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EXISTENCE OF SOLUTION FOR MODIFIED LANDAU-LIFSHITZ MODEL

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Abstract In the present paper, the existence of solutions to Cauchy problem for modified Landau–Lifshitz Model initiated by Augusto Visintin is studied.
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1. Introduction

Ferromagnetic materials can attain a large magnetization under the action of a small applied magnetic field. To explain this phenomenon, in 1907 Weiss suggested that any small portion of the body exhibits a spontaneous magnetization and is magnetically saturated even if no magnetic field is applied. In 1928 Heisenberg explained the spontaneous magnetization postulated by Weiss in terms of the exchange interaction. In 1935 Landau and Lifshitz [1] proposed a quantitative theory, now known as micromagnetics.

In the classical study of 1-dimensional motion of ferromagnetic chain, the so-called Landau-Lifshitz equation for the isotropic Heisenberg chain is a special case of the generalized systems

$$M_t = M \times M_{xx} + f(x,t,M) \tag{1.1}$$

and such an equation usually appears in the study of pure material. In the past years a lot of works contributed to the study of the soliton solution, the interaction of solitary waves and others for the Landau-Lifshitz equation in $[2-5]$. Generally speaking, the existence of global weak solutions for initial-boundary value problems and the Cauchy problem of the generalized system of ferromagnetic have been established in $[6-8]$.

The system of Heisenberg spin chain

$$M_t = M \times M_{xx}, \tag{1.2}$$

also called the Landau-Lifshitz equation, is proposed to describe the evolution of spin field in continuous ferromagnets. In [9], Sulem, Sulem and Bardos studied the well-posedness for the Cauchy problem of the system (1.2). In [10], Zhou, Guo and Tan have gotten existence and uniqueness of smooth solution for the system (1.2).

The above discussion is referred to a perfect crystal and does not allow the presence of magnetic inclusions: impurities, dislocations and other defects. This also covers the case where magnetic inclusions are regularly distributed; a typical example is steel. In [11] Augusto Visintin proposed to describe the effect of defects on evolution by means of modification of the Landau-Lifshitz equation, i.e.

$$\begin{cases} M_t = M \times (M_{xx} - \dfrac{\eta M_t}{|M_t|}), \\ M(x,0) = M^0(x), \end{cases} \tag{1.3}$$

where η is a positive constant to account for the average distribution of defects in the material. For a nonhomogeneous material, η may depend on x , and may be also replaced by a 3×3-tensor to account for anisotropy. In this paper for simplicity $\eta =$constant is discussed. But the argument used here also works for the case $\eta = \eta(x)$.

In order to avoid singularity of (1.3) where $M_t = 0$, with $W = M_{xx} - \dfrac{\eta M_t}{\sqrt{\varepsilon^2 + \mid M_t \mid^2}}$ we first study its regularized problem

$$\begin{cases} M_t = M \times W, \\ M(x,0) = M^0(x). \end{cases} \tag{1.4}$$

Following [12], we introduce Gilbert damping to (1.4) and consider the following problem

$$\begin{cases} M_t = M \times W - \alpha M \times (M \times W), & (t,x) \in (0,T] \times \Omega, \\ M(x,0) = M^0(x), & x \in \Omega, \end{cases} \tag{1.5}$$

where T is a positive constant and $\Omega = [-1,1]$. According to the classical theory of Weiss, $|M(x,t)| = 1$. Hence (1.5) is equivalent to the following problem

$$\begin{cases} M_t = M \times W + \alpha W + \alpha |M_x|^2 M, & (t,x) \in (0,T] \times \Omega, \\ M(x,0) = M^0(x), & x \in \Omega. \end{cases} \tag{1.6}$$

Our sketch is as follows. Firstly, we establish certain a priori α-independent estimates for the solutions of the problem (1.6), which allow us to obtain a sufficient smooth solution to Cauchy problem for the problem (1.4) by passing to the limit $\alpha \to 0$. Secondly, the existence of the sufficient smooth solution for the problem (1.6) is proved by using the fixed point theorem and $\alpha-$independent estimates.

Throughout the present paper all the positive constants depending only on η, $\|M^0\|_{H^k(\Omega)}$, T, independent of α and ε, unless otherwise stated, will be denoted by C and they may be different from line to line.

Theorem *Suppose $M^0(x)$ is in $H^k(\Omega), k \geq 4$ with $|M^0(x)| = 1$ and $M^0(-1) = M^0(1)$, then for any positive constant T,*

(1) *Let α, ε be any positive number and $M(x,t)$ be a sufficiently smooth solution of (1.6). Then*

$$\sup_{0\le t\le T}(\|M_{xx}\|+\|M_t\|+\|M_x\|)\le C,$$

and

$$\sup_{0\le t\le T, 0\le s\le[\frac{k}{2}]}\|M_{t^s x^{k-2s}}\|\le C(\frac{1}{\varepsilon}), k\ge 3.$$

(2) *Let ε be any positive number. Then the initial value problem (1.4) with the periodic boundary condition has a unique periodic solution $M(x,t)\in\bigcap_{s=0}^{[\frac{k}{2}]}W_{loc}^{s,\infty}(R^+;H^{k-2s}(\Omega))$ and $M(x,t)$ satisfies $|M(x,t)|=1$ for any $x\in\Omega$ and for any $t>0$. Especially, if $M^0(x)\in C^\infty(\Omega)$, then $M(x,t)\in C^\infty([0,T]\times\Omega)$.*

(3) *Let $\varepsilon=0$. Then the problem (1.3) admits a solution $M(x,t)$ such that $|M(x,t)|=1$, $M(x,t)\in L^\infty([0,T];H^2(\Omega))\bigcap W^{1,\infty}([0,T];L^2(\Omega))$.*

Finally, it should be pointed out that the uniqueness of the solutions of (1.3) is still open.

In this paper we use $\|\quad\|$ to replace $\|\quad\|_{L^2(\Omega)}$ and denote by $\|\quad\|_p$ the usual $\|\quad\|_{L^p(\Omega)}$ with $2<p\le\infty$.

2. A Priori α-Independent Estimates

Lemma 2.1 *If $u(x)\in H^1(\Omega)$, then $\|u\|_\infty\le C(\|Du\|^{\frac12}\|u\|^{\frac12}+\|u\|)$.*

Proof Denote $\bar{u}=\frac{1}{2}\int_{-1}^{1}udx$ then $|\bar{u}|\le C\|u\|$. By the mean value theorem, $\exists x_0\in\Omega, \forall y\in\Omega$, we can get

$$(u-\bar{u})^2(y)=2\int_{x_0}^{y}u_x(u-\bar{u})dx.$$

Hence

$$\|u-\bar{u}\|_\infty^2\le C\|u_x\|\|u-\bar{u}\|\le C\|u_x\|\|u\|.$$

This completes the proof of Lemma 2.1.

Lemma 2.2 *Let the assumptions in Theorem be fulfilled. Then in the classical sense (1.5) is equivalent to (1.6).*

Proof Let $M(x,t)$ be a classical solution of the problem (1.5). By taking inner product with $M(x,t)$ of (1.5), it is easy to see that

$$\frac{1}{2}\frac{\partial}{\partial t}|M|^2=0.$$

Thus we have $|M(x,t)|=|M^0(x)|=1$. So

$$-\alpha M\times(M\times W)=\alpha|M|^2W-\alpha(M\cdot W)M=\alpha W+\alpha|M_x|^2M.$$

On the other hand, if $M(x,t)$ is a classical solution of (1.6), taking inner product with $M(x,t)$, we have

$$\frac{1}{2}(1+\frac{\alpha\eta}{\sqrt{\varepsilon^2+|M_t|^2}})\frac{\partial}{\partial t}|M|^2=\frac{\alpha}{2}\frac{\partial^2}{\partial x^2}|M|^2+\alpha|M_x|^2(|M|^2-1),$$

$\tilde{M}=|M(x,t)|^2-1$ satisfies

$$\begin{cases}\frac{1}{2}(1+\frac{\alpha\eta}{\sqrt{\varepsilon^2+|M_t|^2}})\frac{\partial}{\partial t}\tilde{M}=\frac{\alpha}{2}\frac{\partial^2}{\partial x^2}\tilde{M}+\alpha|M_x|^2\tilde{M},\\ \tilde{M}(x,0)=0.\end{cases}$$

Using the energy method we know $\tilde{M}(x,t)=0$, i.e. $|M(x,t)|=1,\forall x\in\Omega,t\geq 0$. Obviously $M(x,t)$ is a classical solution of (1.5). This completes the proof of Lemma 2.2.

Lemma 2.3 *Let the assumptions in Theorem be fulfilled and $M(x,t)$ be a sufficiently smooth solution of (1.6). Then we have*

$$\sup_{0\leq t\leq T}\|M_x\|\leq\|M_x^0\|.$$

Proof Taking inner product of (1.6) with W and integrating over Ω, we have

$$-\frac{1}{2}\frac{d}{dt}\|M_x\|^2-\eta\int_{-1}^{1}\frac{|M_t|^2}{\sqrt{\varepsilon^2+|M_t|^2}}dx=\alpha\|W\|^2-\alpha\int_{-1}^{1}|M_x|^2M\cdot Wdx.$$

Consequently,

$$\frac{1}{2}\frac{d}{dt}\|M_x\|^2+\alpha\int_{-1}^{1}|M\times W|^2dx=-\eta\int_{-1}^{1}\frac{|M_t|^2}{\sqrt{\varepsilon^2+|M_t|^2}}dx\leq 0.$$

In getting the last expression we have used $M\cdot W=-|M_x|^2$ and $|M(x,t)|=1$. This ends the proof of Lemma 2.3.

Lemma 2.4 *Let the assumptions in Theorem be fulfilled. Then any sufficiently smooth solution of* (1.6), $M(x,t)$ *satisfies*

$$\sup_{0\leq t\leq T}\|M_{xx}\|\leq C.$$

Proof After differentiating (1.6) with respect to x, taking inner product with W_x and integrating over Ω, we have

$$\begin{aligned}\frac{d}{dt}\|M_{xx}\|^2+2\alpha\|W_x\|^2=&-2\alpha\int_{-1}^{1}(|M_x|^2M)_x\cdot W_xdx-2\int_{-1}^{1}M_x\times W\cdot W_xdx\\&-2\int_{-1}^{1}\eta M_{tx}\cdot(\frac{M_t}{\sqrt{\varepsilon^2+|M_t|^2}})_xdx=I_1+I_2+I_3.\end{aligned}\qquad(2.1)$$

It is easy to see

$$I_3 = 2\int_{-1}^{1} \eta \frac{[-(\varepsilon^2 + |M_t|^2)]M_{tx} \cdot M_{tx} + (M_{tx} \cdot M_t)^2}{\sqrt{\varepsilon^2 + |M_t|^2}^3} dx \le 0 \tag{2.2}$$

and

$$I_1 \le 2\alpha \|M_x\|_6^3 \|W_x\| + 4\alpha \|M_{xx}\|_\infty \|M_x\| \|W_x\|.$$

From Lemma 2.1 and Lemma 2.3, it follows that $\|M_x\|_\infty^2 \le C\|M_{xx}\|\|M_x\|$. So

$$\begin{aligned} I_1 &\le C\alpha \|M_{xx}\|\|W_x\| + C\alpha \|M_{xx}\|_\infty \|W_x\| \\ &\le \frac{\alpha}{2}\|W_x\|^2 + C\alpha\|M_{xx}\|^2 + C\alpha\|M_{xx}\|_\infty^2 \\ &\le \frac{\alpha}{2}\|W_x\|^2 + C\alpha\|M_{xx}\|^2 + C\alpha\left(\|W\|_\infty^2 + \|\frac{\eta M_t}{\sqrt{\varepsilon^2+|M_t|^2}}\|_\infty^2\right) \\ &\le \alpha\|W_x\|^2 + C\alpha\|M_{xx}\|^2 + C\alpha\|W\|^2 + C\alpha \\ &\le \alpha\|W_x\|^2 + C\alpha\|M_{xx}\|^2 + C\alpha. \end{aligned} \tag{2.3}$$

In what follows we are going to handle the term I_2. If $|M_x| \ne 0$, then the vectors $M, M_x, M \times M_x$ form an orthogonal frame of R^3. We let

$$W = (W\cdot M)M + \frac{(W\cdot M_x)}{|M_x|^2}M_x + \frac{W\cdot M\times M_x}{|M_x|^2}M\times M_x.$$

Then

$$\begin{aligned} I_2 &= -2\int_{-1}^{1}[(W_x\cdot M_x\times M)(W\cdot M) + (W\cdot M\times M_x)(W_x\cdot M)]dx \\ &= 4\int_{-1}^{1}|M_x|^2(M_x\cdot(M\times W)_x)dx + \int_{-1}^{1}(|M_x|^2)_x(W\cdot M\times M_x)dx \\ &\quad -2\int_{-1}^{1}\eta\frac{M_t\cdot M_x}{\sqrt{\varepsilon^2+|M_t|^2}}(W\cdot M\times M_x)dx - 2\int_{-1}^{1}|M_x|^2(W\cdot M\times M_{xx})dx \\ &= 5\int_{-1}^{1}[M_x\cdot(M\times W)_x](|M_x|^2)dx + 3\int_{-1}^{1}|M_x|^2(\eta\frac{M_t}{\sqrt{\varepsilon^2+|M_t|^2}}\cdot M\times M_{xx})dx \\ &\quad -2\eta\int_{-1}^{1}\frac{M_t\cdot M_x}{\sqrt{\varepsilon^2+|M_t|^2}}(W\cdot M\times M_x)dx. \end{aligned} \tag{2.4}$$

For the second term on the right hand side of (2.4), we have

$$\begin{aligned} 3\int_{-1}^{1}|M_x|^2(\eta\frac{M_t}{\sqrt{\varepsilon^2+|M_t|^2}}\cdot M\times M_{xx})dx &\le 3\int_{-1}^{1}|M_x|^2\eta|M_{xx}|dx \\ &\le 3\eta\|M_x\|_4^2\|M_{xx}\| \le C\|M_{xx}\|^{\frac{3}{2}} \le C\|M_{xx}\|^2 + C. \end{aligned} \tag{2.5}$$

For the third term on the right hand side of (2.4), we have

$$-2\eta\int_{-1}^{1}\frac{M_t\cdot M_x}{\sqrt{\varepsilon^2+|M_t|^2}}(W\cdot M\times M_x)dx$$
$$\le 2\eta\int_{-1}^{1}|M_x|(|M_{xx}||M_x|+\eta|M_x|)dx\le C\|M_{xx}\|^2+C. \tag{2.6}$$

Integrating the first term on the right hand side of (2.4) with respect to t, we have

$$\begin{aligned}
&5\int_0^t\int_{-1}^{1}[M_x\cdot(M\times W)_x](|M_x|^2)dxd\tau\\
&=5\int_0^t\int_{-1}^{1}|M_x|^2[M_x\cdot(M_t-\alpha|M_x|^2M-\alpha W)_x]dxd\tau\\
&=\frac{5}{4}(\|M_x\|_4^4-\|M_x^0\|_4^4)-5\alpha\int_0^t\int_{-1}^{1}|M_x|^6dxd\tau-5\alpha\int_0^t\int_{-1}^{1}|M_x|^2M_x\cdot W_xdxd\tau\\
&\le\frac{1}{2}\|M_{xx}\|^2+C+\frac{\alpha}{2}\int_0^t\|W_x\|^2d\tau+C\alpha\int_0^t\|M_{xx}\|^2d\tau. \qquad (2.7)
\end{aligned}$$

After inserting (2.2)-(2.6) into (2.1), integrating with t, we can obtain

$$\begin{aligned}
\|M_{xx}\|^2+\alpha\int_0^t\|W_x\|^2d\tau\le C(1+\alpha)\int_0^t\|M_{xx}\|^2d\tau+C&\\
+5\int_0^t\int_{-1}^{1}[M_x\cdot(M\times W)_x](|M_x|^2)dxd\tau.& \qquad (2.8)
\end{aligned}$$

Combining (2.7) with (2.8) gives at once

$$\|M_{xx}\|^2+\frac{\alpha}{2}\int_0^t\|W_x\|^2d\tau\le\frac{1}{2}\|M_{xx}\|^2+C(1+\alpha)\int_0^t\|M_{xx}\|^2d\tau+C(1+\alpha).$$

Using Gronwall's inequality we can obtain

$$\sup_{0\le t\le T}\|M_{xx}\|^2\le C.$$

Remark Let the assumptions in Theorem be fulfilled. Then any sufficiently smooth solution of (1.6), $M(x,t)$ satisfies

$$\sup_{0\le t\le T}\|M_t\|^2\le C.$$

Lemma 2.5 *Let the assumptions in Theorem be fulfilled. Then any sufficiently smooth solution of* (1.6), $M(x,t)$ *satisfies*

$$\sup_{0\le t\le T}\|M_{tx}\|\le C.$$

Proof After differentiating (1.6) with respect to t, taking inner product with W_t and integrating over Ω, we have

$$\begin{aligned}\frac{d}{dt}\|M_{tx}\|^2+2\alpha\|W_t\|^2 = & -2\alpha\int_{-1}^{1}(|M_x|^2M)_t\cdot W_tdx-2\int_{-1}^{1}M_t\times W\cdot W_tdx \\ & -2\int_{-1}^{1}\eta M_{tt}\cdot(\frac{M_t}{\sqrt{\varepsilon^2+\mid M_t\mid^2}})_tdx.\end{aligned}$$

Now we integrate the above equation with respect to t, then we can obtain

$$\begin{aligned}\|M_{tx}\|^2+2\alpha\int_0^t\|W_t\|^2d\tau = & \|M_{tx}^0\|^2-2\alpha\int_0^t\int_{-1}^{1}(|M_x|^2M)_t\cdot W_tdxd\tau \\ & -2\int_0^t\int_{-1}^{1}M_t\times W\cdot W_tdxd\tau \\ & -2\eta\int_0^t\int_{-1}^{1}M_{tt}\cdot(\frac{M_t}{\sqrt{\varepsilon^2+\mid M_t\mid^2}})_tdxd\tau \\ = & \|M_{tx}^0\|^2+J_1+J_2+J_3. \end{aligned}\tag{2.9}$$

It is easy to see that

$$J_3=2\int_0^t\int_{-1}^{1}\eta\frac{[-(\varepsilon^2+|M_t|^2)]M_{tt}\cdot M_{tt}+(M_{tt}\cdot M_t)^2}{\sqrt{\varepsilon^2+|M_t|^2}^3}dxd\tau\le 0. \tag{2.10}$$

The argument used to I_1 in Lemma 2.3 also works for J_1, so we can obtain

$$J_1\le\frac{\alpha}{2}\int_0^t\|W_t\|^2d\tau+C\alpha\int_0^t\|M_{xt}\|^2d\tau+C\alpha. \tag{2.11}$$

In what follows we are going to handle the term J_2. If $|M_t|\ne 0$, then the vectors $M, M_t, M\times M_t$ form an orthogonal basis of R^3. Let

$$W_t=(W_t\cdot M)M+\frac{(W_t\cdot M_t)}{|M_t|^2}M_t+\frac{W_t\cdot M\times M_t}{|M_t|^2}M\times M_t.$$

Then

$$\begin{aligned}J_2= & 2\int_0^t\int_{-1}^{1}|M_x|^2M_t\cdot(M\times W)_tdxd\tau \\ & -2\int_0^t\int_{-1}^{1}(M_t\times W\cdot M)(-2M_x\cdot M_{xt}-W\cdot M_t)dxd\tau \\ \le & 2\int_0^t\int_{-1}^{1}|M_x|^2M_t\cdot(M\times W)_tdxd\tau+C\int_0^t\|M_t\|_\infty\|M_{xt}\|\|W\|d\tau \\ & +C\int_0^t\|M_t\|_\infty^2\|W\|^2d\tau\end{aligned}$$

$$\leq 2\int_0^t\int_{-1}^1 |M_x|^2 M_t\cdot(M\times W)_t dxd\tau + C\int_0^t(\|M_{xt}\|^{\frac{3}{2}}+\|M_{xt}\|)d\tau$$

$$\leq C\int_0^t \|M_{xt}\|^2 d\tau + C + 2\int_0^t\int_{-1}^1 |M_x|^2 M_t\cdot(M\times W)_t dxd\tau. \tag{2.12}$$

In getting the last expression we have used $W_t\cdot M = -2M_x\cdot M_{xt} - W\cdot M_t$.
For the last term on the right hand side of (2.12), we have

$$\begin{aligned} &2\int_0^t\int_{-1}^1 |M_x|^2[M_t\cdot(M_t-\alpha|M_x|^2M-\alpha W)_t]dxd\tau \\ &\quad\leq -\int_0^t\int_{-1}^1(|M_x|^2)_t|M_t|^2dxd\tau + C\alpha\int_0^t\|M_t\|\|W_t\|d\tau \\ &\quad\leq \frac{\alpha}{2}\int_0^t\|W_t\|^2d\tau + C\alpha + C\int_0^t\|M_t\|_\infty^2\|M_{xt}\|d\tau \\ &\quad\leq \frac{\alpha}{2}\int_0^t\|W_t\|^2d\tau + C\alpha + C\int_0^t\|M_{xt}\|^2d\tau. \end{aligned} \tag{2.13}$$

From (2.9) to (2.13), we have

$$\|M_{xt}\|^2 + \alpha\int_0^t\|W_t\|^2d\tau \leq C(\alpha+1)\int_0^t\|M_{xt}\|^2d\tau + C(\alpha+1).$$

Using Gronwall's inequality, we can obtain

$$\|M_{xt}\|^2\leq C, \qquad \forall 0\leq t\leq T. \tag{2.14}$$

Finally

$$M_t = M\times W + \alpha W + \alpha|M_x|^2M$$

and

$$-M\times M_t = -\alpha M\times W + W + |M_x|^2M$$

imply that

$$W = \frac{1}{1+\alpha^2}\{\alpha M_t - M\times M_t\} - |M_x|^2M.$$

Hence

$$M_{xx} = \frac{1}{1+\alpha^2}\{\alpha M_t - M\times M_t\} - |M_x|^2M + \frac{\eta M_t}{\sqrt{\varepsilon^2+|M_t|^2}}, \tag{2.15}$$

$$\begin{aligned} M_{xxx} &= \frac{1}{1+\alpha^2}\{\alpha M_{xt} - (M\times M_t)_x\} - (|M_x|^2M)_x + (\frac{\eta M_t}{\sqrt{\varepsilon^2+|M_t|^2}})_x \\ &= \frac{1}{1+\alpha^2}\{\alpha M_{xt} - (M\times M_t)_x\} - (|M_x|^2M)_x \\ &\quad +\eta\frac{(\varepsilon^2+|M_t|^2)M_{tx} - (M_{tx}\cdot M_t)M_t}{\sqrt{\varepsilon^2+|M_t|^2}^3}. \end{aligned} \tag{2.16}$$

From (2.14) and (2.16), it follows that

$$\sup_{0\le t\le T}\|\sqrt{\varepsilon^2+|M_t|^2}M_{xxx}\|\le C.$$

Lemma 2.6 *Let the assumptions in Theorem be fulfilled. Then any sufficiently smooth solution of* (1.6), $M(x,t)$ *satisfies*

$$\sup_{0\le t\le T}(\|M_{xxxx}\|+\|M_{tt}\|+\|M_{txx}\|)\le C(\frac{1}{\varepsilon}).$$

Proof From the proof of Lemma 2.5, it follows that

$$\|M_{tx}\|^2+2\eta\int_0^t\int_{-1}^1\frac{\varepsilon^2|M_{tt}|^2}{(\sqrt{\varepsilon^2+|M_t|^2})^3}dxd\tau\le C\int_0^t\|M_{tx}\|^2d\tau+C.$$

By Lemma 2.5 and Remark of Lemma 2.4, we can obtain

$$\int_0^T\|M_{tt}\|^2dt\le C(\frac{1}{\varepsilon}). \tag{2.17}$$

From (2.15) we can get

$$\begin{aligned}M_{xxt}&=\frac{1}{1+\alpha^2}\{\alpha M_{tt}-(M\times M_t)_t\}-(|M_x|^2M)_t+(\frac{\eta M_t}{\sqrt{\varepsilon^2+|M_t|^2}})_t,\\ M_{xxxx}&=\frac{1}{1+\alpha^2}\{\alpha M_{txx}-(M\times M_t)_{xx}\}-(|M_x|^2M)_{xx}+(\frac{\eta M_t}{\sqrt{\varepsilon^2+|M_t|^2}})_{xx}.\end{aligned} \tag{2.18}$$

From (2.17) and (2.18), we can get

$$\int_0^T(\|M_{txx}\|+\|M_{xxxx}\|)dt\le C(\frac{1}{\varepsilon}).$$

By the same argument as we have done in the proof of Lemma 2.5 we can complete the proof for Lemma 2.6.

Also we have the following general uniform estimates for arbitrary sufficiently smooth solutions of (1.6).

Lemma 2.7 *Let the assumptions in Theorem be fulfilled and $M(x,t)$ be a sufficiently smooth solution of (1.6). Then*

$$\sup_{0\le t\le T,0\le s\le[\frac{k}{2}],k\ge3}\|M_{t^sx^{k-2s}}\|\le C(\frac{1}{\varepsilon}).$$

3. Existence and Uniqueness

In this section we will prove local the existence of (1.6) with $M^0(x) \in H^4(\Omega)$ and the uniqueness of (1.4) with $M^0(x) \in H^5(\Omega)$.

First of all we consider the following problem

$$\begin{cases} M_t = A(x,t)M_{xx} + F(x,t), & (t,x) \in (0,T] \times \Omega, \\ M(x,0) = 0, & x \in \Omega, \end{cases} \tag{3.1}$$

where $M(x,t)$ is a vector in R^3, $A(x,t)$ is a real matrix.

Hypothesis: $F(x,t), A(x,t) \in H^1([0,T]; L^2(\Omega)) \bigcap L^2([0,T]; H^2(\Omega))$ and $\frac{A+A^t}{2}$ is a uniformly positive matrix, i.e. $\exists \lambda_0 > 0$ satisfying

$$\frac{A+A^t}{2}\xi \cdot \xi \geq \lambda_0 |\xi|^2, \forall \xi \in R^3, (x,t) \in \Omega \times [0,T].$$

[13] tells us that (3.1) has a classical solution.

Lemma 3.1 *Let the above hypothesis be fulfilled and let $M(x,t)$ be a sufficiently smooth solution of (3.1). Then*

$$\sup_{0\leq\tau\leq t} (\|M_t\|_{H^1}^2 + \|M\|_{H^3}^2) + \int_0^t (\|M\|_{H^4}^2 + \|M_t\|_{H^2}^2 + \|M_{tt}\|^2) d\tau$$
$$\leq C \int_0^t (\|F\|_{H^2}^2 + \|F_t\|^2 + \|A_x M_{xxx}\|^2 + \|A_{xx} M_{xx}\|^2 + \|A_t M_{xx}\|^2) d\tau,$$

for some constant C depending only on λ_0.

Proof Taking inner product of (3.1) with M_{xx} and integrating over Ω, we have

$$-\frac{1}{2}\frac{d}{dt}\|M_x\|^2 = \int_{-1}^1 \frac{A+A^t}{2} M_{xx} \cdot M_{xx} dx + \int_{-1}^1 F(x,t) \cdot M_{xx} dx.$$

Integration with respect to t yields

$$\|M_x\|^2 + \lambda_0 \int_0^t \|M_{xx}\|^2 d\tau \leq \frac{1}{\lambda_0} \int_0^t \|F\|^2 d\tau.$$

This ends the proof of Lemma 3.1 by the same steps.

Consider the following problem

$$\begin{cases} M_t = A(x,t)M_{xx} + B(x,t)M, & (t,x) \in (0,T] \times \Omega, \\ M(x,0) = 0, & x \in \Omega. \end{cases} \tag{3.2}$$

Lemma 3.2 *If $\frac{A+A^t}{2}\xi \cdot \xi \geq 0, \forall \xi \in R^3$ and $A(x,t), B(x,t) \in H^1([0,T]; H^1(\Omega))$, then the solution of (3.2) is unique.*

Proof Taking inner product of (3.2) with M_{xx} and integrating over Ω, we have

$$-\frac{1}{2}\frac{d}{dt}\|M_x\|^2 = \int_{-1}^{1}\frac{A+A^t}{2}M_{xx}\cdot M_{xx}dx + \int_{-1}^{1}BM\cdot M_{xx}dx.$$

Using $\frac{A+A^t}{2}\xi\cdot\xi \geq 0$ and integrating by parts, we can obtain

$$\frac{d}{dt}\|M_x\|^2 \leq C(\|M\|_\infty\|M_x\| + \|M_x\|^2). \tag{3.3}$$

Taking inner product of (3.2) with M and integrating over Ω, we have

$$-\frac{1}{2}\frac{d}{dt}\|M\|^2 = \int_{-1}^{1}AM_{xx}\cdot Mdx + \int_{-1}^{1}BM\cdot Mdx.$$

Integration by parts yields

$$\frac{d}{dt}\|M\|^2 \leq C(\|M\|_\infty\|M_x\| + \|M\|^2 + \|M_x\|^2). \tag{3.4}$$

Using (3.3), (3.4) and Lemma 2.1, we can get

$$\frac{d}{dt}(\|M\|^2 + \|M_x\|^2) \leq C(\|M\|^2 + \|M_x\|^2).$$

Using Gronwall's inequality we can prove Lemma 3.2.

Defining

$$|||M|||_t^2 = \sup_{0\leq\tau\leq t}(\|M_t\|_{H^1}^2 + \|M\|_{H^3}^2) + \int_0^t(\|M\|_{H^4}^2 + \|M_t\|_{H^2}^2 + \|M_{tt}\|^2)d\tau,$$

all constants depending only on $\eta, \lambda_0, \|M^0\|_{H^4}, \varepsilon, \alpha$ will be still denoted by C. Set

$$B(M, M_t) = I + \frac{\eta}{\sqrt{\varepsilon^2 + |M_t|^2}}[\alpha I + A(M)],$$

where $M_i = M_i(x,t)$ is a function, $i = 1,2,3$ and I a unit matrix, ε a positive constant.

$$M^t = \begin{pmatrix} M_1 & M_2 & M_3 \end{pmatrix},$$

$$A(M) = \begin{pmatrix} 0 & -M_3 & M_2 \\ M_3 & 0 & -M_1 \\ -M_2 & M_1 & 0 \end{pmatrix}.$$

Then the problem (1.6) is equivalent to the following problem

$$\begin{cases} B(M, M_t)M_t = [\alpha I + A(M)]M_{xx} + \alpha|M_x|^2M, & (t,x) \in (0,T]\times\Omega, \\ M(x,0) = M^0(x), & x \in \Omega. \end{cases} \tag{3.5}$$

Without loss of generality, we may assume $M^0(x) \in C^\infty(\Omega)$. Set

$$M(x,t) = V(x,t) + M^0(x).$$

Then $V(x,t)$ satisfies

$$\begin{cases} B(V, V_t, M^0)V_t = [\alpha I + A(V+M^0)]V_{xx} + F(V, V_x), & (t,x) \in (0,T] \times \Omega, \\ V(x,0) = 0, & x \in \Omega, \end{cases} \tag{3.6}$$

where

$$\begin{aligned} F(V,V_x) &= \left[\alpha I + A(V+M^0)\right] M^0_{xx} + \alpha \left|V_x + M^0_x\right|^2 \left(V + M^0\right) \\ &= F(0,0) + B^0(V) + B^1(V)V_x + B^2(V)|V_x|^2 \end{aligned}$$

for some smooth vector $B^0(V), B^2(V)$ and smooth matrix $B^1(V)$.

Lemma 3.3 *Set* $B = I + \dfrac{\eta}{\sqrt{\varepsilon^2 + |M_t|^2}}[\alpha I + A(M)]$ *and* $D = \alpha I + A(M)$. *Then*

$$\frac{B^{-1}D + (B^{-1}D)^t}{2}\xi \cdot \xi \geq \frac{\alpha\varepsilon}{\varepsilon + \alpha\eta}|\xi|^2, \forall \alpha, \varepsilon, \eta \in R^+, \xi \in R^3.$$

Proof With $a = \dfrac{\sqrt{|M_t|^2 + \varepsilon^2}}{\eta} + \alpha$, we have

$$B = \frac{\eta}{\sqrt{\varepsilon^2 + |M_t|^2}}[aI + A(M)].$$

Hence

$$\begin{aligned} B^{-1}D + (B^{-1}D)^t &= (a-\alpha)(aI + A(M))^{-1}[(\alpha I + A(M))(aI + A(M))^t \\ &\quad + (aI + A(M))(\alpha I + A(M))^t](aI + A(M))^{-t} \\ &= (a-\alpha)(aI + A(M))^{-1}[(\alpha I + A(M))(aI - A(M)) \\ &\quad + (aI + A(M))(\alpha I - A(M))](aI + A(M))^{-t} \\ &= 2(a-\alpha)(aI + A(M))^{-1}[\alpha a I + A(M)A^t(M)](aI + A(M))^{-t}, \end{aligned}$$

then for any $\xi \in R^3$, with $(aI + A(M))^{-t} \cdot \xi = \zeta$, we have

$$\frac{B^{-1}D + (B^{-1}D)^t}{2}\xi \cdot \xi = (a-\alpha)\left[\alpha a|\zeta|^2 + A(M)A^t(M)\zeta \cdot \zeta\right].$$

On the other hand, $\xi = (aI + A(M))^t \cdot \zeta$ and

$$|\xi|^2 = a^2|\zeta|^2 + A(M)A^t(M)\zeta \cdot \zeta,$$

$$\begin{aligned}\frac{B^{-1}D+(B^{-1}D)^t}{2}\xi\cdot\xi &= (a-\alpha)\frac{\alpha}{a}(a^2|\zeta|^2+\frac{a}{\alpha}A(M)A^t(M)\zeta\cdot\zeta)\\ &\geq (a-\alpha)\frac{\alpha}{a}(a^2|\zeta|^2+A(M)A^t(M)\zeta\cdot\zeta)\\ &\geq (a-\alpha)\frac{\alpha}{a}|\xi|^2\geq\frac{\alpha\varepsilon}{\varepsilon+\alpha\eta}|\xi|^2\end{aligned}$$

namely, $\exists\lambda_0(\alpha,\varepsilon)>0$ independent of M, M_t satisfying

$$\frac{B^{-1}D+(B^{-1}D)^t}{2}\xi\cdot\xi\geq\lambda_0|\xi|^2,\qquad\forall\xi\in R^3.$$

Define $\Sigma=\{V\mid |||V|||_{T_0}\leq q\}$, for some constants T_0, q to be specified. For arbitrary $T_0, q>0$ and $V\in\Sigma$, consider the following problem

$$\begin{cases}B(V,V_t,M^0)U_t=\left[\alpha I+A(V+M^0)\right]U_{xx}+F(V,V_x), & (t,x)\in(0,T]\times\Omega,\\ U(x,0)=0, & x\in\Omega.\end{cases}\tag{3.7}$$

Lemma 3.4 *(3.7) has a local solution in* Σ.

Proof Denote

$$B(V)=B(V,V_t,M^0), D(V)=\alpha I+A(V+M^0).$$

For arbitrary $V\in\Sigma$, from Lemma 3.1, it follows that

$$\begin{aligned}|||U|||^2_{T_0} &\leq C\int_0^{T_0}(\|B^{-1}(V)F(V,V_x)\|^2_{H^2}+\|[B^{-1}(V)F(V,V_x)]_t\|^2)dt\\ &\quad +C\int_0^{T_0}(\|(B^{-1}(V)D(V))_xU_{xxx}\|^2+\|(B^{-1}(V)D(V))_{xx}U_{xx}\|^2\\ &\quad +\|(B^{-1}(V)D(V))_tU_{xx}\|^2)dt\\ &\leq Cq^4+Cq^2T_0+Cq^2|||U|||^2_{T_0}\\ &\leq Cq^4+Cq^2T_0+\frac{1}{2}|||U|||^2_{T_0}.\end{aligned}$$

Consequently

$$|||U|||^2_{T_0}\leq Cq^4+Cq^2T_0\leq q^2$$

if $q\leq q_*$ and $T\leq T_*$, for some sufficiently small constants q_*, T_*,depending only on $\varepsilon,\alpha,\eta,\lambda_0,\|M^0\|_{H^4}$.

For arbitrary $V^1, V^2\in\Sigma, U^i(i=1,2)$ are the corresponding solutions of the problem

(3.7), then we have

$$
\begin{aligned}
|||U^1-U^2|||_{T_0}^2 \le & C\int_0^{T_0}\|[B^{-1}(V^1)D(V^1)-B^{-1}(V^2)D(V^2)][U_{xx}^2+F(V^2,V_x^2)]\|_{H^2}^2dt\\
&+C\int_0^{T_0}\|\{[B^{-1}(V^1)D(V^1)-B^{-1}(V^2)D(V^2)][U_{xx}^2+F(V^2,V_x^2)]\}_t\|^2dt\\
&+C\int_0^{T_0}\|B^{-1}(V^1)D(V^1)[F(V^1,V_x^1)-F(V^2,V_x^2)]\|_{H^2}^2dt\\
&+C\int_0^{T_0}\|[B^{-1}(V^1)D(V^1)(F(V^1,V_x^1)-F(V^2,V_x^2))]_t\|^2dt\\
&+C\int_0^{T_0}\{\|[B^{-1}(V^1)D(V^1)]_x(U_{xxx}^1-U_{xxx}^2)\|^2\\
&+\|[B^{-1}(V^1)D(V^1)]_{xx}(U_{xx}^1-U_{xx}^2)\|^2\\
&+\|[B^{-1}(V^1)D(V^1)]_t(U_{xx}^1-U_{xx}^2)\|^2\}dt\\
\le & Cq^2|||V^1-V^2|||_{T_0}^2+Cq^2|||U^1-U^2|||_{T_0}^2\\
\le & Cq^2|||V^1-V^2|||_{T_0}^2+\frac{1}{2}|||U^1-U^2|||_{T_0}^2,
\end{aligned}
$$

i.e.

$$|||U^1-U^2|||_{T_0}^2\le Cq^2|||V^1-V^2|||_{T_0}^2\le\frac{1}{4}|||V^1-V^2|||_{T_0}^2$$

if $q\le q_*$ and $T\le T_*$, for some smaller constants q_*,T_*,depending only on $\varepsilon,\alpha,\eta,\lambda_0$, $\|M^0\|_{H^4}$. By the fixed point theorem, we know (3.6) has a local solution in Σ.

Lemma 3.5 *With $M^0\in H^5(\Omega)$, the solution of (1.4) is unique.*

Proof Let M^1, M^2 be the solutions of (1.4), i.e.

$$M_t^1=A(M^1)(M_{xx}^1-\eta\frac{M_t^1}{\sqrt{\varepsilon^2+|M_t^1|^2}}),$$

$$M_t^2=A(M^2)(M_{xx}^2-\eta\frac{M_t^2}{\sqrt{\varepsilon^2+|M_t^2|^2}}).$$

Let $M=M^1-M^2$, then M satisfies

$$
\begin{cases}
[I+\eta A(M^1)\int_0^1\hat{B}(\hat{M}_t)\mid_{\hat{M}_t=M_t^2+\lambda(M_t^1-M_t^2)}d\lambda]M_t,\\
\qquad\qquad =A(M^1)M_{xx}-A(M^2-\dfrac{\eta M_t^2}{\sqrt{\varepsilon^2+|M_t^2|^2}})M\\
M(x,0)=0,
\end{cases}
\tag{3.8}
$$

with $b = \sqrt{\varepsilon^2 + |\hat{M}_t|^2}, \hat{M}^t = (\hat{M}_1, \hat{M}_2, \hat{M}_3)$. We have

$$\hat{B}(\hat{M}_t) = \frac{1}{b^3} \begin{pmatrix} b^2 - \hat{M}_{t1}^2 & -\hat{M}_{t1}\hat{M}_{t2} & -\hat{M}_{t1}\hat{M}_{t3} \\ -\hat{M}_{t1}\hat{M}_{t2} & b^2 - \hat{M}_{t2}^2 & -\hat{M}_{t1}\hat{M}_{t3} \\ -\hat{M}_{t1}\hat{M}_{t3} & -\hat{M}_{t1}\hat{M}_{t3} & b^2 - \hat{M}_{t3}^2 \end{pmatrix}.$$

It is easy to see $\hat{B}(\hat{M}_t) > \frac{\varepsilon^2}{b^3}$. On the other hand, with

$$R = I + \eta A(M^1) \int_0^1 \hat{B}(\hat{M}_t) d\lambda,$$

$$\begin{aligned} R^{-1}A(M^1) + A^t(M^1)R^{-t} &= R^{-1}[A(M^1)R^t + A^t(M^1)R]R^{-t} \\ &= 2\eta R^{-1}A(M^1) \int_0^1 \hat{B}(\hat{M}_t) d\lambda A^t(M^1)R^{-t} \\ &\geq 0. \end{aligned}$$

Hence, the uniqueness of (3.8) immediately comes from Lemma 3.2 and hence $M^1(x,t) \equiv M^2(x,t)$.

The end of the proof for Theorem: The proof for (1) in Theorem is the immediate consequence of Lemma 2.3, Lemma 2.4, Remark and Lemma 2.7. Although the local existence in Lemma 3.4 depends on α and ε, in view of a priori α-independent estimates, the extension method of local solutions gives: for arbitrary $T > 0$, (1.6) has a periodic solution in $\bigcap_{s=0}^{[\frac{k}{2}]} W^{s,\infty}([0,T]; H^{k-2s}(\Omega))$. Then by passing to the limit $\alpha \to 0$ and Lemma 3.5, (2) in Theorem is proved. At last, passing to the limit $\varepsilon \to 0$ ends the proof of Theorem.

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