial\Omega.\end{aligned} \tag{5}$$

If $\epsilon > 0$ is so small that $a_i - g_i(k_1, ..., k_{i-1}, \epsilon\phi_1, k_{i+1}, ..., k_N) - \lambda_1 > 0$ on $\bar{\Omega}$, where ϕ_1 is the first eigenvector of $-\Delta$ with homogeneous boundary condition, then since

$$\begin{aligned}\Delta\epsilon\phi_1 + \epsilon\phi_1 \; &(a_i - g_i(k_1, ..., k_{i-1}, \epsilon\phi_1, k_{i+1}, ..., k_N))\\ &= \epsilon[\Delta\phi_1 + \phi_1(a_i - g_i(k_1, ..., k_{i-1}, \epsilon\phi_1, k_{i+1}, ..., k_N))]\\ &> \epsilon(\Delta\phi_1 + \phi_1\lambda_1) = 0,\end{aligned}$$

$\epsilon\phi_1$ is a lower solution of (**??**). So, by Lemma **??** and the uniqueness of positive solution, we have

$$\theta_{a_i - g_i(k_1,...,k_{i-1},\cdot,k_{i+1},...,k_N)} < u^i \text{ for } i = 1, ..., N. \tag{6}$$

By (**??**) and (**??**), we find

$$\theta_{a_i - g_i(k_1,...,k_{i-1},\cdot,k_{i+1},...,k_N)} < u^i < \theta_{a_i - g_i(0,...,0,\cdot,0,...,0)}. \tag{7}$$

(B) Without loss of generality, assume $a_1 \leq \lambda_1$. Suppose $(u^1, ..., u^N)$ is a nonnegative solution to (**??**). Then since g_1 is an increasing function with respect to u^i with $i = 1, ..., N$,

$$\begin{aligned}\Delta u^1 + u^1 \; &(a_1 - g_1(u^1, 0, 0, ..., 0))\\ &= \Delta u^1 + u^1(a_1 - g_1(u^1, ..., u^N) + g_1(u^1, ..., u^N) - g_1(u^1, 0, 0, ..., 0))\\ &= u^1(g_1(u^1, ..., u^N) - g_1(u^1, 0, 0, ..., 0)) \geq 0.\end{aligned}$$

Therefore, u^1 is a lower solution to

$$\begin{aligned}&\Delta u + u(a_1 - g_1(u, 0, 0, ..., 0)) = 0 \text{ in } \Omega,\\ &u|_{\partial\Omega} = 0.\end{aligned}$$

Any constant larger than k_1 is an upper solution to

$$\Delta u + u(a_1 - g_1(u,0,0,...,0)) = 0 \text{ in } \Omega,$$
$$u|_{\partial\Omega} = 0.$$

Hence, by Lemma **??**, there is a solution u of

$$\Delta u + u(a_1 - g_1(u,0,0,...,0)) = 0 \text{ in } \Omega,$$
$$u|_{\partial\Omega} = 0,$$

such that $0 \leq u^1 \leq u$. But, since $a_1 \leq \lambda_1$, $u \equiv 0$ by (2) of Lemma **??**, and so $u^1 \equiv 0$.

Theorem 3.2 (Uniqueness) *If* $a_i > \lambda_1 + g_i(k_1,...,k_{i-1},0,k_{i+1},...,k_N)$ *and* $2\inf\left(\frac{\partial g_i}{\partial x_i}\right) > \sum_{j=1,j\neq i}^{N}\left(\sup\left(\frac{\partial g_i}{\partial x_j}\right) + K\sup\left(\frac{\partial g_j}{\partial x_i}\right)\right)$ *for* $i = 1,...,N$, *where* $K = \sup_{i,j\neq i}\frac{\theta_{a_j-g_j(0,...,0,\cdot,0,...,0)}}{\theta_{a_i-g_i(k_1,...,k_{i-1},\cdot,k_{i+1},...,k_N)}}$, *then (**??**) has a unique coexistence state.*

Proof Suppose $(u_1,...,u_N)$ and $(v_1,...,v_N)$ are coexistence state of (**??**) and let $w_i = u_i - v_i$ for $i = 1,...,N$. Then we have

$$\begin{aligned}\Delta w_i + w_i(a_i - g_i(u_1,...,u_N)) &= \Delta u_i - \Delta v_i + (a_i - g_i(u_1,...,u_N))(u_i - v_i)\\ &= -\Delta v_i - (a_i - g_i(u_1,...,u_N))v_i\\ &= -\Delta v_i - (a_i - g_i(v_1,...,v_N) + g_i(v_1,...,v_N)\\ &\qquad -g_i(u_1,...,u_N))v_i\\ &= -v_i(g_i(v_1,...,v_N) - g_i(u_1,...,u_N)).\end{aligned}$$

Therefore,

$$\begin{aligned}\Delta w_i &+ w_i(a_i - g_i(u_1,...,u_N))\\ &- v_i(g_i(u_1,...,u_N) - g_i(v_1,...,v_N)) = 0 \text{ in } \Omega.\end{aligned} \tag{8}$$

Since $\lambda_1(a_i - g_i(u_1,...,u_N)) = 0$, by the Variational Characterization of the first eigenvalue, if $Z \in C^2(\bar{\Omega})$ and $Z|_{\partial\Omega} = 0$, then

$$\int_\Omega Z(-\Delta Z - (a_i - g_i(u_1,...,u_N))Z)dx \geq 0. \tag{9}$$

From (**??**), we have

$$\begin{aligned}-w_i\Delta w_i &- (a_i - g_i(u_1,...,u_N))(w_i)^2\\ &+ v_i w_i(g_i(u_1,...,u_N) - g_i(v_1,...,v_N)) = 0.\end{aligned}$$

By using (**??**), for $i = 1,...,N$,

$$\int_\Omega v_i w_i(g_i(u_1,...,u_N) - g_i(v_1,...,v_N))dx \leq 0.$$

Hence, we obtain

$$\int_\Omega \sum_{i=1}^{N} [v_i w_i (g_i(u_1, ..., u_N) - g_i(v_1, ..., v_N))] dx \leq 0.$$

By the Mean Value Theorem, for each $x \in \Omega$, there exist t^i and z^{ij} such that

$$\begin{aligned} g_i(u_1, .., u_{i-1}, u_i,\ u_{i+1}, ..., u_N) - g_i(u_1, ..., v_i, u_{i+1}, ..., u_N) \\ = \frac{\partial g_i(u_1, ..., u_{i-1}, t^i, u_{i+1}, ..., u_N)}{\partial x_i}(u_i - v_i) \\ = \frac{\partial g_i(u_1, ..., u_{i-1}, t^i, u_{i+1}, ..., u_N)}{\partial x_i} w_i \end{aligned}$$

and

$$\begin{aligned} g_i(v_1, .., v_{j-1}, u_j,\ u_{j+1}, .., u_N) - g_i(v_1, ..., v_{j-1}, v_j, u_{j+1}, ..., u_N) \\ = \frac{\partial g_i(v_1, ..., v_{j-1}, z^{ij}, u_{j+1}, ..., u_N)}{\partial x_j}(u_j - v_j) \\ = \frac{\partial g_i(v_1, ..., v_{v-1}, z^{ij}, u_{j+1}, ..., u_N)}{\partial x_j} w_j \end{aligned}$$

for $i, j = 1, ..., N, j \neq i$. Therefore,

$$\begin{aligned} \int_\Omega \sum_{i=1}^{N} \quad & [\frac{\partial g_i(u_1, ..., u_{i-1}, t^i, u_{i+1}, ..., u_N)}{\partial x_i} v_i (w_i)^2 \\ & + \textstyle\sum_{j=1, j\neq i}^{N} v_i w_i \frac{\partial}{\partial x_j} g_i(v_1, ..., v_{j-1}, z^{ij}, u_{j+1}, ..., u_N) w_j] dx \leq 0. \end{aligned} \tag{10}$$

If the integrand in the left side of (**??**) is positive definite, then (**??**) implies that $w_i \equiv 0$ in Ω for $i = 1, ..., N$, which means the uniqueness of the coexistence state for (**??**). But, for any $\epsilon > 0$,

$$\begin{aligned} \frac{\partial}{\partial x_j} g_i \ & (v_1, ..., v_{j-1}, z^{ij}, u_{j+1}, ..., u_N) v_i w_i w_j \\ & \leq \frac{\partial}{\partial x_j} g_i(v_1, ..., v_{j-1}, z^{ij}, u_{j+1}, ..., u_N) v_i [\frac{(w_i)^2}{2\epsilon} + \frac{\epsilon (w_j)^2}{2}]. \end{aligned}$$

So, we can see that the integrand is positive definite if for $i = 1, ..., N$ and $x \in \Omega$,

$$\begin{aligned} \frac{\partial g_i(u_1, ..., u_{i-1}, t^i, u_{i+1}, ..., u_N)}{\partial x_i} v_i > \textstyle\sum_{j=1, j\neq i}^{N} (& \frac{\frac{\partial}{\partial x_j} g_i(v_1, ..., v_{j-1}, z^{ij}, u_{j+1}, ..., u_N) v_i}{2\epsilon} \\ & + \frac{\epsilon \frac{\partial}{\partial x_i} g_j(v_1, ..., v_{i-1}, z^{ji}, u_{i+1}, ..., u_N) v_j}{2}) \end{aligned}$$

or equivalently,

$$\frac{\partial g_i(u_1,...,u_{i-1},t^i,u_{i+1},...,u_N)}{\partial x_i} > \sum_{j=1,j\neq i}^N \Big(\frac{\frac{\partial}{\partial x_j} g_i(v_1,...,v_{j-1},z^{ij},u_{j+1},...,u_N)}{2\epsilon} + \frac{\epsilon \frac{\partial}{\partial x_i} g_j(v_1,...,v_{i-1},z^{ji},u_{i+1},...,u_N)\frac{v_j}{v_i}}{2}\Big). \tag{11}$$

Since $\theta_{a_i-g_i(k_1,...,k_{i-1},\cdot,k_{i+1},...,k_N)} < v_i < \theta_{a_i-g_i(0,...,0,\cdot,0,...,0)}$ in Ω for all $i=1,...,N$, (**??**) will hold if for $i=1,...,N$,

$$\frac{\partial g_i(u_1,...,u_{i-1},t^i,u_{i+1},...,u_N)}{\partial x_i} > \sum_{j=1,j\neq i}^N \Big(\frac{\sup(\frac{\partial}{\partial x_j}g_i)}{2\epsilon} + \frac{\epsilon\sup(\frac{\partial}{\partial x_i}g_j)}{2}\frac{\theta_{a_j-g_j(0,...,0,\cdot,0,...,0)}}{\theta_{a_i-g_i(k_1,...,k_{i-1},\cdot,k_{i+1},...,k_N)}}\Big).$$

Let $K=\sup_{i,j\neq i}\frac{\theta_{a_j-g_j(0,...,0,\cdot,0,...,0)}}{\theta_{a_i-g_i(k_1,...,k_{i-1},\cdot,k_{i+1},...,k_N)}}$. Then (**??**) holds if

$$\inf(\frac{\partial g_i}{\partial x_i}) > \sum_{j=1,j\neq i}^N \Big(\frac{\sup(\frac{\partial}{\partial x_j}g_i)}{2\epsilon} + \frac{K\epsilon\sup(\frac{\partial}{\partial x_i}g_j)}{2}\Big).$$

Choosing $\epsilon=1$, we have

$$2\inf(\frac{\partial g_i}{\partial x_i}) > \sum_{j=1,j\neq i}^N \Big(\sup(\frac{\partial g_i}{\partial x_j}) + K\sup(\frac{\partial g_j}{\partial x_i})\Big).$$

Biologically, we can interpret the conditions in Theorem **??** and Theorem **??** as follows. The constants a_i's and the functions g_i's describe how species $1(u^1), 2(u^2), ...$, $N(u^N)$ interact among themselves and with each other. Hence, the conditions imply that each species interacts strongly among themselves and weakly with others.

4. Uniqueness in a Neighborhood of Reproduction Rates

We consider the Lotka - Volterra model with general combined self-limitation and competition rates

$$\begin{cases} \Delta u^i + u^i(a_i - g_i(u^1,...,u^N)) = 0 \ \text{ in } \ \Omega, \\ u^i = 0 \ \text{ on } \ \partial\Omega, \end{cases} \tag{12}$$

where Ω is a bounded domain in R^n and

(N1) $g_i \in C^1$ is a strictly increasing function with respect to u_i with $i=1,...,N$, (N2) there exist $k_1, ..., k_N > 0$ such that $g_i(0, ..., 0, x, 0, ..., 0) > a_i$ for $x \geq k_i$.

In this section, we try to get some condition to guarantee the unique coexistence state in a neighborhood of the reproduction rates a_i,s of (**??**).

We know that the Implicit Function Theorem only guarantees the uniqueness of solution locally when Frechet derivative is invertible. The importance of next theorem is the global uniqueness. The techniques used here includes the Implicit Function Theorem and a priori estimates on the solutions of (**??**).

Theorem 4.1 *Suppose*

(A) $a_i > \lambda_1\Big(g_i\Big(\theta_{a_1-g_1(\cdot,0,...,0)}, ..., \theta_{a_{i-1}-g_{i-1}(0,...,0,\cdot,0,...,0)}, 0, \theta_{a_{i+1}-g_{i+1}(0,..,0\cdot,0,...,0)}, \cdots, \theta_{a_N-g_N(0,...,0,\cdot)}\Big)\Big)$ *for* $i = 1, ..., N$,

(B) *(**??**) has a unique coexistence state* $(u_1, ..., u_N)$,

(C) *the Frechet derivative of (**??**) at* $(u_1, ..., u_N)$ *is invertible as a map from* $C^{2,\alpha}$ *to* C^{α}.

Then there is a neighborhood V *of* $(a_1, ..., a_N)$ *in* R^N *such that if* $(a_{10}, ..., a_{N0}) \in V$, *then (**??**) with reproduction rates* $(a_{10}, ..., a_{N0})$ *has a unique coexistence state.*

Here, the condition (A) implies that the rates of self-reproduction are large enough. The condition of invertibility of Frechet derivative also illustrates the similar argument which will be in Theorem **??**.

Proof Since the Frechet derivative of (**??**) at $(u_1, ..., u_N)$ is invertible, by Implicit Function Theorem, there is a neighborhood V of $(a_1, ..., a_N)$ in R^N and a neighborhood W of $(u_1, ..., u_N)$ in $\left[C_0^{2+\alpha}(\bar{\Omega})\right]^N$ such that for all $(a_{10}, ..., a_{N0}) \in V$, there is a unique positive solution $(u_{10}, ..., u_{N0}) \in W$ of (**??**). Suppose the conclusion of the theorem is false. Then there are sequences $(a_{1n}, ..., a_{Nn}, u_{1n}, ..., u_{Nn}), (a_{1n}, ..., a_{Nn}, u^*_{1n}, ..., u^*_{Nn})$ in $V \times \left[C_0^{2+\alpha}(\bar{\Omega})\right]^N$ such that $(u_{1n}, ..., u_{Nn})$ and $(u^*_{1n}, ..., u^*_{Nn})$ are the positive solutions with reproduction rates $(a_{1n}, ..., a_{Nn})$ and $(u_{1n}, ..., u_{Nn}) \neq (u^*_{1n}, ..., u^*_{Nn})$ and $(a_{1n}, ..., a_{Nn}) \to (a_1, ..., a_N)$. By the standard elliptic theory and a priori estimate from the previous section on $(u_{1n}, ..., u_{Nn})$ and $(u^*_{1n}, ..., u^*_{Nn})$, $(u_{1n}, ..., u_{Nn}) \to (\bar{u_1}, ..., \bar{u_N})$, $(u^*_{1n}, ..., u^*_{Nn}) \to (u^*_1, ..., u^*_N)$. Furthermore, $(\bar{u_1}, ..., \bar{u_N})$ and $(u^*_1, ..., u^*_N)$ are the solutions of (**??**) with reproduction rates $(a_1, ..., a_N)$. Claim $\bar{u_1} > 0, ..., \bar{u_N} > 0, u^*_1 > 0, ..., u^*_N > 0$. It is enough to show that $\bar{u_1}, ..., \bar{u_N}$ are not identically zero because of the Maximum Principle. Suppose not. With no loss of generality, suppose $\bar{u_1}$ is identically zero. Let $\widetilde{u_{1n}} = \dfrac{u_{1n}}{\| u_{1n} \|_\infty}$ for all $n \in N$. Then

$$\Delta\widetilde{u_{1n}} + \widetilde{u_{1n}}(a_{1n} - g_1(u_{1n}, u_{2n}, u_{3n}, ..., u_{Nn})) = 0,$$

$$\Delta u_{2n} + u_{2n}(a_{2n} - g_2(u_{1n}, u_{2n}, u_{3n}, u_{4n}, ..., u_{Nn})) = 0,$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$\Delta u_{Nn} + u_{Nn}(a_{Nn} - g_N(u_{1n}, ..., u_{Nn})) = 0.$$

By using elliptic theory again, $\widetilde{u_{1n}} \to \widetilde{u_1}$ in $C^{2,\alpha}$ and

$$\begin{aligned}
&\Delta\widetilde{u_1} + \widetilde{u_1}(a_1 - g_1(0, \bar{u_2}, ..., \bar{u_N})) = 0,\\
&\Delta\bar{u_2} + \bar{u_2}(a_2 - g_2(\bar{u_1}, ..., \bar{u_N})) = 0,\\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\
&\Delta\bar{u_N} + \bar{u_N}(a_N - g_N(\bar{u_1}, ..., \bar{u_N})) = 0.
\end{aligned}$$

Hence, $a_1 = \lambda_1(g_1(0, \bar{u_2}, ..., \bar{u_N}))$.

Let $j = 2, ..., N$. If $\bar{u_j}$ is identically zero, then $\bar{u_j} \equiv 0 \le \theta_{a_j - g_j(0,...,0,\cdot,0,...,0)}$. Suppose $\bar{u_j}$ is not identically zero. Then since

$$\begin{aligned}
&\Delta\bar{u_j} + \bar{u_j}(a_j - g_j(0, ..., 0, \bar{u}_j, 0, ..., 0))\\
&\qquad = \Delta\bar{u}_j + \bar{u}_j(a_j - g_j(\bar{u}_1, ..., \bar{u}_N)\\
&\qquad\quad + g_j(\bar{u}_1, ..., \bar{u}_N) - g_j(0, ..., 0, \bar{u}_j, 0, ..., 0))\\
&\qquad = \bar{u_j}(g_j(\bar{u_1}, ..., \bar{u_N}) - g_j(0, ..., 0, \bar{u}_j, 0, ..., 0)) \ge 0,
\end{aligned}$$

$\bar{u_j}$ is a lower solution of

$$\begin{aligned}
&\Delta Z + Z(a_j - g_j(0, ..., 0, Z, 0, ..., 0)) = 0 \ \text{ in } \ \Omega,\\
&Z = 0 \ \text{ on } \ \partial\Omega.
\end{aligned}$$

Since any constant which is larger than k_j is an upper solution of

$$\begin{aligned}
&\Delta Z + Z(a_j - g_j(0, ..., 0, Z, 0, ..., 0)) = 0 \ \text{ in } \ \Omega,\\
&Z = 0 \ \text{ on } \ \partial\Omega,
\end{aligned}$$

by the uniqueness of positive solution, $\bar{u_j} \le \theta_{a_j - g_j(0,...,0,\cdot,0,...,0)}$. Consequently,

$$\begin{aligned}
a_1 &= \lambda_1(g_1(0, \bar{u_2}, ..., \bar{u_N}))\\
&\le \lambda_1(g_1(0, \theta_{a_2 - g_2(0,\cdot,0,...,0)}, ..., \theta_{a_N - g_N(0,...,0,\cdot)})),
\end{aligned}$$

by the monotonicity of g_1 and the first eigenvalue, which contradicts our assumption. Consequently, $(\bar{u_1}, ..., \bar{u_N})$ and $(u_1^*, ..., u_N^*)$ are coexistence states with reproduction rates $(a_1, ..., a_N)$. But, since the coexistence state in this case is unique by assumption, $(\bar{u_1}, ..., \bar{u_N}) = (u_1^*, ..., u_N^*) = (u_1, ..., u_N)$, which contradicts the Implicit Function Theorem.

The proof of the theorem also tells us that if one of the species becomes extinct, in other word, if one is excluded by others, then that means the reproduction rates are small, i.e. the region condition of reproduction rates (A) is crucial.

Corollary 4.2 *If* $(a_{1n}, ..., a_{Nn}, u_{1n}, ..., u_{Nn}) \to (a_1, ..., a_N, u_1, ..., u_N)$ *and* $u_j \equiv 0$ *for some* $j = 1, ..., N$, *then* $a_j \leq \lambda_1(g_1(\theta_{a_1-g_1(\cdot,0,...,0)}, \theta_{a_2-g_2(0,\cdot,0,...,0)}, ..., \theta_{a_{j-1}-g_{j-1}(0,...,0,\cdot,0,...,0)}, 0, \theta_{a_{j+1}-g_{j+1}(0,...,0\cdot,0,...,0)}, ..., \theta_{a_N-g_N(0,...,0,\cdot)}))$.

From the argument above, it is important to get some condition to guarantee the invertibility of the Frechet derivative of (**??**).

Lemma 4.3 *Suppose*
(A) $(u_1, ..., u_N)$ *is a positive solution to* (**??**),
(B) $2\inf(\frac{\partial g_i(u_1,...,u_N)}{\partial u_i})u_i > \sum_{j=1,j\neq i}^{N}(\sup(\frac{\partial g_i(u_1,...,u_N)}{\partial u_j})u_i + \sup(\frac{\partial g_j(u_1,...,u_N)}{\partial u_i})u_j)$
for $i = 1, ..., N$.
Then the Frechet derivative of (**??**) *at* $(u_1, ..., u_N)$ *is invertible.*

The condition (B) means the rates of self-limitation are relatively larger than those of competition.

Proof The Frechet derivative M of (**??**) at $(u_1, ..., u_N)$ is

$$\begin{pmatrix} -\Delta + g_1(u_1,...,u_N) + u_1\frac{\partial g_1(u_1,...,u_N)}{\partial u_1} - a_1, u_1\frac{\partial g_1(u_1,...,u_N)}{\partial u_2}, ..., u_1\frac{\partial g_1(u_1,...,u_N)}{\partial u_N} \\ u_2\frac{\partial g_2(u_1,...,u_N)}{\partial u_1}, -\Delta + g_2(u_1,...,u_N) + u_2\frac{\partial g_2(u_1,...,u_N)}{\partial u_2} - a_2, u_2\frac{\partial g_2(u_1,...,u_N)}{\partial u_3}, ..., u_2\frac{\partial g_2(u_1,...,u_N)}{\partial u_N} \\ . \\ . \\ . \\ u_N\frac{\partial g_N(u_1,...,u_N)}{\partial u_1}, ..., u_N\frac{\partial g_N(u_1,...,u_N)}{\partial u_{N-1}}, -\Delta + g_N(u_1,...,u_N) + u_N\frac{\partial g_N(u_1,...,u_N)}{\partial x_N} - a_N \end{pmatrix}.$$

By Fredholm Alternative, we need to show that any solution $(\phi_1, ..., \phi_N)$ of $Mx = 0$ is trivial. In fact, from the equations,

$$\begin{aligned} &\int_\Omega |\nabla\phi_1|^2 + \Big(g_1(u_1,...,u_N) + u_1\frac{\partial g_1(u_1,...,u_N)}{\partial u_1} - a_1\Big)\phi_1^2 \\ &\qquad + \Big(\frac{\partial g_1(u_1,...,u_N)}{\partial u_2}\phi_2 + ... + \frac{\partial g_1(u_1,...,u_N)}{\partial u_N}\phi_N\Big)u_1\phi_1 dx = 0, \\ &\int_\Omega |\nabla\phi_2|^2 + \Big(g_2(u_1,...,u_N) + u_2\frac{\partial g_2(u_1,...,u_N)}{\partial u_2} - a_2\Big)\phi_2^2 + \Big(\frac{\partial g_2(u_1,...,u_N)}{\partial u_1}\phi_1 \\ &\qquad + \frac{\partial g_2(u_1,...,u_N)}{\partial u_3}\phi_3 + ... + \frac{\partial g_2(u_1,...,u_N)}{\partial u_N}\phi_N\Big)u_2\phi_2 dx = 0, \\ &\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\ &\int_\Omega |\nabla\phi_N|^2 + \Big(g_N(u_1,...,u_N) + u_N\frac{\partial g_N(u_1,...,u_N)}{\partial u_N} - a_N\Big)\phi_N^2 + \Big(\frac{\partial g_N(u_1,...,u_N)}{\partial u_1}\phi_1 \\ &\qquad + ... + \frac{\partial g_N(u_1,...,u_N)}{\partial u_{N-1}}\phi_{N-1}\Big)u_N\phi_N dx = 0, \end{aligned}$$

since $\lambda_1(g_i(u_1,...,u_N)-a_i)=0$ for $i=1,...,N$, we see that

$$\int_\Omega |\nabla\phi_i|^2+(g_i(u_1,...,u_N)-a_i)\phi_i^2\geq 0$$

for $i=1,...,N$. Hence,

$$\int_\Omega u_1\frac{\partial g_1(u_1,...,u_N)}{\partial u_1}\phi_1^2+\Big(\frac{\partial g_1(u_1,...,u_N)}{\partial u_2}\phi_2+...+\frac{\partial g_1(u_1,...,u_N)}{\partial u_N}\phi_N\Big)u_1\phi_1dx\leq 0,$$

$$\int_\Omega u_2\frac{\partial g_2(u_1,...,u_N)}{\partial u_2}\phi_2^2+\Big(\frac{\partial g_2(u_1,...,u_N)}{\partial u_1}\phi_1+\frac{\partial g_2(u_1,...,u_N)}{\partial u_3}\phi_3$$
$$+...+\frac{\partial g_2(u_1,...,u_N)}{\partial u_N}\phi_N\Big)u_2\phi_2\leq 0,$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$\int_\Omega u_N\frac{\partial g_N(u_1,...,u_N)}{\partial u_N}\phi_N^2+\Big(\frac{\partial g_N(u_1,...,u_N)}{\partial u_1}\phi_1$$
$$+...+\frac{\partial g_N(u_1,...,u_N)}{\partial u_{N-1}}\phi_{N-1}\Big)u_N\phi_N\leq 0.$$

Therefore,

$$\int_\Omega\sum_{i=1}^N u_i\frac{\partial g_i(u_1,...,u_N)}{\partial u_i}\phi_i^2+\sum_{i=1}^N u_i\phi_i\sum_{j=1,j\neq i}^N\frac{\partial g_i(u_1,...,u_N)}{\partial u_j}\phi_j\leq 0.$$

It implies that

$$\int_\Omega\sum_{i=1}^N\Big(u_i\frac{\partial g_i(u_1,...,u_N)}{\partial u_i}\phi_i^2+\sum_{j=1,j\neq i}^N\frac{\partial g_i(u_1,...,u_N)}{\partial u_j}u_i\phi_j\phi_i\Big)\leq 0.$$

But,

$$\frac{\partial g_i(u_1,...,u_N)}{\partial u_j}u_i\phi_i\phi_j\leq\frac{\partial g_i(u_1,...,u_N)}{\partial u_j}u_i(\frac{\phi_i^2}{2}+\frac{\phi_j^2}{2}).$$

If

$$\frac{\partial g_i(u_1,...,u_N)}{\partial u_i}u_i>\sum_{j=1,j\neq i}^N\left(\frac{\frac{\partial g_i(u_1,...,u_N)}{\partial u_j}u_i}{2}+\frac{\frac{\partial g_j(u_1,...,u_N)}{\partial u_i}u_j}{2}\right)\text{ for } i=1,...,N,$$

then the integrand in above inequality is positive definite, which means $(\phi_1,...,\phi_N)$ is trivial. But, it holds if

$$2\inf\left(\frac{\partial g_i(u_1,...,u_N)}{\partial u_i}\right)u_i>\sum_{j=1,j\neq i}^N\left(\sup(\frac{\partial g_i(u_1,...,u_N)}{\partial u_j})u_i+\sup(\frac{\partial g_j(u_1,...,u_N)}{\partial u_i})u_j\right)$$

for $i=1,...,N$.

Corollary 4.4 *Suppose*
(A) $a_i > \lambda_1 + g_i(k_1, ..., k_{i-1}, 0, k_{i+1}, ..., k_N)$ *for* $i = 1, ..., N$,
(B) $2\inf\Big(\frac{\partial g_i}{\partial x_i}\Big) > \sum_{j=1, j\neq i}^{N}\Big(\sup(\frac{\partial g_i}{\partial x_j}) + K\sup(\frac{\partial g_j}{\partial x_i})\Big)$ *for* $i = 1, ..., N$, *where* $K = \sup_{i,j\neq i}\frac{\theta_{a_j - g_j(0,...,0,\cdot,0,...,0)}}{\theta_{a_i - g_i(k_1,...,k_{i-1},\cdot,k_{i+1},...,k_N)}}$.
Then there is a neighborhood V *of* $(a_1, ..., a_N)$ *in* R^N *such that if* $(a_{10}, ..., a_{N0}) \in V$, *then* (**??**) *with* $(a_1, ..., a_N) = (a_{10}, ..., a_{N0})$ *has a unique coexistence state.*

In Theorem **??** we derived a uniqueness result with a fixed reproduction rates $(a_1, ..., a_N)$. But, Corollary **??** implies that we can extend the region of reproduction parameter space with uniqueness to an open ball by perturbation under the same conditions.

Proof From $\theta_{a_i - g_i(0,...,0,\cdot,0,...,0)} < k_i$ and the monotonicity of $g_i(0, ..., 0, \cdot, 0, ..., 0)$ for $i = 1, ..., N$, we have

$$\begin{aligned} a_i >& \lambda_1 + g_i(k_1, ..., k_{i-1}, 0, k_{i+1}, ..., k_N) \\ \geq& \lambda_1\Big(g_i(\theta_{a_1 - g_1(\cdot,0,...,0)}, ..., \theta_{a_{i-1} - g_{i-1}(0,...,0,\cdot,0,...,0)}, 0, \theta_{a_{i+1} - g_i(0,...,\cdot,0,...,0)}, \\ & ..., \theta_{a_N - g_N(0,...,0,\cdot)})\Big) \end{aligned}$$

for $i = 1, ..., N$. The condition already guarantees that there is a unique coexistence state $(u_1, ..., u_N)$ from Theorem **??**. Furthermore, by the estimate of the solution in the proof of Theorem **??**,

$$2\inf\Big(\frac{\partial g_i}{\partial x_i}\Big) > \sum_{j=1, j\neq i}^{N}\Big(\sup(\frac{\partial g_i}{\partial x_j}) + \frac{u_j}{u_i}\sup(\frac{\partial g_j}{\partial x_i})\Big).$$

This gives that

$$2\inf\Big(\frac{\partial g_i}{\partial x_i}\Big)u_i > \sum_{j=1, j\neq i}^{N}\Big(\sup(\frac{\partial g_i}{\partial x_j})u_i + \sup(\frac{\partial g_j}{\partial x_i})u_j\Big).$$

Now, the Frechet derivative of (**??**) is invertible from Lemma **??**. Thus, the theorem follows from Theorem **??**.

5. Uniqueness in a Region of Reproduction Rates

In this section, we find a region of reproduction rates a_is that guarantees the existence of a unique positive solution to

$$\begin{aligned} &\Delta u^i + u^i(a_i - g_i(u^1, ..., u^N)) = 0 \text{ in } \Omega, \\ &u^i = 0 \text{ on } \partial\Omega, \end{aligned} \tag{13}$$

where Ω is a bounded domain in R^n and $g_i \in C^1$ is a strictly increasing function with respect to u^i with $i = 1, ..., N$.

Theorem 5.1 *Suppose*

(A) Γ *is a closed, convex region in* R^N *such that for all* $(a_1,...,a_N)\in\Gamma$, $a_i>\lambda_1(g_i(\theta_{a_1-g_1(\cdot,0,...,0)},...,\theta_{a_{i-1}-g_{i-1}(0,...,0,\cdot,0,...,0)},0,\theta_{a_{i+1}-g_{i+1}(0,...,0,\cdot,0,...,0)},...,\theta_{a_N-g_N(0,...,0,\cdot)}))$, *for* $i=1,...,N$,

(B) *there exist* $k_1,...,k_N>0$ *such that for all* $(a_1,...,a_N)\in\Gamma$, $g_i(0,...,0,x,0,...,0)>a_i$ *for* $x\geq k_i$,

(C) *(**??**) has a unique positive solution for all* $(a_1,...,a_N)\in\partial_L\Gamma$ *and for all* $(a_1,...,a_N)\in\Gamma$, *the Frechet derivative of (**??**) at every positive solution to (**??**) is invertible, where* $\partial_L\Gamma=\{(\lambda_{(a_2,...,a_N)},a_2,...,a_N)\in\Gamma|$*For any fixed* $a_2,...,a_N,\lambda_{(a_2,...,a_N)}=\inf\{a_1|(a_1,...,a_N)\in\Gamma\}\}$.

*Then (**??**) has a unique positive solution for all* $(a_1,...,a_N)\in\Gamma$. *Furthermore, there is an open set* W *in* R^N *such that* $\Gamma\subseteq W$ *and for every* $(a_1,...,a_N)\in W$, *(**??**) has a unique positive solution.*

Theorem **??** goes even further than Theorem **??** which states the uniqueness in an open set containing the whole region of a_i's whenever we have the uniqueness on the left boundary and invertibility of linearized operator at any particular solution inside the domain. We can easily see that this theorem generalizes Theorem **??**.

Proof For each fixed $(a_2,...,a_N)$, let $\lambda^{(a_2,...,a_N)}=\sup\{a_1:(a_1,...,a_N)\in\Gamma\}$. This $\lambda^{(a_2,...,a_N)}$ is defined since a_1 is bounded for fixed $(a_2,...,a_N)$. We need to show that for every a_1 such that $\lambda_{(a_2,...,a_N)}\leq a_1\leq\lambda^{(a_2,...,a_N)}$, (**??**) has a unique positive solution. Since (**??**) with reproduction rates $(\lambda_{(a_2,...,a_N)},a_2,...,a_N)$ has a unique positive solution $(u_1,...,u_N)$ and the Frechet derivative of (**??**) at $(u_1,...,u_N)$ is invertible, Theorem **??** implies that there is an open neighborhood V of $(\lambda_{(a_2,...,a_N)},a_2,...,a_N)$ in R^N such that if $(a_{10},...,a_{N0})\in V$, then (**??**) with reproduction rates $(a_{10},...,a_{N0})$ has a unique positive solution.

Let $\lambda_s=sup\{\lambda_{(a_2,...,a_N)}\leq\lambda\leq\lambda^{(a_2,...,a_N)}:$ (**??**) has a unique coexistence state for $\lambda_{(a_2,...,a_N)}\leq a_1\leq\lambda\}$. We need to show that $\lambda_s=\lambda^{(a_2,...,a_N)}$. Suppose $\lambda_s<\lambda^{(a_2,...,a_N)}$. From the definition of λ_s, there is a sequence $\{\lambda_n\}$ such that $\lambda_n\to\lambda_s^-$ and there is a sequence $(u_{1n},...,u_{Nn})$ of the unique positive solution of (**??**) with reproduction rates $(\lambda_n,a_2,...,a_N)$. Then by the Elliptic theory, there is $(u_{10},...,u_{N0})$ such that $(u_{1n},...,u_{Nn})$ converges to $(u_{10},...,u_{N0})$ uniformly and $(u_{10},...,u_{N0})$ is a solution to (**??**) with reproduction rates $(\lambda_s,a_2,...,a_N)$. We claim that u_{i0} is not identically zero for $i=1,...,N$. Suppose this is false.

(1) Suppose u_{10} is identically zero. Let $\widetilde{u_{1n}}=\dfrac{u_{1n}}{\|u_{1n}\|_\infty}$ for all $n\in N$. Then

$$\begin{aligned}&\Delta\widetilde{u_{1n}}+\widetilde{u_{1n}}(\lambda_n-g_1(u_{1n},...,u_{Nn}))=0,\\&\Delta u_{2n}+u_{2n}(a_2-g_2(u_{1n},...,u_{Nn}))=0,\\&\cdots\cdots\cdots\cdots\cdots\cdots\\&\Delta u_{Nn}+u_{Nn}(a_N-g_N(u_{1n},...,u_{Nn}))=0\end{aligned}$$

and $\widetilde{u_{1n}} \to \widetilde{u_1}$ uniformly in Ω by elliptic theory, and

$$\Delta\widetilde{u_1} + \widetilde{u_1}(\lambda_s - g_1(0, u_{20}, ..., u_{N0})) = 0,$$

$$\Delta u_{20} + u_{20}(a_2 - g_2(u_{10}, ..., u_{N0})) = 0,$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$\Delta u_{N0} + u_{N0}(a_N - g_N(u_{10}, ..., u_{N0})) = 0.$$

It implies that $\lambda_s = \lambda_1(g_1(0, u_{20}, ..., u_{N0}))$. Let $j = 2, ..., N$. If u_{j0} is identically zero, then $u_{j0} \equiv 0 \le \theta_{a_j - g_j(0,...,0,\cdot,0,...,0)}$. Suppose u_{j0} is not identically zero. Then since

$$\begin{aligned}
&\Delta u_{j0} + u_{j0}(a_j - g_j(0, ..., 0, u_{j0}, 0, ..., 0)) \\
&\quad = \Delta u_{j0} + u_{j0}(a_j - g_j(u_{10}, ..., u_{N0}) \\
&\qquad + g_j(u_{10}, ..., u_{N0}) - g_j(0, ..., 0, u_{j0}, 0, ..., 0)) \\
&\quad = u_{j0}(g_j(u_{10}, ..., u_{N0}) - g_j(0, ..., 0, u_{j0}, 0, ..., 0)) \ge 0,
\end{aligned}$$

u_{j0} is a lower solution of

$$\Delta Z + Z(a_j - g_j(0, ..., 0, Z, 0, ..., 0)) = 0 \text{ in } \Omega,$$

$$Z = 0 \text{ on } \partial\Omega.$$

By the uniqueness of positive solution, $u_{j0} \le \theta_{a_j - g_j(0,...,0,\cdot,0,...,0)}$. It gives that

$$\begin{aligned}
\lambda_s &= \lambda_1(g_1(0, u_{20}, ..., u_{N0})) \\
&\le \lambda_1(g_1(0, \theta_{a_2 - g_2(0,\cdot,0,...,0)}, ..., \theta_{a_N - g_N(0,...,0,\cdot)}))
\end{aligned}$$

by the monotonicity of g_1 and the first eigenvalue, which is a contradiction to our assumption since $(\lambda_s, a_2, ..., a_N) \in \Gamma$.

(2) Suppose u_{10} is not identically zero and at least one of $u_{j0}, j = 2, ..., N$ is identically zero. Without loss of generality, assume u_{20} is identically zero.

Let $\widetilde{u_{2n}} = \dfrac{u_{2n}}{\| u_{2n} \|_\infty}$ for all $n \in N$. Then

$$\Delta u_{1n} + u_{1n}(\lambda_n - g_1(u_{1n}, ..., u_{Nn})) = 0,$$

$$\Delta\widetilde{u_{2n}} + \widetilde{u_{2n}}(a_2 - g_2(u_{1n}, ..., u_{Nn})) = 0,$$

$$\Delta u_{3n} + u_{3n}(a_3 - g_3(u_{1n}, ..., u_{Nn})) = 0,$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$\Delta u_{Nn} + u_{Nn}(a_N - g_N(u_{1n}, ..., u_{Nn})) = 0$$

and $\widetilde{u_{2n}} \to \widetilde{u_2}$ uniformly in Ω by elliptic theory, and

$$\begin{aligned}
&\Delta u_{10} + u_{10}(\lambda_s - g_1(u_{10}, ..., u_{N0})) = 0,\\
&\Delta\widetilde{u_2} + \widetilde{u_2}(a_2 - g_2(u_{10}, 0, u_{30}, u_{40}, ..., u_{N0})) = 0,\\
&\Delta u_{30} + u_{30}(a_3 - g_3(u_{10}, ..., u_{N0})) = 0,\\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\
&\Delta u_{N0} + u_{N0}(a_N - g_N(u_{10}, ..., u_{N0})) = 0.
\end{aligned}$$

Hence, $a_2 = \lambda_1(g_2(u_{10}, 0, u_{30}, u_{40}, ..., u_{N0}))$. Since

$$\begin{aligned}
\Delta u_{10} &+ u_{10}(\lambda_s - g_1(u_{10}, 0, ..., 0))\\
&= \Delta u_{10} + u_{10}(\lambda_s - g_1(u_{10}, ..., u_{N0})\\
&\quad + g_1(u_{10}, ..., u_{N0}) - g_1(u_{10}, 0, ..., 0))\\
&= u_{10}(g_1(u_{10}, ..., u_{N0}) - g_1(u_{10}, 0, ..., 0)) \geq 0,
\end{aligned}$$

u_{10} is a lower solution of

$$\begin{aligned}
&\Delta Z + Z(\lambda_s - g_1(Z, 0, ..., 0)) = 0 \text{ in } \Omega,\\
&Z = 0 \text{ on } \partial\Omega.
\end{aligned}$$

Since u_{10} is not identically zero, by the uniqueness of the positive solution, $u_{10} \leq \theta_{\lambda_s - g_1(\cdot,0,...,0)}$. Let $j = 3, ..., N$. If u_{j0} is identically zero, then $u_{j0} \equiv 0 \leq \theta_{a_j - g_j(0,...,0,\cdot,0,...,0)}$. If u_{j0} is not identically zero, then since

$$\begin{aligned}
\Delta u_{j0} &+ u_{j0}(a_j - g_j(0, ..., 0, u_{j0}, 0, ..., 0))\\
&= \Delta u_{j0} + u_{j0}(a_j - g_j(u_{10}, ..., u_{N0})\\
&\quad + g_j(u_{10}, ..., u_{N0}) - g_j(0, ..., 0, u_{j0}, 0, ..., 0))\\
&= u_{j0}(g_j(u_{10}, ..., u_{N0}) - g_j(0, ..., 0, u_{j0}, 0, ..., 0)) \geq 0,
\end{aligned}$$

u_{j0} is a lower solution of

$$\begin{aligned}
&\Delta Z + Z(a_j - g_j(0, ..., 0, Z, 0, ..., 0)) = 0 \text{ in } \Omega,\\
&Z = 0 \text{ on } \partial\Omega.
\end{aligned}$$

By the uniqueness of positive solution, $u_{j0} \leq \theta_{a_j - g_j(0,...,0,\cdot,0,...,0)}$. Consequently, we obtain

$$a_2 = \lambda_1(g_2(u_{10}, 0, u_{30}, u_{40}, ..., u_{N0}))$$

$$\leq \lambda_1(g_2(\theta_{\lambda_s - g_1(\cdot,0,...,0)}, 0, \theta_{a_3 - g_3(0,0,\cdot,0,...,0)}, ..., \theta_{a_N - g_N(0,...,0,\cdot)}))$$

by the monotonicity of g_2 and the first eigenvalue, which is a contradiction to our assumption since $(\lambda_s, a_2, ..., a_N) \in \Gamma$. Thus, u_{i0} is not identically zero for $i = 1, ..., N$. We claim that (**??**) has a unique coexistence state with reproduction rates $(\lambda_s, a_2, ..., a_N)$. In fact, if not, assume that $(\bar{u_{10}}, ..., \bar{u_{N0}}) \neq (u_{10}, ..., u_{N0})$ is another coexistence state. By Implicit Function Theorem, there exists a $\lambda_{(a_2,...,a_N)} < \tilde{a} < \lambda_s$ and very close to λ_s, (**??**) has a coexistence state very close to $(\bar{u_{10}}, ..., \bar{u_{N0}})$ which means that (**??**) has more than one coexistence state for reproduction rates $(\tilde{a}, a_2, ..., a_N)$. This is a contradiction to the definition of λ_s. But since (**??**) has a unique coexistence state with reproduction rates $(\lambda_s, a_2, ..., a_N)$ and Frechet derivative is invertible, Theorem **??** concluded that λ_s can not be as defined. Therefore, for each $(a_1, ..., a_N) \in \Gamma$, (**??**) has a unique coexistence state $(u_1, ..., u_N)$. Furthermore, by the assumption, for each $(a_1, ..., a_N) \in \Gamma$, the Frechet derivative of (**??**) at the unique solution $(u_1, ..., u_N)$ is invertible. Hence, Theorem **??** concludes that there is an open neighborhood $V_{(a_1,...,a_N)}$ of $(a_1, ..., a_N)$ in R^N such that if $(a_{10}, ..., a_{N0}) \in V_{(a_1,...,a_N)}$, then (**??**) with reproduction rates $(a_{10}, ..., a_{N0})$ has a unique coexistence state. Let $W = \bigcup_{(a_1,...,a_N)\in\Gamma} V_{(a_1,...,a_N)}$. Then W is an open set in R^N such that $\Gamma \subseteq W$ and for each $(a_{10}, ..., a_{N0}) \in W$, (**??**) has a unique coexistence state.

Corollary 5.2 *Suppose*

(A) Γ *is a closed, convex region in* R^N,

(B) *there exist* $k_1, ..., k_N > 0$ *such that for all* $(a_1, ..., a_N) \in \Gamma$, $a_i > \lambda_1 + g_i(k_1, ..., k_{i-1}, 0, k_{i+1}, ..., k_N)$ *and* $g_i(0, ..., 0, x, 0, ..., 0) > a_i$ *for* $x \geq k_i$.

(C) $2\inf(\frac{\partial g_i}{\partial x_i}) > \sum_{j=1,j\neq i}^{N} (\sup(\frac{\partial g_i}{\partial x_j}) + K\sup(\frac{\partial g_j}{\partial x_i}))$ *for* $i = 1, ..., N$, *where* $K = \sup_{(a_1,...,a_N)\in\Gamma} \sup_{i,j\neq i} \frac{\theta_{a_j - g_j(0,...,0,\cdot,0,...,0)}}{\theta_{a_i - g_i(k_1,...,k_{i-1},\cdot,k_{i+1},...,k_N)}}$.

Then there is an open set W *in* R^N *such that* $\Gamma \subseteq W$ *and for every* $(a_1, ..., a_N) \in W$, *(**??**) has a unique positive solution.*

Proof From $\theta_{a_i - g_i(0,...,0,\cdot,0,...,0)} < k_i$ and the monotonicity of $g_i(0, ..., 0, \cdot, 0, ..., 0)$, we have

$$\begin{aligned} a_i &> \lambda_1 + g_i(k_1, ..., k_{i-1}, 0, k_{i+1}, ..., k_N) \\ &\geq \lambda_1(g_i(\theta_{a_1 - g_1(\cdot,0,...,0)}, ..., \theta_{a_{i-1} - g_{i-1}(0,...,0,\cdot,0,...,0)}, 0, \theta_{a_{i+1} - g_{i+1}(0,...,0,\cdot,0,...,0)}, \\ &\quad ..., \theta_{a_N - g_N(0,...,0,\cdot)})) \end{aligned}$$

for all $(a_1, ..., a_N) \in \Gamma$. The condition already guarantees that for all $(a_1, ..., a_N) \in \partial_L\Gamma$ and $i = 1, ..., N$,

$$2\inf(\frac{\partial g_i}{\partial x_i}) > \sum_{j=1,j\neq i}^{N} (\sup(\frac{\partial g_i}{\partial x_j}) + K_{(a_1,...,a_N)}\sup(\frac{\partial g_j}{\partial x_i})),$$

where $K_{(a_1,...,a_N)} = \sup_{j\neq i} \dfrac{\theta_{a_j - g_j(0,...,0,\cdot,0,...,0)}}{\theta_{a_i - g_i(k_1,...,k_{i-1},\cdot,k_{i+1},...,k_N)}}$, and so by Theorem **??**, (**??**) has a unique positive solution for all $(a_1, ..., a_N) \in \partial_L\Gamma$. Furthermore, by the estimate of the solution in the proof of Theorem **??**, if $(u_1, ..., u_N)$ is a positive solution for $(a_1, ..., a_N) \in \Gamma$, then

$$2\inf(\frac{\partial g_i}{\partial x_i})u_i > \sum_{j=1,j\neq i}^{N} (\sup(\frac{\partial g_i}{\partial x_j})u_i + \sup(\frac{\partial g_j}{\partial x_i})u_j)$$

for $i = 1, ..., N$. Hence, by Lemma **??**, if $(u_1, ..., u_N)$ is a positive solution of (**??**) for $(a_1, ..., a_N) \in \Gamma$, then the Frechet derivative at $(u_1, ..., u_N)$ is invertible. Therefore, the theorem follows from Theorem **??**.

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