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EXISTENCE AND UNIQUENESS OF BV SOLUTIONS FOR THE POROUS MEDIUM EQUATION WITH DIRAC MEASURE AS SOURCES

Yuan Hongjun
(Department of Mathematics, Jilin University, Changchun 130012, China)
(E-mail: hjy@mail.jlu.edu.cn)
Jin Yang
(School of Statistics, Renmin University of China, Beijing 100872, China)
(E-mail: jinyange_mail@sina.com)
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Abstract The aim of this paper is to discuss the existence and uniqueness of solutions for the porous medium equation

$$u_t - (u^m)_{xx} = \mu(x) \qquad \text{in} \quad (x,t) \in \mathbb{R} \times (0,+\infty)$$

with initial condition

$$u(x,0) = u_0(x) \qquad x \in (-\infty,+\infty),$$

where $\mu(x)$ is a nonnegative finite Radon measure, $u_0 \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$ is a nonnegative function, and $m > 1$, and $\mathbb{R} \equiv (-\infty,+\infty)$.
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1. Introduction

In this paper we consider the porous medium equation

$$u_t - (u^m)_{xx} = \mu(x) \qquad \text{in} \quad Q \tag{1.1}$$

with initial condition

$$u(x,0) = u_0(x) \qquad x \in \mathbb{R}, \tag{1.2}$$

where $\mu(x)$ is a nonnegative finite Radon measure, $u_0 \in L^1(\mathbb{R})) \cap L^\infty(\mathbb{R}))$ is a nonnegative function, $m > 1$,$Q \equiv \mathbb{R} \times (0,+\infty)$.

We denote

$$M_0 \equiv ||u_0||_{L^\infty(\mathbb{R})} + 1, \quad M_1 \equiv \int_{\mathbb{R}} d\mu$$

in this paper.

Clearly, the Cauchy problem (1.1)–(1.2) has no classical solutions in general. Therefore we consider its weak solutions.

Definition 1.1 *A nonnegtive function* $u : Q \longmapsto \mathbb{R}$ *is said to be a solution of (1.1) if* u *satisfies the following conditions* [H1] *and* [H2]*:*

[H1] *For all* $T \in (0, +\infty)$*, we have*

$$u \in L^\infty(0,T; L^1(\mathbb{R})) \cap BV_t(Q_T \setminus Q_s),$$

and

$$u(\cdot, t) \in C^\alpha(R) \quad \forall t \in (0,T)$$

with $s \in (0,T)$*, where* $Q_T \equiv \mathbb{R} \times (0,T)$*.*

[H2] *For any* $\phi \in C_0^\infty(Q_T)$*, we have*

$$\int\int_{Q_T} (-u\phi_t - u^m \phi_{xx})dxdt = \int\int_{Q_T} \phi(x,t)\mu(x)dxdt.$$

Definition 1.2 *A nonnegative function* $u : Q \longmapsto (0, +\infty)$ *is said to be a solution of (1.1)-(1.2) if* u *is a solution of (1.1) and satisfies the initial condition (1.2) in the following sense:*

$$ess \lim_{t \to 0^+} \int_{\mathbb{R}} \psi(x)u(x,t)dx = \int_{\mathbb{R}} \psi(x)u_0(x)dx, \quad \forall \psi \in C_0^\infty(\mathbb{R}).$$

Our main results are the following theorems.

Theorem 1.1 *The Cauchy problem (1.1)–(1.2) has a unique a solution* $u = u(x,t)$ *satisfying*

$$u(x,t) \le \frac{C}{t^{m-\delta}} \qquad \forall (x,t) \in Q_T$$

for all $\delta \in (0,1)$*, where* C *is a positive constant depending only on* δ*,* m*,* M_0 *and* M_1*.*

Such kind of results has been obtained by a number of authours, for example, see [1–11].

Remark 1.1 The proof of the existence in Theorem 1.1 is different from that of [1–8],it is based some BV estimates. In particular, the uniqueness in Theorem 1.1 is very intereting and is also different from that of [9–11].

In addition, we have

Theorem 1.2 *Assume that* u *is the solution of (1.1)-(1.2). Then we have*

$$|u(x_1,t) - u(x_2,t)| \le C|x_1 - x_2|^\beta$$

for all $x_i \in \mathbb{R}(i = 1,2)$ *and all* $t \in (\tau, +\infty)$*,where* $\beta \in (0,1)$ *and* $C > 0$ *are some positive constants depending only on* τ*,* M_0 *and* M_1*.*

The proofs of Theorem 1.1–1.2 are completed in Section 3–5. In proving process we shall use some uniform estimates in Section 2.

2. Some Estimates of Approximate Solutions

In order to discuss the existence of solutions for the Cauchy Problem (1.1)-(1.2), we consider the following equations of the form

$$u_t - (u^m)_{xx} = \mu_\varepsilon(x) \quad \text{in} \quad Q \tag{2.1}$$

with initial condition

$$u(x,0) = u_{0\varepsilon}(x) \tag{2.2}$$

where $\mu_\varepsilon = j_\varepsilon * (\mu(x)+\varepsilon)$, $u_{0\varepsilon}(x) = j_\varepsilon * u_0(x) + \varepsilon$, and

$$j_\varepsilon(x) = \frac{1}{\varepsilon} j\left(\frac{x}{\varepsilon}\right), \quad 0<\varepsilon<1, \tag{2.3}$$

$$j(x) = \begin{cases} \delta_0 \exp\left\{\dfrac{x^2}{x^2-1}\right\}, & |x|<1, \\ 0, & |x|\geq 1. \end{cases} \tag{2.4}$$

$$\delta_0 \int_{-\infty}^{+\infty} j(x)dx = 1. \tag{2.5}$$

It is well known that the Cauchy problem (2.1)-(2.2) has a unique nonnegative bounded solution $u_\varepsilon \in C^\infty(\overline{Q_T})$ with $u_\varepsilon \geq \varepsilon$ in Q_T for all $T\in(0,+\infty)$.

In addition, we have the following estimates on u_ε.

Lemma 2.1 *We have*

$$\frac{\partial u_\varepsilon}{\partial t} \geq -\frac{k u_\varepsilon}{t}, \tag{2.6}$$

where $k = \dfrac{1}{m-1}$.

Remark 2.1 Such estimates have been obtained by a number of authors, for some quasilinear degenerate pralolic equations or quasilinear hyperbolic equations, see [12-20].

Proof of Lemma 2.1 Denote

$$u_{\varepsilon r}(x,t) = r^{\frac{1}{m-1}} u_\varepsilon(x,rt), \quad \forall r \in \left(\frac{1}{2},1\right).$$

By (1.1) we compute

$$(u_{\varepsilon r})_t - (u_{\varepsilon r}^m)_{xx} = r^{\frac{m}{m-1}}\mu_\varepsilon(x) \leq \mu_\varepsilon(x), \quad \forall r\in\left(\frac{1}{2},1\right), \tag{2.7}$$

$$u_{\varepsilon r}(x,0) = r^{\frac{1}{m-1}} u_{0\varepsilon}(x) \leq u_{0\varepsilon}(x), \quad \forall x\in\mathbb{R}. \tag{2.8}$$

Applying the comparison principle and (2.1)-(2.2) with (2.7)-(2.8),we obtain

$$u_{\varepsilon r}(x,t) \leq u_\varepsilon(x,t), \quad \forall (x,t)\in Q.$$

By the definition of $u_{\varepsilon r}$ we have

$$r^{\frac{1}{m-1}}u_\varepsilon(x,rt)\le u_\varepsilon(x,t),\quad \forall(x,t)\in Q,$$

which implies that

$$\frac{u_\varepsilon(x,t)-u_\varepsilon(x,rt)}{t-rt}\ge\frac{r^{\frac{1}{m-1}}-1}{t(1-r)}u_\varepsilon(x,t),\quad \forall(x,t)\in Q,\forall r\in\left(\frac{1}{2},1\right)$$

Letting $r\to 1^-$,we get

$$\frac{\partial u_\varepsilon}{\partial t}\ge -\frac{ku_\varepsilon}{t}.$$

Thus the proof is completed.

Lemma 2.2 *We have*

$$\int_{\mathbb{R}}(u_\varepsilon(x,t)-M_0)_+dx\le TM_1$$

for all $t\in(0,T)$.

Proof Choose a number of functions $\xi_R\in C_0^\infty(R)$ with $R>2$ such that

$$\xi_R=1\quad\text{in}\quad(-R,R);\quad \xi_R=0\quad\text{in}\quad\mathbb{R}\setminus(-2R,2R),$$

$$0\le\xi_R\le 1,\quad |\xi_R^{'}|\le\frac{C}{R},\quad |\xi_R^{''}|\le\frac{C}{R^2}\quad\text{in}\quad\mathbb{R}. \tag{2.9}$$

We multiply (2.1) by $\dfrac{\xi_R(u_\varepsilon-M_0)_+^m}{(u_\varepsilon-M_0)_+^m+\eta}$ $(0<\eta<1)$ and integrate over in Q_T to obtain

$$\begin{aligned}&\int\int_{Q_T}u_{\varepsilon t}\cdot\frac{\xi_R(u_\varepsilon-M_0)_+^m}{(u_\varepsilon-M_0)_+^m+\eta}dxdt-\int\int_{Q_T}u_\varepsilon^m)_{xx}\cdot\frac{\xi_R(u_\varepsilon-M_0)_+^m}{(u_\varepsilon-M_0)_+^m+\eta}dxdt\\&\quad=\int\int_{Q_T}\mu_\varepsilon(x)\cdot\frac{\xi_R(u_\varepsilon-M_0)_+^m}{(u_\varepsilon-M_0)_+^m+\eta}dxdt.\end{aligned} \tag{2.10}$$

We compute

$$\begin{aligned}&\int\int_{Q_T}u_{\varepsilon t}\cdot\frac{\xi_R(u_\varepsilon-M_0)_+^m}{(u_\varepsilon-M_0)_+^m+\eta}dxdt\\&=\int\int_{Q_T}\frac{\partial}{\partial t}\left(\xi_R\int_0^{(u_\varepsilon-M_0)_+}\frac{s^mds}{s^m+\eta}\right)dxdt\\&=\int_{\mathbb{R}}\xi_R\left(\int_0^{(u_\varepsilon(x,T)-M_0)_+}\frac{s^mds}{s^m+\eta}\right)dx-\int_{\mathbb{R}}\xi_R\left(\int_0^{(u_{0\varepsilon}(x)-M_0)_+}\frac{s^mds}{s^m+\eta}\right)dx\\&=\int_{\mathbb{R}}\xi_R\left(\int_0^{(u_\varepsilon(x,T)-M_0)_+}\frac{s^mds}{s^m+\eta}\right)dx\end{aligned}$$

which implies that

$$\int\int_{Q_T} u_{\varepsilon t}\cdot\frac{\xi_R(u_\varepsilon-M_0)_+^m}{(u_\varepsilon-M_0)_+^m+\eta}dxdt=\int_{\mathbb{R}}\xi_R\left(\int_0^{(u_\varepsilon(x,T)-M_0)_+}\frac{s^m ds}{s^m+\eta}\right)dx. \tag{2.11}$$

In addition, we have

$$\begin{aligned}
&-\int\int_{Q_T}(u_\varepsilon^m)_{xx}\cdot\frac{\xi_R(u_\varepsilon-M_0)_+^m}{(u_\varepsilon-M_0)_+^m+\eta}dxdt\\
&=\int\int_{Q_T} mu_\varepsilon^{m-1}u_{\varepsilon x}\left(\frac{\xi_R(u_\varepsilon-M_0)_+^m}{(u_\varepsilon-M_0)_+^m+\eta}\right)_x dxdt\\
&=\int\int_{Q_T} mu_\varepsilon^{m-1}u_{\varepsilon x}\left(\frac{\eta\xi_R[(u_\varepsilon-M_0)_+^m]_x}{[(u_\varepsilon-M_0)_+^m+\eta]^2}\right)dxdt\\
&\quad+\int\int_{Q_T} mu_\varepsilon^{m-1}u_{\varepsilon x}\left(\frac{\xi_{Rx}(u_\varepsilon-M_0)_+^m}{(u_\varepsilon-M_0)_+^m+\eta}\right)dxdt\\
&=\int\int_{Q_T}\frac{\eta\xi_R\cdot m(u_\varepsilon-M_0)_+^{m-1}\cdot mu_\varepsilon^{m-1}\cdot|[(u_\varepsilon-M_0)_+]_x|^2}{[(u_\varepsilon-M_0)_+^m+\eta]^2}dxdt\\
&\quad+\int\int_{Q_T}\xi_{Rx}\frac{\partial}{\partial x}\left(\int_0^{u_\varepsilon}\frac{ms^{m-1}(s-M_0)_+^m}{(s-M_0)_+^m+\eta}ds\right)dxdt\\
&\geq-\int\int_{Q_T}\xi_{Rxx}\left(\int_0^{u_\varepsilon}\frac{ms^{m-1}(s-M_0)_+^m}{(s-M_0)_+^m+\eta}ds\right)dxdt\\
&\geq-\frac{CT||u_\varepsilon||^m_{L^\infty(Q_T)}}{R},
\end{aligned}$$

which implies that

$$-\int\int_{Q_T}(u_\varepsilon^m)_{xx}\cdot\frac{\xi_R(u_\varepsilon-M_0)_+^m}{(u_\varepsilon-M_0)_+^m+\eta}dxdt\geq-\frac{CT||u_\varepsilon||^m_{L^\infty(Q_T)}}{R}, \tag{2.12}$$

where C is a positive constant depending only on m.

Clearly, we have

$$\int\int_{Q_T}\xi_R\frac{(u_\varepsilon-M_0)_+^m}{(u_\varepsilon-M_0)_+^m+\eta}\mu_\varepsilon(x)dxdt\leq T\int_{\mathbb{R}}d\mu(x). \tag{2.13}$$

Combining (2.12)-(2.13) with (2.11) we get

$$\int_{\mathbb{R}}\xi_R\left(\int_0^{(u_\varepsilon(x,T)-M_0)_+}\frac{s^m}{s^m+\eta}\right)dx\leq\frac{CT||u_\varepsilon||^m_{L^\infty(Q_T)}}{R}+T\int_{\mathbb{R}}d\mu(x).$$

Letting $\eta \to +0$ and $R \to +\infty$ we have

$$\int_{\mathbb{R}} (u_\varepsilon(x,T) - M_0)_+ dx \le T \int_{\mathbb{R}} d\mu(x)$$

for all $T \in (0, +\infty)$. Thus the proof is completed.

Theorem 2.1 *Assume that u_ε is the solution of (1.1)-(1.2). Then we have*

$$u_\varepsilon(x,t) \le \frac{\gamma_1}{t^{m-\delta}} \qquad \forall (x,t) \in Q_T$$

for all $\delta \in (0,1)$, where γ_1 is a constant depending only on δ, m, T, M_0 and M_1.

Proof For any $\delta \in (0,1)$, we can find a number $\alpha \in (0, 1/m)$ such that

$$m\alpha = \delta.$$

Clearly, we have

$$m(1-\alpha) > m-1. \tag{2.14}$$

Assume that

$$M_* = \sup_{(x,t)\in Q_T} [t^{1/[m(1-\alpha)]} u_\varepsilon(x,t)].$$

Therefore, there exists a point $(x_*, t_*) \in Q_T$ such that

$$u_\varepsilon(x_*, t_*) \geqq M_* - 1. \tag{2.15}$$

Denote

$$\eta_R(x) = \xi_R(x - x_*)$$

for all $x \in \mathbb{R}$ and all $R \ge 1$,where ξ_R is defined by (2.9).

By Lemma 2.1 and (2.1) we get

$$-(u_\varepsilon^m)_{xx} \le \frac{k u_\varepsilon}{t} + \mu_\varepsilon. \tag{2.16}$$

We multiply (2.16) by $\dfrac{\eta_R u_\varepsilon^{m\alpha}}{u_\varepsilon^{m\alpha}+1}$ $(0 < \alpha < 1/m)$ and integrate over $\mathbb{R} \times (s,t)$ to obtain

$$-\int_s^t \int_{\mathbb{R}} (u_\varepsilon^m)_{xx} \frac{\eta_R u_\varepsilon^{m\alpha}}{u_\varepsilon^{m\alpha}+1} dx d\tau \le \int_s^t \int_{\mathbb{R}} \frac{k u_\varepsilon}{\tau} \frac{\eta_R u_\varepsilon^{m\alpha}}{u_\varepsilon^{m\alpha}+1} dx d\tau + \int_s^t \int_{\mathbb{R}} \mu_\varepsilon \frac{\eta_R u_\varepsilon^{m\alpha}}{u_\varepsilon^{m\alpha}+1} dx d\tau. \tag{2.17}$$

We compute

$$\begin{aligned} &-\int_s^t \int_{\mathbb{R}} (u_\varepsilon^m)_{xx} \frac{\eta_R u_\varepsilon^{m\alpha}}{u_\varepsilon^{m\alpha}+1} dx d\tau \\ &\quad = \int_s^t \int_{\mathbb{R}} (u_\varepsilon^m)_x \left[\frac{\eta_R u_\varepsilon^{m\alpha}}{u_\varepsilon^{m\alpha}+1}\right]_x dx d\tau \end{aligned}$$

$$
\begin{aligned}
&=\int_s^t\int_{\mathbb{R}}\eta_R\frac{(u_\varepsilon^{m\alpha})_x(u_\varepsilon^m)_x}{(u_\varepsilon^{m\alpha}+1)^2}dxd\tau+\int_s^t\int_{\mathbb{R}}(u_\varepsilon^m)_x\frac{\eta_R' u_\varepsilon^{m\alpha}}{u_\varepsilon^{m\alpha}+1}dxd\tau\\
&=\frac{4\alpha}{(1-\alpha)^2}\int_s^t\int_{\mathbb{R}}\eta_R\frac{u_\varepsilon^{2m\alpha}}{(u_\varepsilon^{m\alpha}+1)^2}\left|\left[u_\varepsilon^{\frac{m(1-\alpha)}{2}}\right]_x\right|^2dxd\tau\\
&\quad+\int_s^t\int_{\mathbb{R}}\eta_R'\frac{\partial}{\partial x}\left(\int_0^{u_\varepsilon^m}\frac{s^\alpha}{s^\alpha+1}ds\right)dxd\tau\\
&=\frac{4\alpha}{(1-\alpha)^2}\int_s^t\int_{\mathbb{R}}\eta_R\frac{u_\varepsilon^{2m\alpha}}{(u_\varepsilon^{m\alpha}+1)^2}\left|\left[u_\varepsilon^{\frac{m(1-\alpha)}{2}}\right]_x\right|^2dxd\tau\\
&\quad-\int_s^t\int_{\mathbb{R}}\eta_R''\left(\int_0^{u_\varepsilon^m}\frac{s^\alpha}{s^\alpha+1}ds\right)dxd\tau\\
&\geq\frac{4\alpha}{(1-\alpha)^2}\int_s^t\int_{\mathbb{R}}\eta_R\frac{u_\varepsilon^{2m\alpha}}{(u_\varepsilon^{m\alpha}+1)^2}\left|\left[u_\varepsilon^{\frac{m(1-\alpha)}{2}}\right]_x\right|^2dxd\tau-\int_s^t\int_{\mathbb{R}}|\eta_R''|u_\varepsilon^m dxd\tau,
\end{aligned}
$$

which implies that

$$
\begin{aligned}
&-\int_s^t\int_{\mathbb{R}}(u_\varepsilon^m)_{xx}\frac{\eta_R u_\varepsilon^{m\alpha}}{u_\varepsilon^{m\alpha}+1}dxd\tau\\
&\quad\leq\frac{4\alpha}{(1-\alpha)^2}\int_s^t\int_{\mathbb{R}}\eta_R\frac{u_\varepsilon^{2m\alpha}}{(u_\varepsilon^{m\alpha}+1)^2}\left|\left[u_\varepsilon^{\frac{m(1-\alpha)}{2}}\right]_x\right|^2dxd\tau+\int_s^t\int_{\mathbb{R}}|\eta_R''|u_\varepsilon^m dxd\tau. \qquad (2.18)
\end{aligned}
$$

Denote

$$
I_R=(x^*-2R,x^*+2R). \qquad (2.19)
$$

By (2.19) and (2.14), we compute

$$
\begin{aligned}
\int_s^t\int_{\mathbb{R}}|\eta_R''|u_\varepsilon^m dxd\tau&=\int_s^t\int_{\mathbb{R}}|\eta_R''|\tau^{1/(m(1-\alpha))}u_\varepsilon|^{m-1}\cdot\tau^{-(m-1)/(m(1-\alpha))}u_\varepsilon dxd\tau\\
&\leq s^{-(m-1)/(m(1-\alpha))}M_*^{m-1}\int_s^t\int_{\mathbb{R}}|\eta_R''|u_\varepsilon dxd\tau\\
&\leq s^{-1}M_*^{m-1}\int_s^t\int_{\mathbb{R}}|\eta_R''|u_\varepsilon dxd\tau\\
&=s^{-1}M_*^{m-1}\int_s^t\int_{I_R\cap\{u_\varepsilon>M_0\}}|\eta_R''|u_\varepsilon dxd\tau\\
&\quad+s^{-1}M_*^{m-1}\int_s^t\int_{I_R\cap\{u_\varepsilon\leq M_0\}}|\eta_R''|u_\varepsilon dxd\tau\\
&=s^{-1}M_*^{m-1}\int_s^t\int_{I_R\cap\{u_\varepsilon>M_0\}}|\eta_R''|(u_\varepsilon-M_0)dxd\tau
\end{aligned}
$$

$$+ s^{-1}M_*^{m-1}\int_s^t\int_{I_R\cap\{u_\varepsilon>M_0\}}|\eta_R''|M_0dxd\tau$$
$$+ s^{-1}M_*^{m-1}\int_s^t\int_{I_R\cap\{u_\varepsilon\le M_0\}}|\eta_R''|u_\varepsilon dxd\tau$$
$$\le \frac{CM_1TM_*^{m-1}(t-s)}{sR^2}+\frac{CM_0M_*^{m-1}(t-s)}{sR}$$

which implies that

$$\int_s^t\int_{\mathbb{R}}|\eta_R''|u_\varepsilon^m dxd\tau \le \frac{CM_*^{m-1}(t-s)}{sR^2}+\frac{CM_*^{m-1}(t-s)}{sR}, \tag{2.20}$$

where C is a positive constant depending only on m,T,M_0 and M_1. Combining (2.18)-(2.19) with (2.20) we get

$$-\int_s^t\int_{\mathbb{R}}(u_\varepsilon^m)_{xx}\frac{\eta_R u_\varepsilon^{m\alpha}}{u_\varepsilon^{m\alpha}+1}dxd\tau$$
$$\ge\frac{4\alpha}{(1-\alpha)^2}\int_s^t\int_{\mathbb{R}}\eta_R\frac{u_\varepsilon^{2m\alpha}}{(u_\varepsilon^{m\alpha}+1)^2}\left|\left[u_\varepsilon^{\frac{m(1-\alpha)}{2}}\right]_x\right|^2dxd\tau-\frac{CM_*^{m-1}(t-s)}{sR} \tag{2.21}$$

for all $R\ge 2$,where C is a positive constant depending only on m,T,M_0 and M_1.

We have

$$\int_s^t\int_{\mathbb{R}}\frac{ku_\varepsilon}{\tau}\frac{\eta_R u_\varepsilon^{m\alpha}}{u_\varepsilon^{m\alpha}+1}dxd\tau$$
$$\le\frac{k}{s}\int_s^t\int_{\mathbb{R}}\eta_R(u_\varepsilon-M_0)_+dxd\tau+\frac{k}{s}\int_s^t\int_{\mathbb{R}}\eta_R M_0dxd\tau$$
$$\le CRs^{-1}(t-s) \tag{2.22}$$

for all $R\ge 2$, where C is a positive contant depending only on m,T,M_0 and M_1. On the other hand, we also have

$$\int_s^t\int_{\mathbb{R}}\mu_\varepsilon\frac{\eta_R u_\varepsilon^{m\alpha}}{u_\varepsilon^{m\alpha}+1}dxd\tau\le M_1(t-s). \tag{2.23}$$

Combining (2.21)-(2.23) with (2.17) we conclude that

$$\frac{1}{t-s}\int_s^t\int_{\mathbb{R}}\frac{\eta_R u_\varepsilon^{2m\alpha}}{(u_\varepsilon^{m\alpha}+1)^2}\left|\left[u_\varepsilon^{\frac{m(1-\alpha)}{2}}\right]_x\right|^2dxd\tau]\le\frac{(1-\alpha)^2}{4\alpha}\left[\frac{CM_*^{m-1}}{sR}+CRs^{-1}+M_1\right]$$

for all s,t such that $0<s<t<T$. Let $s\uparrow t$ and obtain

$$\int_{\mathbb{R}}\frac{\eta_R u_\varepsilon^{2m\alpha}}{(u_\varepsilon^{m\alpha}+1)^2}\left|\left[u_\varepsilon^{\frac{m(1-\alpha)}{2}}\right]_x\right|^2(x,t)dx\le\frac{CR(1-\alpha)^2}{\alpha t}\left[M_*^{m-1}+1\right] \tag{2.24}$$

for all $t\in(0,T)$ and all $R\ge 2$.

On the other hand, we also have

$$\begin{aligned}
&\int_{\mathbb{R}} \frac{\eta_R u_\varepsilon^{2m\alpha}}{(u_\varepsilon^{m\alpha}+1)^2}\left|\left[u_\varepsilon^{\frac{m(1-\alpha)}{2}}\right]_x\right|^2 (x,t)dx \\
&\qquad\geq \int_{\mathbb{R}\cap\{u_\varepsilon>1\}} \frac{\eta_R u_\varepsilon^{2m\alpha}}{(u_\varepsilon^{m\alpha}+1)^2}\left|\left[u_\varepsilon^{\frac{m(1-\alpha)}{2}}\right]_x\right|^2 (x,t)dx \\
&\qquad\geq \frac{1}{2}\int_{\mathbb{R}\cap\{u_\varepsilon>1\}} \eta_R\left|\left[u_\varepsilon^{\frac{m(1-\alpha)}{2}}\right]_x\right|^2 (x,t)dx \\
&\qquad= \frac{1}{2}\int_{\mathbb{R}\cap\{u_\varepsilon>1\}} \eta_R\left|\left[(u_\varepsilon^{\frac{m(1-\alpha)}{2}}-1)_+\right]_x\right|^2 (x,t)dx \\
&\qquad= \frac{1}{2}\int_{\mathbb{R}} \eta_R\left|\left[(u_\varepsilon^{\frac{m(1-\alpha)}{2}}-1)_+\right]_x\right|^2 (x,t)dx
\end{aligned}$$

By (2.24), we get

$$\int_{\mathbb{R}} \eta_R(x)\left|\left[(u_\varepsilon^{\frac{m(1-\alpha)}{2}}(x,t)-1)_+\right]_x\right|^2 dx \leq Ct^{-1}(M_*^{m-1}+1)$$

and then

$$\int_{\mathbb{R}} t\xi_R(x)\left|\left[(u_\varepsilon^{\frac{m(1-\alpha)}{2}}(x+x_*,t)-1)_+\right]_x\right|^2 dx \leq C(M_*^{m-1}+1) \tag{2.25}$$

for all $t\in(0,T)$ and $\alpha\in(0,1/m)$,where C is a positive constant depending only on R,α,m,M_0 and M_1.

Using (2.15) we compute

$$\begin{aligned}
M_*^{m(1-\alpha)/2} &\leq \left(t_*^{1/[m(1-\alpha)]}u_\varepsilon(x_*,t_*)+1\right)^{m(1-\alpha)/2} \\
&\leq C\left(t_*^{1/2}u_\varepsilon^{m(1-\alpha)/2}(x_*,t_*)+1\right) \\
&\leq C\left\{t_*^{1/2}[(u_\varepsilon^{m(1-\alpha)/2}(x_*,t_*)-1)_+ +1]+1\right\} \\
&\leq C\left\{t_*^{1/2}(u_\varepsilon^{m(1-\alpha)/2}(x_*,t_*)-1)_+ +1\right\} \\
&= C\left\{t_*^{1/2}\xi_R(0)(u_\varepsilon^{m(1-\alpha)/2}(x_*,t_*)-1)_+ +1\right\} \\
&= C\left\{\int_{-2R}^{0}\frac{\partial}{\partial x}\left(t_*^{1/2}\xi_R(x)(u_\varepsilon^{m(1-\alpha)/2}(x+x_*,t_*)-1)_+\right)dx+1\right\} \\
&= C\left\{1+\int_{-2R}^{0}t_*^{1/2}\xi_R(x)\frac{\partial}{\partial x}\left((u_\varepsilon^{m(1-\alpha)/2}(x+x_*,t_*)-1)_+\right)dx\right.
\end{aligned}$$

$$
\begin{aligned}
&+\int_{-2R}^{0} t_*^{1/2}\xi_R'(x)\left((u_\varepsilon^{m(1-\alpha)/2}(x+x_*,t_*)-1)_+\right)dx\Bigg\} \\
\le & C\left\{1+t_*^{1/2}\left[\int_{-2R}^{0}\xi_R(x)\left|\frac{\partial}{\partial x}\left((u_\varepsilon^{m(1-\alpha)/2}(x+x_*,t_*)-1)_+\right)\right|^2dx\right]^{1/2}\right. \\
&\left.\cdot\left(\int_{-2R}^{0}\xi_R(x)dx\right)^{1/2}+\int_{-2R}^{0}t_*^{1/2}|\xi_R'(x)|(u_\varepsilon^{m(1-\alpha)/2}(x+x_*,t_*)dx\right\} \\
\le & C\left\{1+[C(M_*^{m-1}+1)]^{1/2}\cdot R^{1/2}+\int_{-2R}^{0}t_*^{1/2}|\xi_R'(x)|u_\varepsilon^{m(1-\alpha)/2}(x+x_*,t_*)dx\right\} \\
\le & C\left\{1+M_*^{(m-1)/2}+\int_{-2R}^{0}t_*^{1/2}|\xi_R'(x)|u_\varepsilon^{m(1-\alpha)/2}(x+x_*,t_*)dx\right\},
\end{aligned}
$$

which implies that

$$
M_*^{m(1-\alpha)/2}\le C\left\{1+M_*^{(m-1)/2}+\int_{-2R}^{0}t_*^{1/2}|\xi_R'(x)|u_\varepsilon^{m(1-\alpha)/2}(x+x_*,t_*)dx\right\}, \quad (2.26)
$$

where C is a positive constant depending only on R,α, T, m,M_0 and M_1.

On the other hand, for $m(1-\alpha)/2<1$, we have

$$
\begin{aligned}
&\int_{-2R}^{0}t_*^{1/2}|\xi_R'(x)|u_\varepsilon^{m(1-\alpha)/2}(x+x_*,t_*)dx \\
&\quad\le CR^{-1}\int_{-2R}^{0}u_\varepsilon^{m(1-\alpha)/2}(x+x_*,t_*)dx \\
&\quad\le C\left(\int_{-2R}^{0}u_\varepsilon(x+x_*,t_*)dx\right)^{2/[m(1-\alpha)]}\left(\int_{-2R}^{0}dx\right)^{1-2/[m(1-\alpha)]} \\
&\quad\le C\left(\int_{-2R}^{0}u_\varepsilon(x+x_*,t_*)dx\right)^{2/[m(1-\alpha)]} \\
&\quad\le C\left(\int_{-2R}^{0}(u_\varepsilon(x+x_*,t_*)-M_0)_+dx+\int_{-2R}^{0}M_0dx\right)^{2/[m(1-\alpha)]} \\
&\quad\le C,
\end{aligned}
$$

which implies that

$$
\int_{-2R}^{0}t_*^{1/2}|\xi_R'(x)|(u_\varepsilon^{m(1-\alpha)/2}(x+x_*,t_*)dx\le C \quad (2.27)
$$

for $m(1-\alpha)/2<1$, where C is a positive constant depending only on R,α, m,T,M_0 and M_1.

For $m(1-\alpha)/2 \geq 1$, we have

$$\begin{aligned}\int_{-2R}^{0} &|\xi_R'(x)| u_\varepsilon^{\frac{m(1-\alpha)}{2}}(x+x_*,t_*)dx \\ &\leq M_*^{m(1-\alpha)/2-1}\int_{-2R}^{0}|\xi_R'(x)|u_\varepsilon(x+x_*,t_*)dx + \int_{-2R}^{0}|\xi_R'(x)|M_0 dy \\ &\leq C\left(1+M_*^{m(1-\alpha)/2-1}\right),\end{aligned}$$

which implies that

$$\int_{-2R}^{0}|\xi_R'(x)|u_\varepsilon^{\frac{m(1-\alpha)}{2}}(x+x_*,t_*)dx \leq C\left(1+M_*^{m(1-\alpha)/2-1}\right) \tag{2.28}$$

for $m(1-\alpha)/2 \geq 1$, where C is a positive constant depending only on R,α, m,T,M_0 and M_1.

By (2.27) and (2.28), we obtain

$$\int_{-2R}^{0}|\xi_R'(x)|u_\varepsilon^{\frac{m(1-\alpha)}{2}}(x+x_*,t_*)dx \leq C\left(1+M_*^{(m(1-\alpha)/2-1)_+}\right), \tag{2.29}$$

where C is a positive constant depending only on R,α, m,T,M_0 and M_1.

By (2.26) and (2.29), we get

$$M_*^{m(1-\alpha)/2} \leq C\left\{1+M_*^{(m-1)/2}+C(1+M_*^{(m(1-\alpha)/2-1)_+})\right\},$$

where C is a positive constant depending only on R,α, m,T,M_0 and M_1. Applying Young inequality we conclude that

$$M_* \leq C,$$

where C is a positive constant depending only on R,α, m,T,M_0 and M_1. Thus the proof is completed.

Theorem 2.2 *Assume that u_ε is the solution of (1.1)-(1.2). For any $R \in (0,+\infty)$, we have*

$$|u_\varepsilon(x_1,t)-u_\varepsilon(x_2,t)| \leq \gamma_2|x_1-x_2|^\beta$$

for all $(x_1,t) \in (-R,R)\times(\tau,T)$ and all $(x_2,t) \in (-R,R)\times(\tau,T)$, where γ_2 and $\beta \in (0,1)$ are some positive constants depending only on τ, T, R,m,M_0 and M_1.

Proof We multiply (2.16) by $\xi_R u_\varepsilon^\alpha(x,t)$ $(0<\alpha<1)$ and integrate over $\mathbb{R}$ to obtain

$$-\int_{\mathbb{R}}(u_\varepsilon^m)_{xx}\cdot\xi_R u_\varepsilon^\alpha(x,t)dx \leq \int_{\mathbb{R}}kt^{-1}u_\varepsilon\cdot\xi_R u_\varepsilon^\alpha(x,t)dx + \int_{\mathbb{R}}\mu_\varepsilon\cdot\xi_R u_\varepsilon^\alpha(x,t)dx \tag{2.30}$$

for all $t \in (\tau,T)$.

By Theorem 2.1, we compute

$$
\begin{aligned}
&-\int_{\mathbb{R}}(u_\varepsilon^m)_{xx}\cdot\xi_R u_\varepsilon^\alpha(x,t)dx\\
&\qquad=\int_{\mathbb{R}}(u_\varepsilon^m)_x(\xi_R u_\varepsilon^\alpha(x,t))_x dx\\
&\qquad=\int_{\mathbb{R}}\xi_R(u_\varepsilon^m)_x(u_\varepsilon^\alpha(x,t))_x dx+\int_{\mathbb{R}}\xi_R'(u_\varepsilon^m)_x u_\varepsilon^\alpha(x,t)dx\\
&\qquad=\frac{4m\alpha}{(m+\alpha)^2}\int_{\mathbb{R}}\xi_R|(u_\varepsilon^{(m+\alpha)/2})_x|^2dx+\int_{\mathbb{R}}\xi_R'\left(\int_0^{u_\varepsilon(x,t)}ms^{m+\alpha-1}ds\right)dx\\
&\qquad\geq\frac{4m\alpha}{(m+\alpha)^2}\int_{\mathbb{R}}\xi_R|(u_\varepsilon^{(m+\alpha)/2})_x|^2dx-C,
\end{aligned}
$$

which implies that

$$
-\int_{\mathbb{R}}(u_\varepsilon^m)_{xx}\cdot\xi_R u_\varepsilon^\alpha(x,t)dx\geq\frac{4m\alpha}{(m+\alpha)^2}\int_{\mathbb{R}}\xi_R|(u_\varepsilon^{(m+\alpha)/2})_x|^2dx-C \tag{2.31}
$$

for all $t\in(\tau,T)$, where C is a positive constant depending only on τ, T, R,m,M_0 and M_1.

In addition, by Theorem 2.1, we also have

$$
\int_{\mathbb{R}}kt^{-1}u_\varepsilon\cdot\xi_R u_\varepsilon^\alpha(x,t)dx\leq C \tag{2.32}
$$

and

$$
\int_{\mathbb{R}}\mu_\varepsilon\cdot\xi_R u_\varepsilon^\alpha(x,t)dx\leq C \tag{2.33}
$$

where C is a positive constant depending only on τ, T, R,m, M_0 and M_1.

Combining (2.31)-(2.33) with (2.30) we obtain

$$
\int_{\mathbb{R}}\xi_R|(u_\varepsilon^{(m+\alpha)/2})_x|^2dx\leq C \tag{2.34}
$$

for all $t\in(\tau,T)$, where C is a positive constant depending only on τ, T, R,m,M_0 and M_1.

Choose

$$
\alpha=\begin{cases}1-\dfrac{m}{4} & \text{if } 1<m<2\\[2mm] \dfrac{1}{2} & \text{if } m\geq 2\end{cases}
$$

and have

$$
(m+\alpha)/2>1. \tag{2.35}
$$

For $(x_1,t)\in(-R,R)\times(\tau,T)$ and $(x_2,t)\in(-R,R)\times(\tau,T)$, using (2.34), (2.35) we compute

$$
|u_\varepsilon(x_1,t)-u_\varepsilon(x_2,t)|\leq|u_\varepsilon^{(m+\alpha)/2}(x_1,t)-u_\varepsilon^{(m+\alpha)/2}(x_2,t)|^{2/(m+\alpha)}
$$

$$
\begin{aligned}
&= \left| \int_{x_1}^{x_2} [u_\varepsilon^{(m+\alpha)/2}(x,t)]_x dx \right|^{2/(m+\alpha)} \\
&\le \left| \int_{x_1}^{x_2} |[u_\varepsilon^{(m+\alpha)/2}(x,t)]_x|^2 dx \right|^{1/(m+\alpha)} \left| \int_{x_1}^{x_2} 1 dx \right|^{1/(m+\alpha)} \\
&\le C\,|x_1 - x_2|^{1/(m+\alpha)},
\end{aligned}
$$

which implies that

$$
|u_\varepsilon(x_1,t) - u_\varepsilon(x_2,t)| \le C\,|x_1 - x_2|^{1/(m+\alpha)}
$$

for all $t \in (\tau, T)$, where C is a positive constant depending only on τ, T, R,m, M_0 and M_1. Thus the proof is completed.

Theorem 2.3 *Assume that u_ε is the solution of (1.1)-(1.2). For any $R \in (0,+\infty)$, and any $s \in (0,T)$ we have*

$$
\int_s^T \int_{-R}^{R} |u_{\varepsilon t}| dxdt \le \gamma_3
$$

where γ_3 is a positive constant depending only on R, s, T,m,M_0 and M_1.

Proof Applying (2.6) we get

$$
\frac{\partial}{\partial t}\left(t^k u_\varepsilon(x,t)\right) = kt^{k-1}u_\varepsilon + t^k u_{\varepsilon t} \ge 0. \tag{2.36}
$$

On the other hand, by (2.1),we have

$$
\frac{\partial}{\partial t}\left(t^k u_\varepsilon\right) = t^k (u_\varepsilon^m)_{xx} + kt^{k-1}u_\varepsilon + t^k \mu_\varepsilon. \tag{2.37}
$$

We multiply (2.37) by ξ_R in (2.9) and integrate over in $\mathbb{R} \times (s,T)$ $(0 < s < T < +\infty)$ to obtain

$$
\begin{aligned}
\int_s^T \int_{\mathbb{R}} \xi_R \frac{\partial}{\partial t}\left(t^k u_\varepsilon\right) dxdt &= \int_s^T \int_{\mathbb{R}} \xi_R t^k (u_\varepsilon^m)_{xx} dxdt \\
&\quad + \int_s^T \int_{\mathbb{R}} \xi_R k t^{k-1} u_\varepsilon dxdt + \int_s^T \int_{\mathbb{R}} \xi_R t^k \mu_\varepsilon dxdt.
\end{aligned} \tag{2.38}
$$

Applying Theorem 2.1 we compute

$$
\int_s^T \int_{\mathbb{R}} \xi_R t^k (u_\varepsilon^m)_{xx} dxdt = \int_S^T \int_{\mathbb{R}} \xi_R'' t^k u_\varepsilon^m dxdt \le C, \tag{2.39}
$$

where C is a positive constant depending only on s,T,R,m,M_0 and M_1. In addition, we have

$$
\int_s^T \int_{\mathbb{R}} \xi_R k t^{k-1} u_\varepsilon dxdt + \int_s^T \int_{\mathbb{R}} \xi_R t^k \mu_\varepsilon dxdt \le C, \tag{2.40}
$$

where C is a positive constant depending only on s, T, R m,M_0 and M_1. Combining (2.39) and (2.40) with (2.38) we conclude that

$$\int_s^T \int_{\mathbb{R}} \xi_R \frac{\partial}{\partial t}\left(t^k u_\varepsilon\right) dxdt \le C. \tag{2.41}$$

Using (2.39) and (2.41) we have

$$\begin{aligned}\int_s^T \int_{\mathbb{R}} \xi_R |u_{\varepsilon t}| dxdt \le & \int_s^T \int_{\mathbb{R}} \xi_R t^{-k} \left|\frac{\partial}{\partial t}\left(t^k u_\varepsilon\right) - kt^{k-1}u_\varepsilon\right| dxdt \\ \le & s^{-k}\int_s^T \int_{\mathbb{R}} \xi_R \frac{\partial}{\partial t}\left(t^k u_\varepsilon\right) dxdt + ks^{-1}\int_s^T \int_{\mathbb{R}} \xi_R u_\varepsilon dxdt \\ \le & Cs^{-k} + Cs^{-1}.\end{aligned}$$

Thus the proof is completed.

3. The Proof of the Existence in Theorem 1.1

In order to prove the existence in Theorem 1.1, we cosider the following problem

$$u_t - (u^m)_{xx} = \mu_\varepsilon(x) \quad \text{in} \quad Q, \tag{3.1}$$

$$u(x,0) = u_{0\varepsilon}(x) \quad \text{for } x \in \mathbb{R}, \tag{3.2}$$

where μ_ε and $u_{0\varepsilon}$ are defined by (2.1)-(2.2).Clearly,the Cauchy problem (3.1)-(3.2) has a unique nonnegtive bounded smooth solution u_ε. By Theorem 2.1 - Theorem 2.3, there exists a subsequent $\{u_{\varepsilon_j}\}$ of $\{u_\varepsilon\}$ such that, for any compact subset $K \subset Q$,

$$u_{\varepsilon_j} \to u \quad a.e. \text{in } K \quad \text{as } \varepsilon_j \to 0^+. \tag{3.3}$$

In addition,we also have

$$u \in L^\infty(0,T; L^1(\mathbb{R})) \tag{3.4}$$

and

$$u \in L^\infty(\mathbb{R} \times (s,T)), \quad u \in BV_t(\mathbb{R} \times (s,T)) \qquad \forall 0 < s < T < +\infty, \tag{3.5}$$

and

$$u(\cdot,t) \in C^\beta(\mathbb{R}) \quad \text{for some } \beta \in (0,1) \qquad \forall t \in (s,T) \quad \text{with} \quad 0 < s < T < +\infty. \tag{3.6}$$

For any $\varphi \in C_0^\infty(Q_T)$, we multipy (3.1) by φ and integrate over Q_T to obtain

$$\int\int_{Q_T} \varphi u_{\eta t} dxdt - \int\int_{Q_T} \varphi (u_\varepsilon^m)_{xx} dxdt = \int\int_{Q_T} \varphi \mu_\varepsilon(x) dxdt,$$

which implies that

$$\int\int_{Q_T} (-u_\varepsilon \varphi_t - \varphi_{xx} u_\varepsilon^m) dxdt = \int\int_{Q_T} \varphi \mu_\varepsilon(x) dxdt.$$

By (3.3), letting $\varepsilon=\varepsilon_j\to 0^+$ we have

$$\int\int_{Q_T}(-u\varphi_t-u^m\varphi_{xx})dxdt=\int\int_{Q_T}\varphi d\mu dt.$$

By Definition 1.1 we conclude that u is a solution of (1.1).

In addition, we shall prove that u is a solution of (1.1)–(1.2).

In fact, For any $\varphi\in C_0^\infty(\mathbb{R})$, we multiply (3.1) by ψ and integrate over Q_t to obtain

$$\int\int_{Q_t}\psi u_{\varepsilon t}dxd\tau-\int\int_{Q_t}\psi\left(u_\varepsilon^m\right)_{xx}dxd\tau=\int\int_{Q_t}\psi\mu_\varepsilon(x)dxd\tau. \tag{3.7}$$

We compute

$$\int\int_{Q_t}\psi u_{\eta t}dxd\ =\int_{\mathbb{R}}\psi u_\varepsilon(x,t)dx-\int_{\mathbb{R}}\psi u_{0\varepsilon}(x)dx. \tag{3.8}$$

By Theorem 2.1 and Lemma 2.2, we have

$$\begin{aligned}&\left|-\int\int_{Q_t}\psi(u_\varepsilon^m)_{xx}dxd\tau\right|\\ &=\left|-\int\int_{Q_t}\psi_{xx}u_\varepsilon^m dxd\tau\right|\\ &\le\int_0^t\left\|\psi_{xx}u_\varepsilon^{m-1}(\cdot,\tau)\right\|_{L^\infty(supp\psi)}\left(\int_{supp\psi}u_\varepsilon dx\right)d\tau\\ &\le C\int_0^t\left(C\tau^{1/(m-\delta)}\right)^{m-1}\left(\int_{supp\psi}(u_\varepsilon(x,\tau)-M_0)_+dx+\int_{supp\psi}(u_\varepsilon M_0dx\right)d\tau\\ &\le C\int_0^t\tau^{-(m-1)/(m-\delta)}d\tau\\ &\le Ct^{(1-\delta)/(m-\delta)}\end{aligned} \tag{3.9}$$

for all $\delta\in(0,1)$. In addition, we also have

$$\int\int_{Q_t}\psi\mu_\varepsilon(x)dxd\tau\le Ct. \tag{3.10}$$

Combining (3.8)-(3.10) with (3.7) we conclude that

$$\left|\int_{\mathbb{R}}\psi u_\varepsilon(x,t)dx-\int_{\mathbb{R}}\psi u_{0\varepsilon}(x)dx\right|\le Ct+Ct^{(1-\delta)/(m-\delta)}.$$

Letting $\varepsilon=\varepsilon_j\to 0^+$ and $t\to 0^+$ we get

$$\lim_{t\to 0^+}\int_{\mathbb{R}}\psi u(x,t)dx=\int_{\mathbb{R}}\psi u_0(x)dx.$$

By Definition 1.2, the proof is completed.

4. The Proof of the Uniqueness in Theorem 1.1

In this section we shall prove the uniqueness in Theorem 1.1.

We assume that u and v are the solutions of (1.1)-(1.2).

We define

$$A(x,t)=\begin{cases}\dfrac{u^m(x,t)-v^m(x,t)}{u(x,t)-v(x,t)} & \text{if}\quad u(x,t)\neq v(x,t)\\ 0, & \text{otherwise}\end{cases}$$

and

$$A_\varepsilon(x,t)=A(x,t)+\varepsilon,\quad 0<\varepsilon<1$$

and

$$A_{\varepsilon,\rho}(x,t)=(J_\rho * A_\varepsilon)(x,t),\quad 0<\rho<1,$$

where J_ρ is defined by

$$J_\rho\in C^\infty(\mathbb{R}\times\mathbb{R}),\int_{\mathbb{R}}\int_{\mathbb{R}}J_\rho(x,t)dxdt=1$$

with

$$\text{supp}J_\rho\subset\{(x,t):|x|<\rho,\qquad |t|<\rho\}.$$

Clearly, we have

$$\varepsilon\le A_{\varepsilon,\rho}(x,t)\le M$$

for all $(x,t)\in\mathbb{R}\times(s,T)$, where M is a positive constant depending only on $||u||_{L^\infty(\mathbb{R}\times(s,T))}$ and $||v||_{L^\infty(\mathbb{R}\times(s,T))}$.

For $\theta\in C_0^\infty(\mathbb{R})$ with $|\theta|\le 1$, we choose a positive number R such that

$$\theta\in C_0^\infty(B_{R-1}),$$

where $B_R\equiv\{x:|x|<R\}$ with $R>0$.

Now consider the following equations

$$\frac{\partial\psi}{\partial t}+A_{\varepsilon,\rho}\Delta\psi=0 \tag{4.1}$$

in $B_R\times(0,T)$ with the following initial-boundary values

$$\psi(x,t)=0\qquad\forall(x,t)\in\partial B_R\times(0,T) \tag{4.2}$$

and

$$\psi(x,T)=\theta(x)e^{-|x|}\qquad x\in B_R. \tag{4.3}$$

It is known that the problem (4.1)-(4.3) has a unique smooth solution $\psi_{\varepsilon,\rho}$. In order to prove the uniqueness in Theorem 1.1 we need the following lemmas.

Lemma 4.1 *The solution $\psi_{\varepsilon,\rho}$ of (4.1)-(4.3) satisfies the following inequalities*

$$|\psi_{\varepsilon,\rho}(x,t)| \le 1 \tag{4.4}$$

for all $(x,t) \in B_R \times (0,T)$, and

$$\int_{B_R} |\nabla\psi_{\varepsilon,\rho}(x,t)|^2 dx \le C \tag{4.5}$$

for all $t \in (0,T)$, and

$$\int_0^T \int_{B_R} A_{\varepsilon,\rho}(\Delta\psi_{\varepsilon,\rho})^2 dxdt \le C_0, \tag{4.6}$$

where C_0 is a positive constant depending only on θ.

Proof The inequality (4.4) follows from the maximum principle. In order to prove (4.5) and (4.6) we multiply (4.1) by $\Delta\psi_{\varepsilon,\rho}$ and integrate in $B_R \times (t,T)$ to obtain

$$\int_t^T \int_{B_R} \left\{(\Delta\psi_{\varepsilon,\rho})(\psi_{\varepsilon,\rho})_t + A_{\varepsilon,\rho}[\Delta\psi_{\varepsilon,\rho}]^2\right\} dxd\tau = 0$$

for all $t \in (0,T)$.

We compute

$$\int_t^T \int_{B_R} (\Delta\psi_{\varepsilon,\rho})(\psi_{\varepsilon,\rho})_t dxd\tau = -\frac{1}{2}\int_{B_R} |\nabla(\theta e^{-|x|})|^2 dx + \frac{1}{2}\int_{B_R} |\nabla\psi_{\varepsilon,\rho}|^2 dx$$

and then have

$$\frac{1}{2}\int_{B_R} |\nabla\psi_{\varepsilon,\rho}|^2 dx + \int_t^T \int_{B_R} A_{\varepsilon,\rho}[\Delta\psi_{\varepsilon,\rho}]^2 dxd\tau = \frac{1}{2}\int_{B_R} |\nabla(\theta e^{-|x|})|^2 dx,$$

which implies (4.5) and (4.6). Thus the proof is completed.

Lemma 4.2 *The solution $\psi_{\varepsilon,\rho}$ of (4.1)-(4.3) satisfies*

$$\psi_{\varepsilon,\rho}(x,t)| \le C_1 e^{-|x|} \tag{4.7}$$

for all $(x,t) \in B_R \times (s,T)$ with $s \in (0,T)$, where C_1 is a positive constant depending only on θ and $N(s,T)$, and

$$N(s,T) = ||u||_{L^\infty(\mathbb{R} \times (s,T))} + ||v||_{L^\infty(\mathbb{R} \times (s,T))}.$$

Proof We consider the following functions

$$w^{\pm}(x,t) = \mp\psi_{\varepsilon,\rho}(x,t) + e^{1-|x|+\nu(T-t)},$$

where $\nu > 0$ will be determined later.

From (4.1)-(4.3) and Lemma 4.1, we have

$$w^{\pm}(x,t) \ge 0$$

on $|x| = 1$ and $|x| = R$, and

$$w^{\pm}(x,T) = \mp\theta e^{-|x|} + e^{1-|x|+\nu(T-T)} \geq 0,$$

and

$$\begin{aligned}\frac{\partial w^{\pm}}{\partial t} + A_{\varepsilon,\rho}\Delta w^{\pm} =& \frac{\partial e^{1-|x|} + \nu(T-t)}{\partial t} + A_{\varepsilon,\rho}\Delta e^{1-|x|} + \nu(T-t) \\ =& e^{1-|x|} + \nu(T-t)\left\{A_{\varepsilon,\rho} - \nu\right\}.\end{aligned}$$

Therefore, we can choose ν depending only on $N(s,T)$ such that

$$\frac{\partial w^{\pm}}{\partial t} + A_{\varepsilon,\rho}\Delta w^{\pm} < 0$$

for all $(x,t) \in \{(-R,-1)\times(s,T)\} \cap \{(1,R)\times(s,T)\}$ with $s \in (0,T)$. Applying comparison principle, we have

$$w^{\pm}(x,t) \geq 0$$

for all $(x,t) \in \{(-R,-1)\times(s,T)\} \cap \{(1,R)\times(s,T)\}$ with $s \in (0,T)$. This implies (4.7). Thus the proof is completed.

Lemma 4.3 *The solution $\psi_{\varepsilon,\rho}$ of (4.1)-(4.3) satisfies*

$$|\nabla\psi_{\varepsilon,\rho}(x,t)| \leq C_2 e^{-R} \tag{4.8}$$

for all $(x,t) \in \partial B_R \times (s,T)$ with $s \in (0,T)$, where C_1 is a positive constant depending only on θ and $N(s,T)$.

Proof We consider the functions

$$z^{\pm}(x,t) = \mp\psi_{\varepsilon,\rho}(x,t) + K_1 e^{-R}[e^{K_2(|x|-R)} - 1]$$

for all $(x,t) \in \{(-R,-1)\times(0,T)\} \cap \{(1,R)\times(0,T)\}$.

Clearly, we have

$$z^{\pm}(x,t) = 0$$

for $|x| = R$, and

$$z^{\pm}(x,T) = \mp\theta e^{-|x|} + K_1 e^{-R}[e^{K_2(|x|-R)} - 1] < 0$$

for $x \in B_R \setminus B_{R-1}$.

Using Lemma 4.2 we can choose K_1 and K_2 large enough such that

$$z^{\pm}(x,t) = \mp\psi_{\varepsilon,\rho}(x,t) + K_1 e^{-R}[e^{-K_2} - 1] < 0$$

for $|x| = R-1$. Clearly,

$$\frac{\partial z^{\pm}}{\partial t} + A_{\varepsilon,\rho}\Delta z^{\pm} = K_1 e^{-R}\cdot e^{K_2(|x|-R)} A_{\varepsilon,\rho} K_2^2 > 0$$

for $R-1 \le |x| \le R$. Therefore, by maximum principle, we have

$$z^{\pm}(x,t) \le 0$$

for all $(x,t) \in [(-R,-(R-1)) \times (s,T)] \cap [(R-1,R) \times (s,T)]$ with $s \in (0,T)$, and

$$\frac{\partial z^{\pm}}{\partial x} \ge 0$$

on $\partial B_R \times (s,T)$ with $s \in (0,T)$. This implies

$$\mp \frac{\partial z^{\pm}}{\partial x} \ge -K_1 K_2 e^{-R}$$

on $\partial B_R \times (s,T)$ with $s \in (0,T)$. Thus the proof is completed.

Proof of the uniqueness in Theorem 1.1 We choose $\eta_\alpha \in C_0^\infty(B_R)$ such that

$$0 \le \eta_\alpha(x,t) \le 1 \quad \forall (x,t) \in B_R; \qquad \eta_\alpha(x,t) = 1 \quad \forall (x,t) \in B_{R-\alpha}$$

and

$$|\nabla \eta_\alpha(x,t)| \le C\alpha^{-1}, \qquad |\Delta \eta_\alpha(x,t)| \le C\alpha^{-2}$$

for all $x \in B_R$, where $0 < \alpha < R$ and C is a positive constant independent of R and α. Using Definition 1.1 we get

$$\begin{aligned}\int_{B_R} \eta_\alpha \psi_{\varepsilon,\rho} u(x,t)dx = & \int_{B_R} \eta_\alpha \psi_{\varepsilon,\rho} u(x,s)dx + \int_s^t \int_{B_R} \eta_\alpha [\psi_{\varepsilon,\rho}]_t u(x,\tau)dxd\tau \\ & + \int_s^t \int_{B_R} u^m \Delta[\eta_\alpha \psi_{\varepsilon,\rho}]dxd\tau \end{aligned} \tag{4.9}$$

and

$$\begin{aligned}\int_{B_R} \eta_\alpha \psi_{\varepsilon,\rho} v(x,t)dx = & \int_{B_R} \eta_\alpha \psi_{\varepsilon,\rho} v(x,s)dx + \int_s^t \int_{B_R} \eta_\alpha [\psi_{\varepsilon,\rho}]_t v(x,\tau)dxd\tau \\ & + \int_s^t \int_{B_R} v^m \Delta[\eta_\alpha \psi_{\varepsilon,\rho}]dxd\tau \end{aligned} \tag{4.10}$$

for a. e. s,t with $0 < s < t < T$, where $\psi_{\varepsilon,\rho}$ is a solution of (4.1)-(4.3).

By (4.9) and (4.10), we have

$$\begin{aligned} & \int_{B_R} \eta_\alpha \psi_{\varepsilon,\rho}(u(x,t) - v(x,t))dx \\ & = \int_{B_R} \eta_\alpha \psi_{\varepsilon,\rho}(u(x,t) - v(x,s))dx + \int_s^t \int_{B_R} \eta_\alpha [\psi_{\varepsilon,\rho}]_t (u(x,\tau) - v(x,\tau))dxd\tau \\ & \quad + \int_s^t \int_{B_R} (u^m - v^m)\Delta[\eta_\alpha \psi_{\varepsilon,\rho}]dxd\tau \end{aligned}$$

for a. e. s, t with $0 < s < t < T$. By (4.1), we have

$$\begin{aligned}&\int_{B_R} \eta_\alpha \psi_{\varepsilon,\rho}(u(x,t)-v(x,t))dx\\&\qquad=\int_{B_R} \eta_\alpha \psi_{\varepsilon,\rho}(u(x,t)-v(x,s))dx\\&\qquad\quad+\int_s^t\int_{B_R} \eta_\alpha[(u^m-v^m)-A_{\varepsilon,\rho}(u-v)]\Delta\psi_{\varepsilon,\rho}dxd\tau\\&\qquad\quad+\int_s^t\int_{B_R}(u^m-v^m)[2\nabla\eta_\alpha\nabla\psi_{\varepsilon,\rho}+\psi_{\varepsilon,\rho}\Delta\eta_\alpha]dxd\tau\end{aligned}$$

for a. e. s, t with $0 < s < t < T$. In addition, we also have

$$\begin{aligned}&\int_{B_R} \eta_\alpha \theta e^{-|x|}(u(x,T)-v(x,T))dx\\&\qquad=\int_{B_R} \eta_\alpha \psi_{\varepsilon,\rho}(u(x,s)-v(x,s))dx\\&\qquad\quad+\int_s^t\int_{B_R} \eta_\alpha[(u^m-v^m)-A_{\varepsilon,\rho}(u-v)]\Delta\psi_{\varepsilon,\rho}dxd\tau\\&\qquad\quad+\int_s^T\int_{B_R}(u^m-v^m)[2\nabla\eta_\alpha\nabla\psi_{\varepsilon,\rho}+\psi_{\varepsilon,\rho}\Delta\eta_\alpha]dxd\tau\end{aligned}\tag{4.11}$$

for a. e. s, t with $0 < s < t < T$.

By Lemma 4.1, we compute

$$\begin{aligned}&\int_s^T\int_{B_R} \eta_\alpha[(u^m-v^m)-A_{\varepsilon,\rho}(u-v)]\Delta\psi_{\varepsilon,\rho}dxd\tau\\&\quad=\int_s^T\int_{B_R}\eta_\alpha(u-v)(A_\varepsilon-A_{\varepsilon,\rho})\Delta\psi_{\varepsilon,\rho}dxd\tau\\&\quad\quad-\varepsilon\int_s^T\int_{B_R}\eta_\alpha(u-v)\Delta\psi_{\varepsilon,\rho}dxd\tau\\&\quad\le\left\{\int_s^T\int_{B_R}[\eta_\alpha(u-v)\Delta\psi_{\varepsilon,\rho}]^2dxd\tau\right\}^{1/2}\cdot\left\{\int_s^T\int_{B_R}|A_\varepsilon-A_{\varepsilon,\rho}|^2dxd\tau\right\}^{1/2}\\&\quad\quad+\varepsilon\int_s^T\left\{\int_{B_R}|\eta_\alpha(u-v)|^2dxd\tau\right\}^{1/2}\cdot\int_s^T\left\{\int_{B_R}|\Delta\psi_{\varepsilon,\rho}|^2dxd\tau\right\}^{1/2}\\&\quad\le C\varepsilon^{-1/2}\left\{\int_s^T\int_{B_R}|A_\varepsilon-A_{\varepsilon,\rho}|^2dxd\tau\right\}^{1/2}+C\varepsilon^{1/2},\end{aligned}$$

which implies that

$$\begin{aligned}&\int_s^T\int_{B_R}\eta_\alpha\left[(u^m-v^m)-A_{\varepsilon,\rho}(u-v)\right]\Delta\psi_{\varepsilon,\rho}dxd\tau\\&\qquad\le C_3\varepsilon^{-1/2}\left\{\int_s^T\int_{B_R}|A_\varepsilon-A_{\varepsilon,\rho}|^2dxd\tau\right\}^{1/2}+C_3\varepsilon^{1/2},\end{aligned}\tag{4.12}$$

where C_3 is a positive constant independent of ε and ρ.

Using Lemma 4.3 we compute

$$\int_s^T \int_{B_R} (u^m - v^m)\left[2\nabla\eta_\alpha \nabla\psi_{\varepsilon,\rho} + \psi_{\varepsilon,\rho}\Delta\eta_\alpha\right] dxd\tau$$

$$= \int_s^T \int_{B_R} (u^m - v^m)[2\nabla\eta_\alpha \nabla\psi_{\varepsilon,\rho}]dxd\tau + \int_s^T \int_{B_R} (u^m - v^m)[\psi_{\varepsilon,\rho}\Delta\eta_\alpha]dxd\tau$$

$$\leq C\alpha^{-1} \int_s^T \int_{B_R \backslash B_{R-\alpha}} |\nabla\psi_{\varepsilon,\rho}|dxd\tau + C\alpha^{-2} \int_s^T \int_{B_R \backslash B_{R-1}} |\psi_{\varepsilon,\rho}|dxd\tau$$

$$\leq C \sup_{(x,t)\in[B_R\backslash B_{R-1}]\times(s,T)} |\nabla\psi_{\varepsilon,\rho}| + C\alpha^{-1} \sup_{(x,t)\in[B_R\backslash B_{R-1}]\times(s,T)} |\psi_{\varepsilon,\rho}|,$$

which implies that

$$\int_s^T \int_{B_R} (u^m - v^m)[2\nabla\eta_\alpha \nabla\psi_{\varepsilon,\rho} + \psi_{\varepsilon,\rho}\Delta\eta_\alpha]dxd\tau$$

$$\leq C_4 \sup_{(x,t)\in[B_R\backslash B_{R-1}]\times(s,T)} |\nabla\psi_{\varepsilon,\rho}| + C_4\alpha^{-1} \sup_{(x,t)\in[B_R\backslash B_{R-1}]\times(s,T)} |\psi_{\varepsilon,\rho}|, \tag{4.13}$$

where C_4 is a positive constant independent of R, ε and ρ. By Lemma 4.1-Lemma 4.3, there exists a subsequence $\{\psi_{\varepsilon,\rho_i}\}$ of $\{\psi_{\varepsilon,\rho}\}$ such that

$$\psi_{\varepsilon,\rho_i}(\cdot,t) \to \psi_\varepsilon(\cdot,t) \tag{4.14}$$

as $\rho_i \to 0^+$ in $C(B_R)$, where ψ_ε satisfies

$$|\psi_\varepsilon(x,t)| \leq 1 \quad (x,t) \in B_R \times (0,T) \tag{4.15}$$

and

$$\int_{B_R} |\nabla\psi_\varepsilon(x,t)|^2 dx \leq C_0 \quad t \in (0,T). \tag{4.16}$$

By (4.15) and (4.16), there exists a subsequence $\{\psi_{\varepsilon_j}\}$ of $\{\psi_\varepsilon\}$ such that

$$\psi_{\varepsilon_j}(\cdot,t) \to \psi_R(\cdot,t) \tag{4.17}$$

as $\varepsilon_j \to 0^+$ in $C(B_R)$, where ψ_R satisfies

$$|\psi_R(x,t)| \leq 1 \quad (x,t) \in B_R \times (0,T) \tag{4.18}$$

and

$$\int_{B_R} |\nabla\psi_R(x,t)|^2 dx \leq C_0 \quad t \in (0,T). \tag{4.19}$$

By (4.18) and (4.19), there exists a subsequence $\{\psi_{R_k}\}$ of $\{\psi\}$ such that

$$\psi_{R_k}(\cdot,t) \to \psi(\cdot,t) \tag{4.20}$$

as $R_k \to +\infty$ in $C_{loc}(\mathbb{R})$, where ψ satisfies

$$|\psi(x,t)| \le 1 \quad (x,t) \in \mathbb{R} \times (0,T) \tag{4.21}$$

and

$$\int_{\mathbb{R}} |\nabla \psi(x,t)|^2 dx \le C_0 \quad t \in (0,T). \tag{4.22}$$

Combining (4.12)-(4.13) with (4.11) we conclude that

$$\begin{aligned}
&\int_{B_R} \eta_\alpha \psi_{\varepsilon,\rho}(u(x,t)-v(x,t))dx \\
&\qquad = \int_{B_R} \eta_\alpha \psi_{\varepsilon,\rho}(u(x,s)-v(x,s))dx \\
&\qquad + C_3 \varepsilon^{-1/2} \left\{ \int_s^T \int_{B_R} |A_\varepsilon - A_{\varepsilon,\rho}|^2 dx d\tau \right\}^{1/2} + C_3 \varepsilon^{1/2} \\
&\qquad + C_4 \sup_{(x,t)\in[B_R\setminus B_{R-1}]\times(s,T)} |\nabla \psi_{\varepsilon,\rho}| + C_4 \alpha^{-1} \sup_{(x,t)\in[B_R\setminus B_{R-1}]\times(s,T)} |\psi_{\varepsilon,\rho}|.
\end{aligned}$$

Letting $\rho = \rho_i \to 0+$ and $\varepsilon = \varepsilon_j \to 0^+$ and using (4.14) and (4.17), we get

$$\begin{aligned}
\int_{B_R} \theta e^{-|x|}(u(x,T)-v(x,T))dx \le & \int_{B_R} \psi_R(u(x,s)-v(x,s))dx \\
\le & C_4 e^{-R} + C_4 e^{-R}.
\end{aligned}$$

Letting $R = R_k \to +\infty$ and using (4.20) we get

$$\int_{\mathbb{R}} \theta e^{-|x|}(u(x,T)-v(x,T))dx \le \int_{\mathbb{R}} \psi(u(x,s)-v(x,s))dx.$$

Leting $s \to 0^+$ we have

$$\int_{\mathbb{R}} \theta e^{-|x|}(u(x,T)-v(x,T))dx \le 0$$

for all $\theta \in C_0^\infty(\mathbb{R})$ with $|\theta| \le 1$. This implies that

$$\int_{\mathbb{R}} e^{-|x|}|u(x,T)-v(x,T)|dx \le 0$$

for a.e. $T \in (0,+\infty)$. Therefore, we have

$$u(x,t) = v(x,t)$$

for a. e. $(x,t) \in Q_T$. Thus the proof is completed.

5. Proof of Theorem 1.2

In this section we shall prove Theorem 1.2.

Proof of Theorem 1.2 By Theorem 2.2, there exist two positive constants $\beta \in (0,1)$ and C independent of ε such that

$$|u_\varepsilon(x_1,t) - u_\varepsilon(x_2,t)| \leq C|x_1 - x_2|^\beta$$

for all $x_i \in \mathbb{R}(i=1,2)$ and all $t \in (0,+\infty)$. Letting $\varepsilon = \varepsilon_j \to 0^+$ and using (3.3) we get

$$|u(x_1,t) - u(x_2,t)| \leq C|x_1 - x_2|^\beta$$

for all $x_i \in \mathbb{R}(i=1,2)$ and all $t \in (\tau,+\infty)$. Thus, by the proof of Theorem 1.1, the proof is completed.

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